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ON SEMIPRIME NOETHERIAN PI-RINGS

Dedicated to Professor Yukio Tsushima on his 60th birthday

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Abstract. Let R be a semiprime Noetherian PI-ring and $\mathcal{Q}(R)$ the semisimple Artinian ring of fractions of R. We shall prove the following conditions are equivalent: (1) the Krull dimention of R is at most one, (2) Any ring between R and $\mathcal{Q}(R)$ is again right Noetherian, (3) Let a, b be central regular elements of $\mathcal{Q}(R)$. Then the subring $R + aR[b]$ of $\mathcal{Q}(R)$ is right Noetherian.

Throughout this note all rings will have a unit element. Let R be a ring. We denote by $\dim(R)$ the Krull dimention of R, i.e., the supremum of the lengths of chains of distinct prime ideals in R, and by $Z(R)$ the center of R. Cauchon showed that a semiprime PI-ring with the ascending chain condition on two-sided ideals is left and right Noetherian (See [5, II, p.174]). Therefore we simply say semiprime Noetherian PI-rings for semiprime left and right Noetherian PI-rings. Let R be a semiprime Noetherian PI-ring. As is well known, the ring of fractions of R with respect to the set of central regular elements of R is a semisimple Artinian ring (See [5, II, p.174]). The main result of this note is the following:

Theorem. *Let R be a semiprime Noetherian* PI*-ring and $\mathcal{Q} = \mathcal{Q}(R)$ the semisimple Artinian ring of fractions of R. Then the following conditions are equivalent:*

(1) $\dim(R) \leq 1$.
(2) *Any ring between R and $\mathcal{Q}(R)$ is again right Noetherian.*
(3) *Let a, b be central regular elements of $\mathcal{Q}(R)$. Then the subring $R + aR[b]$ of $\mathcal{Q}(R)$ is right Noetherian.*

Remark. Let R and $\mathcal{Q}(R)$ be as in the above Theorem, and let T be a ring between R and $\mathcal{Q}(R)$. If T satisfies the conditions of the above theorem, then T is semiprime, therefore T is left and right Noetherian.

Let S be a ring and R a subring of S. We say that S is an *extension* of R if $S = RS^R$, where $S^R = \{s \in S \ ; \ sr = rs \ \text{ for all } \ r \in R\}$ and that S is *integral* over R if each element s of S satisfies an equation of the form $s^n + r_1 s^{n-1} + r_2 s^{n-2} + \cdots + r_n = 0$, where $r_i \in R$ for all $i = 1, 2, \cdots, n$.

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Let R be a prime PI-ring and $\mathcal{Q}(R)$ the central simple ring of fractions of R. The ring obtained by adjoining to R all elements of $Z(\mathcal{Q}(R))$ which are integral over R is called centrally integral closure of R. The proof of Theorem is similar to that in the commutative case as given by Kaplansky [2,Theorem 93 and Exercise 20, p.64]. For the proof of Theorem we need several lemmas.

Lemma 1 ([6, Theorem 1]). *Let S be a* PI-*ring and R a subring of S. If S is an extension of R, i.e. $S = RS^R$ and integral over R, then the following hold;*

(1) *For any prime ideal P in R there exists a prime ideal $\mathcal{Q}$ in S with $\mathcal{Q} \cap R = P$ (lying over).*
(2) *For any pair of prime ideals $P \subset P_1$ in R and a prime ideal $\mathcal{Q}$ in S with $\mathcal{Q} \cap R = P$, then there exists a prime ideal $\mathcal{Q}_1$ in S satisfying $\mathcal{Q} \subset \mathcal{Q}_1$ and $\mathcal{Q}_1 \cap R = P_1$ (going up).*
(3) *Two different primes in S with the same contraction in R cannot be comparable (incomparability).*

Lemma 2 ([7, Theorem 3]). *If R is a prime* PI-*ring and integral over an integrally closed central subring A of R, then $A \subset R$ has going down, i.e. , given prime ideals $P_0 \subset P$ in A and a prime ideal $\mathcal{Q}$ in R with $\mathcal{Q} \cap A = P$ then there exists a prime ideal $\mathcal{Q}_0$ in R satisfying $\mathcal{Q}_0 \subset \mathcal{Q}$ and $\mathcal{Q}_0 \cap A = P_0$.*

Lemma 3 ([6, Theorem 2]). *If R is a prime* PI-*ring with ascending chain condition on centrally generated ideals, then the coefficients of the reduced characteristic polynomial of any element of R are integral over R.*

As a corollary of Lemma 3 we shall prove the following lemma.

