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NONLINEAR SEMIGROUPS ANALYTIC ON SECTORS

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0. Introduction.

Of concern in this paper is a typical class of nonlinear evolution equations of the form

(EE) $$(d/d\xi)u(\xi) = Au(\xi), \qquad \xi \in \Sigma \subset \mathbb{C}$$

in a complex Banach space $(X, |\cdot|)$. Here the Banach space X stands for an appropriate space of complex-valued functions and A represents a possibly nonlinear differential operator in X. Also, Σ denotes an open sector in the complex plain $\mathbb{C}$

$$\Sigma = \{se^{i\phi} + te^{i\psi} : s, t > 0\},$$

where $-\pi/2 < \phi < 0 < \psi < \pi/2$. The closure of Σ is denoted by Σ^*, namely,

$$\Sigma^* = \{se^{i\phi} + te^{i\psi} : s, t \geq 0\}.$$

We here discuss the generation and characterization of a semigroup of the solution operators to (EE) which provide solutions analytic in Σ.

In this paper the totality of admissible initial data is denoted by D and is classified in terms of a lower semicontinuous functional $\Phi : X \to [0, +\infty]$ in such a way that $D \subset D(\Phi) = \{v \in X : \Phi(v) < +\infty\}$ and $D = \bigcup_{\alpha>0} D_\alpha$, where

$$D_\alpha = \{v \in D : \Phi(v) \leq \alpha\}, \quad \alpha > 0.$$

Evolution equation (EE) is considered under the initial condition

(IC) $$\lim_{\xi\in\Sigma,\ \xi\to 0} u(\xi) = v \in D$$

and the solution $u(\xi; v)$ to the evolution problem (EE)-(IC) is sought as an analytic function in Σ. Now the growth of the analytic solution $u(\xi; v)$ is restricted by means of the nonnegative-valued function $\Phi(u(\xi; v))$ on Σ^*. Also, the solution operators $W(\xi)$, $\xi \in \Sigma^*$, are constructed on D in such

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a way that to each $\alpha > 0$ and each $\tau > 0$ there corresponds a constant $M(\alpha, \tau) > 0$ such that

$$|S(\xi)v - S(\xi)w| \leq M(\alpha, \tau)|v - w| \tag{LL}$$

for $v, w \in D_\alpha$ and $\xi \in \Sigma^*$ with $|\xi| \leq \tau$.

The closed sector Σ^* is an additive semigroup in the sense that 0 is the apex of Σ^* and $\eta, \xi \in \Sigma^*$ imply $\eta + \xi \in \Sigma^*$. Therefore the solution operators $\{W(\xi) : \xi \in \Sigma^*\}$ forms a nonlinear operator semigroup on D such that $u(\xi; v) = W(\xi)v$ is a solution of (EE)-(IC) and analytic in Σ.

Nonlinear semigroups analytic on sectors as mentioned above was first discussed by Hayden and Massey Ⅲ [3]. Our results are concerned with solutions analytic on sectors. These results can be extended to the case of solutions analytic in neighborhoods of the real half line. Early work in this direction was done in Massey Ⅲ [5], Ōuchi [8] and Promislow [9]. An extension in this direction and generation of real analytic semigroups will be treated in the forthcoming paper [6].

This paper is organized as follows: Section 1 is devoted to the study of semigroups analytic in Σ. In this section a class of analytic semigroups on Σ is introduced and the characterization of those analytic semigroups is discussed in terms of three different types of conditions. Section 2 is concerned with the proof of the above characterization theorem. A characteristic feature of this argument is to apply Morera's theorem for vector-valued analytic functions. Application of Morera's theorem to nonlinear analytic semigroups was first made by K. Furuya [1]. It should be noted that the class D of initial-data is subdivided by means of the lower semicontinuous functional Φ, and that our analytic semigroup is Lipschitz continuous on each D_α.

1. Analytic semigroups on sectors.

In this section we introduce the notion of nonlinear semigroup on a subset D of X which provides analytic solutions of the evolution problem (EE)-(IC) and discuss the analyticity in time of the semigroups. Here A is a possibly nonlinear operator in a complex Banach space $(X, |\cdot|)$, $D \equiv \overline{D(A)}$ stands for a given class of initial data in X, and ϕ, ψ are an arbitrary but fixed pair of angles satisfying

$$-\pi/2 < \phi < 0 < \psi < \pi/2. \tag{1}$$

The union of Σ and its boundary lines

$$\Gamma_\phi \equiv \{se^{i\phi} : s \geq 0\} \quad \text{and} \quad \Gamma_\psi \equiv \{te^{i\psi} : t \geq 0\}$$

is its closure Σ^*. In what follows, the space of continuous functions from a subset Δ of $\mathbb{C}$ into D is denoted by $\mathrm{C}(\Delta; D)$.

