Math. J. Okayama Univ. **42** (2000), 1-5

SOME CONDITIONS FOR SOLUBILITY

JARADEN JEHAD

Abstract. In this paper some new conditions are given under which a finite group is soluble.

1. Introduction

In the paper [1] the author introduced the concept of c-normal subgroups and using the c-normality of maximal subgroups he gave some conditions of solubility for finite groups. As a continuation of [1], in this paper we consider the concept of c-subnormality and we give some new conditions under which a finite group is soluble.

2. Preliminaries

All groups considered in this paper are finite. Our definitions and notations are taken mainly from [2].

Remind [1] that a subgroup H of a group G is said to be c-normal in G if there exists a normal subgroup T of G such that $TH = G$ and $T \cap H \subseteq H_G$.

Definition . Let H be a subgroup of a group G. We say that H is *c-subnormal* in G if there exists a subnormal subgroup T of G such that $HT = G$ and $H \cap T \subseteq H_G$.

We shall need the following well known facts about subnormal subgroups (see [3] or also the section 14 in [2, A]).

Lemma 1. *Let H be a subgroup of a group G.*

(a) *If H is subnormal in G and $T \leq G$, then $H \cap T$ is a subnormal subgroup of T;*
(b) *If $N \trianglelefteq G$ and H is subnormal in G, then HN/N is subnormal in G/N.*

Lemma 2. *Let L be a minimal normal subgroup of a group G and T be a subnormal subgroup of G. Then $L \subseteq N_G(T)$.*

3. Some new conditions for solubility of groups

Lemma 3. *Let $K \trianglelefteq G$, $H \leq G$. Assume that $K \subseteq H$. Then*

$$(H/K)_{G/K} = H_G/K.$$

Proof. Let $(H/K)_{G/K} = T/K$ and $L = H_G$. Then $T/K \subseteq H/K$, and so $T \subseteq H_G$. On the other hand $L/K \trianglelefteq G/K$, $L/K \subseteq H/K$, and so $L/K \subseteq T/K$. Hence $T/K = L/K$. The lemma is proved. □

Lemma 4. *Let G be a group, H be some subgroup of G. Then the following statements are true:*

(1) *If H is c-subnormal in G and $H \leq K \leq G$, then H is c-subnormal in K;*
(2) *Let $K \trianglelefteq G$ and $K \leq H$. Then H is c-subnormal in G if and only if H/K is c-subnormal in G/K.*

Proof. Let H be c-subnormal in G and let $H \leq K \leq G$. We shall prove that H is c-subnormal in K. By hypothesis there is a subnormal subgroup T of G such that $TH = G$ and $T \cap H \subseteq H_G$. Let $\bar{T} = T \cap K$. Then by Lemma 1 $\bar{T}$ is a subnormal subgroup of K. Applying the Dedekind Law we obtain $K = K \cap HT = H(K \cap T) = H\bar{T}$. On the other hand

$$\bar{T} \cap H = K \cap (T \cap H) \subseteq K \cap H_G \subseteq K \cap H_K \subseteq H_K,$$

by Lemma 1. Hence H is c-subnormal in K.

Now we prove (2). Assume that H is c-subnormal in G. And let T be a subnormal subgroup of G such that $TH = G$ and $T \cap H \subseteq H_G$. By Lemma 1 TK/K is a subnormal subgroup of G/K. Clearly $(TK/K)(H/K) = G/K$. Applying the Dedekind Law we have

$$H \cap TK = K(H \cap T) \subseteq KH_G \subseteq H_G.$$

Hence by Lemma 3

$$(TK/K) \cap (H/K) \subseteq (H/K)_{G/K}.$$

Thus H/K is c-subnormal in G/K.