Lemma 4. *Let R be a Noetherian prime* PI-*ring with the central simple ring of fractions $\mathcal{Q}(R)$ and let R^* be its centrally integral closure, the ring obtained by adjoining to R all elements of $Z(\mathcal{Q}(R))$ which are integral over R, then:*

(1) *R^* is integral over R.*
(2) *R^* is integral over $Z(R^*)$.*
(3) *$Z(R^*)$ is integrally closed in its field of fractions.*

Proof. (1) Let $x \in R^*$ then there are finite elements $t_1, \cdots, t_k \in Z(\mathcal{Q}(R))$ which are integral over R and $x \in R[t_1, \cdots, t_k]$. Clearly, $R[t_1, \cdots, t_k]$ is a finitely generated R-module. Since R is Noetherian, x is integral over R. (2) If $x \in R^*$, then there are finite elements $t_1, \cdots, t_k \in Z(\mathcal{Q}(R))$ such that $x \in R[t_1, \cdots, t_k]$ as in the proof of (1). Let θ be a coefficient of the reduced characteristic polynomial of x. It is enough to show that θ is an element of $Z(R^*)$. Since $R[t_1, \cdots, t_k]$ is a Noetherian prime PI-ring, by

Lemma 3, θ is integral over $R[t_1, \cdots, t_k]$, hence $R[t_1, \cdots, t_k, \theta]$ is a finite R-module, therefore θ is integral over R. This shows that θ is an element of $Z(R^*)$. (3) Let t be an element of $Z(\mathcal{Q}(R))$ and integral over $Z(R^*)$. Then, as in (1), t is integral over $R[t_1, \cdots, t_k]$, where $t_1, \cdots, t_k \in Z(\mathcal{Q}(R))$ are integral over R, hence t is integral over R, therefore, by the definition of R^*, $t \in Z(R^*)$. $\square$

Lemma 5. *Let R be a* PI*-ring. Then R is a right Noetherian ring with* $\dim(R) = 0$ *if and only if R is a right Artinian ring.*

Proof. Suppose R is a right Noetherian ring with $\dim(R) = 0$. Then all its prime ideals are both minimal and maximal prime ideals and there are only finitely many such prime idals, say $M_1, M_2, \cdots, M_n$. We have $J(R) = M_1 \cap M_2 \cap \cdots \cap M_n$, where $J(R)$ is the Jacobson radical of R. Since $J(R)$ is nil, by Levitzki's theorem [1, Theorem 1.4.5], $J(R)$ is nillpotent and hence $(M_1 \cdots M_n)^k = 0$ for some k. Since R/M_i $(i = 1, 2, \cdots, n)$ are simple Artinian rings, we can refine the series of R-modules $R \supset M_1 \supset M_1M_2 \supset \cdots \supset M_1 \cdots M_n \supset \cdots \supset 0$ and then we have a composition series of the right R-module R. Thus R is a right Artinian ring. The converse is well known. $\square$

Lemma 6. *Let R be a semiprime Noetherian* PI*-ring with finite Krull dimention* $\dim(R)$. *Let a be a non-unit central regular element of R. Then* $\dim(R/aR) < \dim(R)$.

Proof. Since R is a semiprime Noetherian PI-ring, R has a finite set of minimal prime ideals, say $P_1, P_2, \cdots, P_n$. We show that the canonical image of a in the factor ring R/P_i is regular for each i. Suppose $\bar{a}$ is not regular in R/P_i for some i, where $\bar{a}$ denote the canonical image of a in R/P_i, then there is an element b in R such that $\bar{a}\bar{b} = 0$ and $\bar{b} \neq 0$ in R/P_i. Since $\bigcap_{j \neq i} P_j \not\subset P_i$, there is a non-zero central element $\bar{c}$ in R/P_i such that $\bar{c} \in \bigcap_{j \neq i} P_j - P_i$ by [4, Theorem 2]. Then $abc \in \bigcap P_i = 0$ so $bc = 0$, implying $\bar{b} = 0$, a contradiction. Now, let $\mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_n$ be a chain of prime ideals of R such that $a \in \mathcal{Q}_0$, then $\mathcal{Q}_0$ is not a minimal prime ideal of R. This shows that $\dim(R/aR) < \dim(R)$. $\square$

Now we shall prove Theorem:

Proof of Theorem. (1) implies (2). Let T be a ring between R and $\mathcal{Q}$ and let X be a right ideal of T. Then $X\mathcal{Q} \oplus Y = \mathcal{Q}$ for some right ideal Y of $\mathcal{Q}$. Since $\mathcal{Q}$ is the central localization of R, we have $xa^{-1} + ya^{-1} = 1$, for suitable elements $a \in Z(R)$, $x \in X \cap R$ and $y \in Y \cap R$. Hence the right ideal $X \oplus (Y \cap T)$ of T contains a regular central element a of R. It suffices to show that T/aT is a finite generated right R-module. Since the Krull dimention of R is at most 1, by Lemma 6 it follows that the Krull dimention of R/aR is 0. By Lemma 5, R/aR is right Artinian. Thus the descending chain of ideals $\{a^mT \cap R + aR | m = 1, 2, \cdots\}$ in R becomes stable, say at $a^nT \cap R + aR$. For this n we assert that $T/aT \subseteq (a^{-n}R + aT)/aT$. Let $t = zc^{-1}$ be an element of T, where $z \in R$ and c is a regular central element of R. Since R/cR is right Artinian, then $a^kR \subseteq a^{k+1}R + cR$, for some k, so $a^kt = a^{k+1}rt + cr_1t$ $(r, r_1 \in R)$, whence $t \in a^{-h}R + aT$ for some h. Let us suppose that the equation $t \in a^{-h}R + aT$ has been arranged with the smallest possible value of h. We shall prove that $h \leq n$. Suppose $h > n$. We write $t = a^{-h}u + at_1$, $u \in R$ and $t_1 \in T$. Then $u = a^h(t - at_1) \in a^hT \cap R \subseteq a^{h+1}T \cap R + aR$. So we can write $u = a^{h+1}t_2 + au_1$, where $t_2 \in T$ and $u_1 \in R$. Thus we have $t = a^{-(h-1)}u_1 + a(t_1 + t_2)$. This contradicts the minimal choice of h.