In order to introduce a class of operators which are Lipschitz continuous in a local sense on D, we employ a lower semicontinuous functional $\Phi : X \to [0, +\infty]$ such that $D \subset \{v \in X : \Phi(v) < +\infty\}$. For each $\alpha > 0$ we put

$$D_\alpha = \{v \in D : \Phi(v) \le \alpha\} \tag{2}$$

and the class D of initial data is subdivided in the sense that $D = \bigcup_{\alpha>0} D_\alpha$. Throughout this paper we assume that for each $\alpha > 0$ there exists $\beta > 0$ such that $D_\alpha \subset \overline{D(A) \cap D_\beta}$.

Our aim in this section is twofold. First, we introduce a class of operator semigroups $\mathrm{W} = \{W(\xi)\}$ on D such that given an initial-value $v \in D$, $u(\xi; v) \equiv W(\xi)v$ is a unique analytic solution of (EE)-(IC) on Σ. Secondly, we discuss the characterization of such semigroup on D by means of the boundary semigroups defined as below.

Our class of nonlinear analytic semigroups is formulated as follows:

Definition. A one-parameter family $\mathrm{W} \equiv \{W(\xi) : \xi \in \Sigma^*\}$ of nonlinear operators from D into itself is called *a semigroup in the class* $\mathrm{H}\,(D, \Sigma^*)$, if the following conditions hold:

(W1) For $v \in D$, $W(0)v = v$, $W(\cdot)v \in \mathrm{C}(\Sigma^*, D)$, and $W(\cdot)v$ is analytic in Σ.

(W2) For $v \in D$ and $\eta, \xi \in \Sigma^*$,

$$W(\eta + \xi)v = W(\eta)W(\xi)v.$$

(W3) For $\alpha, \tau > 0$ there exist $M_{\alpha,\tau} > 0$ and $\beta > 0$ such that

$$|W(\xi)v - W(\xi)w| \le M_{\alpha,\tau}|v - w| \quad \text{and} \quad \Phi(W(\xi)v) \le \beta$$

for $v, w \in D_\alpha$ and $\xi \in \Sigma^*$ with $|\xi| \le \tau$.

Let W be a semigroup in the class $\mathrm{H}\,(D, \Sigma^*)$. The operators on the boundary lines Γ_ϕ and Γ_ψ defined by

$$S(s) \equiv W(se^{i\phi}) \quad \text{and} \quad T(t) \equiv W(te^{i\psi}) \quad \text{for } s, t \ge 0 \tag{3}$$

play an important role in this paper. The one-parameter families $\mathrm{S} \equiv \{S(s) : s \ge 0\}$ and $\mathrm{T} \equiv \{T(t) : t \ge 0\}$ form semigroups of possibly nonlinear operators on D satisfying the three conditions below:

(L1) For each $v \in D$, $S(0)v = T(0)v = v$ and $S(\cdot)v, T(\cdot)v \in \mathrm{C}([0, \infty); D)$.

(L2) For $v \in D$ and $s, t \ge 0$, $S(s + t)v = S(t)S(s)v$, and $T(s + t)v = T(t)T(s)v$.

(L3) For $\alpha, \tau > 0$ there exist $L_{\alpha,\tau} > 0$ and $\beta > 0$ such that both $S(s)$ and $T(t)$ map D into itself and

$$\begin{aligned} |S(s)v - S(s)w| \vee |T(t)v - T(t)w| &\leq L_{\alpha,\tau}|v - w| \\ \Phi(S(s)v) \vee \Phi(T(t)v) &\leq \beta \end{aligned}$$

for $v, w \in D_\alpha$ and $s, t \in [0, \tau]$, where $a \vee b = \max\{a, b\}$.

Definition. Given a semigroup W in the class $\mathrm{H}\,(D, \Sigma^*)$, two nonlinear semigroups S and T are defined by (3). These semigroups are called the *boundary semigroups* of W.