Next suppose that H/K is c-subnormal in G/K and let T/K be a subnormal subgroup of G/K such that $(T/K)(H/K) = G/K$ and

$$(T/K) \cap (H/K) \subseteq (H/K)_{G/K}.$$

Then T is subnormal in G and evidently $TH = G$. Besides

$$(T/K) \cap (H/K) = (T \cap H)/K \subseteq H_G/K.$$

Thus $T \cap H \subseteq H_G$, and so H is c-subnormal in G. The lemma is proved. □

Theorem 1. *If each Sylow subgroup of a group G is c-subnormal in G, then G is soluble.*

Proof. Let P be a Sylow p-subgroup of G. Assume that $P_G \neq 1$ and let L be a minimal normal subgroup of G contained in P_G. By hypothesis P is c-subnormal in G. Hence by Lemma 4 the subgroup P/L is c-subnormal in G/L. Clearly P/L is a Sylow p-subgroup of G/L. Hence if P_1 is an arbitrary Sylow p-subgroup of G/L then $P_1 = (P/L)^x$ for some element $x \in G/L$. Thus P_1 is c-subnormal in G/L.

Now let $\tilde{Q}$ be a Sylow q-subgroup of G/L where $q \neq p$. And let Q be a Sylow q-subgroup of the group D where $D/L = \tilde{Q}$. Then Q is a Sylow q-subgroup of G. Hence by hypothesis there is a subnormal subgroup T of G such that $TQ = G$ and $T \cap Q \subseteq Q_G$. By Lemma 1 TL/L is a subnormal subgroup of G/L and clearly $(TL/L)\tilde{Q} = G/L$. We show that $(TL/L) \cap \tilde{Q} \subseteq \tilde{Q}_{G/L}$. Clearly $\tilde{Q} = QL/L$. Hence by Lemma 3 $\tilde{Q}_{G/L} = (QL)_G/L$. Thus we have only to show that $TL \cap QL = L(T \cap QL) \subseteq (QL)_G$. Suppose that $L \subseteq T$. Then $T \cap QL = L(Q \cap T) \subseteq LQ_G$, and so $L(T \cap QL) \subseteq LQ_G \subseteq (LQ)_G$. Let $L \nsubseteq T$. Since $TQ = G$, then every Sylow p-subgroup of T is a Sylow p-subgroup of G. Let P_1 be a Sylow p-subgroup of T. Then $P_1 = P^x$ for some $x \in G$, and so $L \subseteq P_1 \subseteq T$. This contradiction shows that $L \subseteq T$. Hence $(TL/L) \cap \tilde{Q} \subseteq \tilde{Q}_{G/L}$. Thus QL/L is c-subnormal in G/L. Therefore each Sylow subgroup of G/L is c-subnormal in G/L. By induction we may conclude that G/L is a soluble group, and so the group G is soluble as well.

Now assume that $P_G = 1$ for every Sylow subgroup P of the group G. Then by hypothesis there is a subgroup T of G such that $TP = G$ and $T \cap P = 1$. Then $|G| = |T||P|$, and so every Sylow subgroup of T is a Sylow subgroup of G. Hence every Sylow subgroup of T is c-subnormal in G. Hence by Lemma 4 every Sylow subgroup of T is c-subnormal in T. But $|T| < |G|$, and so by induction T is soluble. Let $\pi = \{p_1, p_2, \ldots, p_t\}$ be the set of all primes dividing the order $|G|$ of the group G.

We have shown that for each $p_i \in \pi$ the group G has a soluble Hall p_i'-subgroup. Using now results of the section 3 in [2, I] we see that G is a soluble group. □

Theorem 2. *A group G is soluble if and only if every its maximal subgroup M is c-subnormal in G.*

Proof. We shall prove that G is soluble. Assume that this is false and let G be a group of minimal order such that G is not soluble but every its maximal subgroup M is c-subnormal. Then the group G is not simple. Indeed if G is a simple group and M is a maximal subgroup of G, then by hypothesis M is c-subnormal in G. And there exists a subnormal subgroup T of G such that $MT = G$ and $M \cap T \subseteq M_G$. Since G is a simple group,

$M_G = 1$. Hence $T \neq G$. But then $T = 1$ and $TM = M \neq G$. This contradiction shows that G is not a simple group.