(2) implies (3). Trivial.

(3) implies (1). Assume first that R is prime. Let R^* be the centrally integral closure of R, then $\dim(R^*) = \dim(R)$ by Lemma 4 (1) and [6, Corollary 1, p.247]. It suffice to show that $\dim(R^*) = 1$. By Lemma 4 (3), R^* is integral over $Z(R^*)$ and $Z(R^*)$ is integrally closed. Therefore, we may assume that R is integral over $Z(R)$ and $Z(R)$ is integrally closed. If $\dim(R) > 1$ then there exist prime ideals $0 \neq \mathcal{Q} \subset P$ in R. By Lemma 1 (3), $Z(R) \cap \mathcal{Q} \subset Z(R) \cap P$ are distinct primes in $Z(R)$. Take $x \in Z(R) \cap \mathcal{Q}$, $x \neq 0$. Since R is Noetherian, there are only finitely many prime ideals minimal over xR, say $P_1, \cdots, P_n$ by [3, Corollary 2.4, p.108]. By Lemma 2, $P_k \cap Z(R)$ is a minimal prime ideal over x in $Z(R)$ and thus $P \cap Z(R) \not\subset P_k \cap Z(R)$ for any k. Hence we have $P \cap Z(R) \not\subset (P_1 \cap Z(R)) \cup \cdots \cup (P_n \cap Z(R))$ by [2, Theorem 81]. Take $y \in P \cap Z(R)$ with $y \notin (P_1 \cap Z(R)) \cup \cdots \cup (P_n \cap Z(R))$. Let $T = R + xR[y^{-1}]$ and $I = xR[y^{-1}]$. We assert that I is not a finitely generated ideal in T. If I is a finitely generated ideal then I is generated by xy^{-i} for some i. Then we have $xR[y^{-1}] = xy^{-i}T$, and so $R[y^{-1}] = T$. Let $y^{-1} = a + xby^{-j}$, $a, b \in R$, $j \geq 1$. Then $y^{j-1}(1 - ay) = xb \in xR \subset P_k$ for all k. Therefore $1 - ay \in P_k$ for all k, and then, we have $(1 - ay)^m \in xR$ for some m. Expanding $(1 - ay)^m = xc$, $c \in R$, we have the relation $1 = yd + xc$ where $d \in R$, which leads a contradiction,$1 \in P$.

Suppose R is semiprime. Since R is Noetherian, there are finitely many minimal prime ideals, say $\mathcal{Q}_1, \cdots, \mathcal{Q}_r$. Put $R_i = R/\mathcal{Q}_i$. We show that the Krull dimention of R_i is at most 1 for any i. Let $\mathcal{Q}(R_i)$ $(i = 1, 2, \cdots, r)$ be the ring of fractions of R_i. Suppose $\dim(R/\mathcal{Q}_k) > 1$ for some k. By the above argument, there are regular central elements $\bar{x}, \bar{y}$ of R_k such that the subring $R_k + \bar{x}R_k[\bar{y}^{-1}]$ in $\mathcal{Q}(R_k)$ is non-Noetherian, where $\bar{x}, \bar{y}$ are the canonical image of x, y $(\in R)$ in R_k. Using the injections

$R \to \prod_{i=1}^{r} R_i \subseteq \prod_{i=1}^{r} \mathcal{Q}(R_i)$, the ring $\prod_{i=1}^{r} \mathcal{Q}(R_i)$ is considered as the ring of fractions of R (See [5, I, Theorem 3.2.27 and II, p.174]). Let $x_k = (1+\mathcal{Q}_1, \cdots, x+\mathcal{Q}_k, \cdots, 1+\mathcal{Q}_r)$, $y_k = (1+\mathcal{Q}_1, \cdots, y+\mathcal{Q}_k, \cdots, 1+\mathcal{Q}_r)$ be the elements of $\prod_{i=1}^{r} \mathcal{Q}(R_i)$. By the hypotheses, the subring $R + x_k R[y_k^{-1}]$ of $\prod_{i=1}^{r} \mathcal{Q}(R_i)$ is Noetherian and any homomorphic image of $R + x_k R[y_k^{-1}]$ is Noetherian, hence the subring $R_k + \bar{x}R[\bar{y}^{-1}]$ of $\mathcal{Q}(R_k)$ is Noetherian, which is a contradiction. This completes the proof. $\square$

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