We then define an operator by

$$A_+v = \lim_{r \in \mathbb{R}, r \downarrow 0} r^{-1}(W(r)v - v), \tag{4}$$

whenever the limit exists in X. Hence the domain $D(A_+)$ is the set of all elements $v \in X$ for which the limits (4) exist. Since $W(\cdot)v$ is analytic in Σ for each $v \in D$, the complex derivative exists at each $\xi \in \Sigma$ and the identity

$$(d/d\xi)W(\xi)v = A_+W(\xi)v \tag{5}$$

holds for $\xi \in \Sigma$ and $v \in D$. This means that $D^\circ \equiv \{W(\xi)v : \xi \in \Sigma, v \in D\} \subset D(A_+)$. Now the set D° is dense in D by (W1) and hence A_+ is densely defined in D. In this sense the limit operator A_+ may be called the infinitesimal generator of W. Likewise, the infinitesimal generators of the boundary semigroups S and T are defined by

$$\begin{aligned} A_\phi v &= e^{-i\phi} \lim_{r \downarrow 0} r^{-1}(S(r)v - v), \\ A_\psi v &= e^{-i\psi} \lim_{r \downarrow 0} r^{-1}(T(r)v - v), \end{aligned} \tag{6}$$

respectively. In view of the definitions of S and T and equation (5), we make the following assumption:

(A) $D(A) = D(A_\phi) = D(A_\psi) \neq \emptyset$ and

$$A = A_\phi = A_\psi \quad \text{in } X,$$

where A is the possibly nonlinear operator in (EE).

Condition (A) is essential in the subsequent discussions. In order to characterize the structure of analytic semigroups in the class $\mathrm{H}\,(D, \Sigma^*)$ in terms of their boundary semigroups, we impose the following regularity assumptions on the boundary semigroups:

(D) For $v \in D(A)$, $S(\cdot)v$ and $T(\cdot)v$ are strongly absolutely continuous on bounded closed subintervals of $[0,\infty)$,

$$S(s)v \in D(A), \quad (d/ds)^{+}S(s)v = e^{i\phi}AS(s)v \quad \text{for a.e. } s,$$

$$T(t)v \in D(A), \quad (d/dt)^{+}T(t)v = e^{i\psi}AT(t)v \quad \text{for a.e. } t,$$

where $(d/ds)^{+}S(s)v$ and $(d/dt)^{+}T(t)v$ stand for the right-hand side strong derivatives of $S(\cdot)v$ at s and $T(\cdot)v$ at t, respectively.

Our characterization problem may be reformulated as follows: we first consider two evolution problems:

$$(\mathrm{EP};\phi)_{+} \qquad \begin{aligned} (d/ds)^{+}u(s) &= e^{i\phi}Au(s), \quad s > 0, \\ \lim_{s\downarrow 0} u(s) &= y \in D(A), \end{aligned}$$

$$(\mathrm{EP};\psi)_{+} \qquad \begin{aligned} (d/dt)^{+}v(t) &= e^{i\psi}Av(t), \quad t > 0, \\ \lim_{t\downarrow 0} v(t) &= z \in D(A). \end{aligned}$$

By condition (D) and (6), D-valued functions $u(\cdot) \equiv S(\cdot)y$ and $v(\cdot) \equiv T(\cdot)z$ provide solutions to the problems $(\mathrm{EP};\phi)_{+}$ and $(\mathrm{EP};\psi)_{+}$, respectively. Using the semigroups S and T and applying conditions (L1) through (L3), (A) and (D), we seek necessary and sufficient conditions under which the problem (EE)-(IC) admits solutions analytic in Σ and the solutions depend continuously upon initial-data. Although the relation between the operator A and the infinitesimal generator A_{+} of W can not be explicitly obtained in general, it is shown in the next section that given a pair of semigroups S and T on D satisfying (A) and (D), an analytic semigroup W in the class $\mathrm{H}(D,\Sigma^{*})$ can be constructed in such a way that $A \subset A_{+}$. In this case the analytic semigroup W is nothing but the family of a solution operators to (EE)-(IC).

We here need a lemma concerning weak-star derivatives of strongly absolutely continuous functions.

Lemma. *Let $u : [a,b] \to X$ be strongly absolutely continuous. Then:*

(a) *There exists an X^{**}-valued, weakly-star integrable function $\nu(\cdot)$ on $[a,b]$ such that*

$$\langle u(t) - u(s), f\rangle = \int_s^t \langle \nu(r), f\rangle \, dr$$

for $a \le s \le t \le b$ and $f \in X^{*}$.