Let R be a minimal normal subgroup of G. And let M/R be a maximal subgroup of the group G/R. Then M is a maximal subgroup of G. Since M is c-subnormal in G, M/R is c-subnormal in G/R, by Lemma 4. But $|G/R| < |G|$. Hence by the choice of the group G we have to conclude that G/R is a soluble group and R is not an abelian group. If $R \subseteq \Phi(G)$, then R is nilpotent by Theorem 9.3 of [2, A]. Hence R is abelian, a contradiction. Therefore $R \nsubseteq \Phi(G)$. If the group G has at least two minimal normal subgroups R and L, then from above we have known that G/R and G/L are soluble groups. Hence $G \cong G/1 = G/L \cap R$ is a soluble group. This contradicts the choice of the group G. Hence R is the unique minimal normal subgroup of the group G.

Let p be a prime dividing the order $|R|$ of the group R. Let P be a Sylow p-subgroup of R. Then $G = RN_G(P)$ (see Remarks 6.3 in [2, A]). Clearly $N_G(P) \neq G$. Let M be a maximal subgroup of G such that $N_G(P) \subseteq M$. Then evidently R is not contained in M. We show that p does not divide the index $|G : M|$ of M in G. Indeed let P_1 be a Sylow p-subgroup of G such that $P \subseteq P_1$. Then by Theorem 6.4 in [2, A] $P_1 \cap R$ is a Sylow p-subgroup of R. But $P \subseteq P_1 \cap R$. Hence $P = P_1 \cap R$. Then $P \trianglelefteq P_1$, and so $P_1 \subseteq N_G(P)$. Thus the prime p does not divide $|G : M|$.

As M is c-subnormal in G, there is a subnormal subgroup T of G such that $MT = G$ and $T \cap M \subseteq M_G$. Since $R \nsubseteq M$ and R is the unique minimal normal subgroup of G, then $M_G = 1$, and so $T \cap M = 1$. Let L be a minimal subnormal subgroup of G contained in T. Let L^G be the normal closure of L in G. Then $L^G \neq 1$, and so $R \subseteq L^G$. Assume that $L \nsubseteq R$. Then by Lemma 1 $L \cap R$ is a subnormal subgroup of G and $1 \subseteq L \cap R \subseteq L$. Hence $L \cap R = 1$, since L is a minimal subnormal subgroup of G. By Lemma 2 $R \subseteq N_G(L)$. Hence $< L, R >= LR = L \times R$. But then $L \subseteq C_G(R)$. Since $C_G(R) \trianglelefteq G$ and R is the unique minimal normal subgroup of G, then $R \subseteq C_G(R)$. Therefore R is an abelian group. This contradiction shows that $L \subseteq R$. Since R is a minimal normal subgroup of G, $R = A_1 \times \ldots \times A_t$ where $A_1 \cong A_2 \cong \ldots \cong A_t \cong A$ where A is a simple non-abelian group. Hence $L \cong A$ (see Theorem 3.2 in [2, A]). Clearly p divides the order $|A|$ of the group A. Hence p divides the order $|L|$ of the group L. By Lagrange's theorem the order $|L|$ of the group L divides the order $|T|$ of the group T. Hence the prime p divides $|T|$. We have known that $G = TM$ and $T \cap M = 1$. Hence $|G| = |T||M| = |G : M||M|$, and so $|T| = |G : M|$. But the prime p does not divide the index $|G : M|$ of M in G. Hence p does not divide $|T|$. This contradiction shows that G is a soluble group.

Now let G be a soluble group. Let M be a maximal subgroup of G, and let H/M_G be a chief factor of G. Then evidently $HM = G$ and $H \cap M \subseteq M_G$. Hence M is c-subnormal in G. □

From the proof of Theorem 2, we have the following:

Corollary 1 ([1])**.** *A group G is soluble if and only if every its maximal subgroup M is c-normal in G.*

References

[1] Y. Wang: C-normality of groups and its properties, J. Algebra **180** (1996), 954-965.

[2] K. Doerk and T. Hawkes: Finite Soluble Groups, Walter de Gruyter, Berlin, New York, 1992.

[3] H. Wilandt: Eine Verallgemeinerung der envarianten Untergruppen, Math. Z. **45** (1939), 209-244.

[4] V. E. Deskins: On maximal subgroups, Proc. Symp Pure Math. **1** (1959), 100-104.

Jaraden Jehad
Al-Hussein Bin Talal University
Ma'an Jordan P.O.Box ((20))

(*Received April 13, 2000*)