(b) $|\nu(\cdot)|$ *is integrable over* $[a, b]$ *and the total variation of* u *is given by* $TV[u] = \int_a^b |\nu(r)|\, dr$.

(c) *If in particular,* u *has a strong right-hand side derivative* $(d^+/dt)u(t)$ *at a.e.* $t \in [a, b]$, *then* $|(d^+/dt)u(t)| = |\nu(t)|$ *for a.e.* $t \in [a, b]$ *and* $TV[u] = \int_a^b |(d^+/dt)u(t)|\, dt$.

This result is obtained in a more general setting, as shown in Hashimoto and Oharu [4]. The main point here is that for a strongly absolutely continuous function u, the strong right-derivative is integrable over $[a, b]$ and its L^1-norm gives the total variation of u. We are now in a position to state the main theorem of this paper.

Theorem. *Let* S *and* T *satisfy* (L1) *through* (L3), (A) *and* (D). *Then the following three conditions are equivalent:*

(I) *There is an analytic semigroup* W *such that its boundary semigroups are* S *and* T.

(II) *For* $v \in D(A)$, *the function* $T(t)S(s)v$ *is continuously differentiable with respect to* s, $t > 0$, *and* $T(t)S(s)v = S(s)T(t)v$ *for* $s, t \geq 0$.

(III) *For* $v \in D(A)$ *there exists a null set* $\mathrm{N} \subset [0, \infty)$ *such that for each pair of numbers* $s_0, t_0 \in (0, \infty) \setminus \mathrm{N}$ *and for each pair of functions* $g, h : [0, 1] \to D_\gamma$ *for some* $\gamma > 0$ *satisfying*

$$g(\eta) = T(t_0)v + e^{i\psi}\eta AT(t_0)v + o(\eta) \quad \text{as } \eta \downarrow 0,$$

$$h(\mu) = S(s_0)v + e^{i\phi}\mu AS(s_0)v + o(\mu) \quad \text{as } \mu \downarrow 0,$$

we have

$$S(s)g(\eta) = S(s)T(t_0)v + e^{i\psi}\eta AS(s)T(t_0)v + o(\eta) \quad \text{for} \quad s > 0,$$

and

$$T(t)h(\mu) = T(t)S(s_0)v + e^{i\phi}\mu AT(t)S(s_0)v + o(\mu) \quad \text{for} \quad t > 0.$$

Remark. (a) For linear analytic semigroups, the corresponding result is found in a recent book Engel and Nagel [1], Theorem 4.6 on page 101.
(b) If D is convex, and if $S(s)$ and $T(t)$ are complex Fréchet differentiable at $v \in D(A)$ for $s, t \in [0, \infty)$, then condition (III) holds. See also Oharu [7].

2. Morera's theorem and analyticity.

A proof of the main theorem is given in five steps.

Step 1. We first demonstrate that for $\alpha > 0$ the functions $(s, v) \mapsto S(s)v$ and $(t, v) \mapsto T(t)v$ are continuous from $[0, \infty) \times D_\alpha$ into D. In fact, let $\tau > 0$ and $v \in D_\alpha$. Then, by (L1), $S(\cdot)v$ is strongly continuous on $[0, \infty)$. Hence

$$\begin{aligned} |S(s)v - S(\hat{s})\hat{v}| &\leq |S(s)v - S(\hat{s})v| + |S(\hat{s})v - S(\hat{s})\hat{v}| \\ &\leq L_{\alpha,\tau}|v - \hat{v}| + |S(\hat{s})v - S(s)v| \end{aligned}$$

so far as $v, \hat{v} \in D_\alpha$, $s, \hat{s} \in [0, \tau]$.

Step 2. Implication (III) $\Rightarrow$ (I): Set $W(\xi)v = T(t)S(s)v$ for $v \in D$ and $\xi = se^{i\phi} + te^{i\psi} \in \Sigma^*$. Then $W(\cdot)v$ is strongly continuous in $\xi \in \Sigma^*$. Now it is seen from Step 1 that (W1) and (W3) follow from conditions (L1) and (L3). Let v be an arbitrary element of $D(A)$. Then, by condition (D), there exists a null set N_1 such that

$$S(s + r)v = S(s)v + e^{i\phi} rAS(s)v + o(r) \tag{7}$$

for $s \in (0, \infty) \setminus \mathrm{N}_1$ and $r \geq 0$. For each $s \in (0, \infty) \setminus \mathrm{N}_1$ there exists a null set $\mathrm{N}_2(s)$ such that

$$\partial_t^+ T(t)S(s)v = e^{i\psi} AT(t)S(s)v \tag{8}$$

for $t \in [0, \infty) \setminus \mathrm{N}_2(s)$. By (III) and (7) we have

$$T(t)S(s + r)v = T(t)S(s)v + e^{i\phi} rAT(t)S(s)v + o(r)$$

for $t > 0$. Hence

$$e^{-i\phi} \partial_s^+ T(t)S(s)v = AT(t)S(s)v \tag{9}$$

for $(s, t) \in ((0, \infty) \setminus \mathrm{N}_1) \times (0, \infty)$ and $AT(t)S(s)v$ is strongly measurable with respect to $(s, t) \in (0, \infty) \times (0, \infty)$.

Let $0 < a < b$ and $0 < c < d$. We wish to show the identity

$$\begin{aligned} & e^{i\phi} \left[\int_a^b T(d)S(s)v \, ds - \int_a^b T(c)S(s)v \, ds \right] \\ = \; & e^{i\psi} \left[\int_c^d T(t)S(b)v \, dt - \int_c^d T(t)S(a)v \, dt \right]. \end{aligned} \tag{10}$$

Let $a \leq s \leq b$ and $c \leq t \leq d$. It follows from the Lemma that $\partial_s^+ T(t)S(s)v$ is integrable with respect to $(s, t) \in [a, b] \times [c, d]$. We infer from (8) and (9) that

$$
\begin{aligned}
T(t)S(b)v - T(t)S(a)v &= \int_a^b \partial_s^+ T(t)S(s)v\,ds \\
&= e^{i\phi} \int_a^b AT(t)S(s)v\,ds,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
T(d)S(s)v - T(c)S(s)v &= \int_c^d \partial_t^+ T(t)S(s)v\,dt \\
&= e^{i\psi} \int_c^d AT(t)S(s)v\,dt.
\end{aligned} \tag{12}
$$

By (L3) and the assertion of Step 1, $T(t)S(s)v$ is strongly continuous over $[0,\infty) \times [0,\infty)$. The relations (11) and (12) then imply the identities

$$
\begin{aligned}
\int_c^d T(t)S(b)v\,dt - \int_c^d T(t)S(a)v\,dt &= e^{i\phi} \int_c^d\!\!\int_a^b AT(t)S(s)v\,ds\,dt, \\
\int_a^b T(d)S(s)v\,ds - \int_a^b T(c)S(s)v\,ds &= e^{i\psi} \int_a^b\!\!\int_c^d AT(t)S(s)v\,dt\,ds.
\end{aligned}
$$

Using Fubini's theorem and combining the above relations, we get the desired identity (10). This implies

$$
\begin{aligned}
&\int_a^b \left[W(se^{i\phi} + de^{i\psi})v - W(se^{i\phi} + ce^{i\psi})v\right] d(se^{i\phi}) \\
= &\int_c^d \left[W(be^{i\phi} + te^{i\psi})v - W(ae^{i\phi} + te^{i\psi})v\right] d(te^{i\psi}).
\end{aligned}
$$

Now one can apply Morera's theorem to the continuous function $W(\cdot)v$ on Σ^* to conclude that $W(\cdot)v$ is analytic on the connected domain Σ. Next, let $\alpha > 0$ and $v \in D_\alpha$. Then, there exist $\beta > 0$ and a sequence $\{v_n\} \subset D(A) \cap D_\beta$ such that $v_n \to v$ as $n \to \infty$. Hence (W3) implies that for any $\tau > 0$ $W(\xi)v_n$ converges to $W(\xi)v$ uniformly for $\xi \in \Sigma^*$ with $|\xi| \leq \tau$. Hence $W(\cdot)v$ is analytic in Σ and S and T are its boundary semigroups. It now remains to show that W $\equiv \{W(\xi) : \xi \in \Sigma^*\}$ forms a D-valued semigroup on Σ^*. To this end, we set $V(\xi)v = S(s)T(t)v$ for $\xi = se^{i\phi} + te^{i\psi} \in \Sigma^*$. Then it is seen in the same way as above that $V(\cdot)v$ is continuous on Σ^* and analytic on Σ. Moreover,

$$W(se^{i\phi})v = V(se^{i\phi})v = S(s)v \quad \text{for } s \geq 0 \text{ and } v \in D.$$

Put $U(\xi)v = W(\xi)v - V(\xi)v \quad$ for $\xi \in \Sigma^*$. Then $U(se^{i\phi})v = 0 \quad$ for $s \geq 0$. By the reflection principle and $f(U(se^{i\phi})v) = 0 \quad$ for $s \geq 0$ and $f \in X^*$, X^* being the dual space of X, we have $U(\xi)v = 0 \quad$ for $\xi \in \Sigma^*$. This means that $S(s)T(t)v = T(t)S(s)v$ for $s, t \geq 0$ and $v \in D$. Hence W satisfies (W2).

Step 3. Implication (I) $\Rightarrow$ (II): (II) follows directly from (W2) and the definition of the boundary semigroups.

Step 4. Implication (I) $\Rightarrow$ (III): By (W2) and the definition of boundary semigroups, we have

$$W(\xi)v = T(t)S(s)v = S(s)T(t)v$$

for $\xi = se^{i\phi} + te^{i\psi} \in \Sigma^*$ and $v \in D$. Let $v \in D(A)$. Then, by (D), there exists a null set N_3 such that $T(t)v$ is differentiable at any point $t = t_0 \in (0, \infty) \setminus \mathrm{N}_3$. Let $t_0 \in (0, \infty) \setminus \mathrm{N}_3$ and let $g : [0, 1] \to D_\gamma$ be such that $g(t) = T(t_0)v + e^{i\psi}tAT(t_0)v + o(t)$ as $t \to 0$. Since $T(t + t_0)v = T(t_0)v + e^{i\psi}tAT(t_0)v + o(t)$ as $t \downarrow 0$, (L3) implies that

$$S(s)g(t) - S(s)T(t + t_0)v = o(t) \quad \text{as } t \downarrow 0. \tag{13}$$

Since (I) implies that $S(s)T(t + t_0)v = T(t + t_0)S(s)v$ and that it is continuously differentiable with respect to t, we have

$$\begin{aligned} \partial_t^+ S(s)T(t + t_0)v|_{t=0} &= \partial_t T(t + t_0)S(s)v|_{t=0} \\ &= e^{i\psi}AT(t_0)S(s)v \\ &= e^{i\psi}AS(s)T(t_0)v, \end{aligned}$$

and so

$$S(s)T(t + t_0)v = S(s)T(t_0)v + te^{i\psi}AS(s)T(t_0)v + o(t) \quad \text{as } t \downarrow 0. \tag{14}$$

Combining (13) with (14) gives

$$S(s)g(t) = S(s)T(t_0)v + te^{i\psi}AS(s)T(t_0)v + o(t) \quad \text{as } t \downarrow 0.$$

Thus the property of S in (III) is obtained. The corresponding property of T is obtained in the same way.

Step 5. Implication (II) $\Rightarrow$ (I): We define

$$W(\xi)v = T(t)S(s)v \quad \text{for } \xi = se^{i\phi} + te^{i\psi} \in \Sigma^* \text{ and } v \in D. \tag{15}$$

By definition, (II) implies (W2). (W1) follows from (L1) and the assertion of Step 1. (W3) follows from (15) and (L3). Thus $\mathrm{W} = \{W(\xi) : \xi \in \Sigma^*\}$ forms a D-valued semigroup. Next, let $v \in D(A)$ and $s, t > 0$. Then, by

(II), we have $\partial_t T(t)S(s)v = e^{i\psi}AT(t)S(s)v$ and

$$\begin{aligned}\partial_s T(t)S(s)v &= \partial_s S(s)T(t)v \\ &= e^{i\phi}AS(s)T_\alpha(t)v \\ &= e^{i\phi}AT(t)S(s)v.\end{aligned}$$

Therefore we have

$$e^{-i\phi}\partial_s T(t)S(s)v = e^{-i\psi}\partial_t T(t)S(s)v = AT(t)S(s)v.$$

Once this is obtained, one can show in the same way as in Step 2 that W is an analytic semigroup with the boundary semigroups S and T. The proof of the main theorem is now complete.

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