NAGATA CRITERION FOR SERRE’S
\((R_n)\) AND \((S_n)\)-CONDITIONS

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1. Introduction

Throughout the present paper, we assume that all rings are noetherian commutative rings.

First of all, we recall Serre’s \((R_n)\) and \((S_n)\)-conditions for a ring \(A\). These are defined as follows. Let \(n\) be an integer.

\((R_n)\) : If \(p \in \text{Spec}(A)\) and \(\text{ht}(p) \leq n\), then \(A_p\) is regular.

\((S_n)\) : \(\text{depth}(A_p) \geq \inf(n, \text{ht}(p))\) for all \(p \in \text{Spec}(A)\).

Let \(P\) be a property of local rings. For a ring \(A\) we put

\[ P(A) = \{ p \in \text{Spec}(A) \mid P \text{ holds for } A_p \} \]

and call it the \(P\)-locus of \(A\). The following statement is called the (ring-theoretic) Nagata criterion for the property \(P\), and we abbreviate it to \((NC)\).

\((NC)\) : If \(A\) is a ring and if \(P(A/p)\) contains a non-empty open subset of \(\text{Spec}(A/p)\) for every \(p \in \text{Spec}(A)\), then \(P(A)\) is open in \(\text{Spec}(A)\).

This statement was invented by Nagata in 1959. In algebraic geometry, there is a problem asking when the regular locus (that is, the non-singular locus) of a ring is open. He proposed the above criterion to consider this problem, and he proved that \((NC)\) holds for \(P = \text{regular}\) ([6]).

There are some other properties \(P\) for which \((NC)\) holds, for example, \(P = \text{Cohen-Macaulay}\) ([3], [4]), \(Gorenstein\) ([2], [4]), and complete intersection ([2]). On the other hand, it is easy to see that \((NC)\) holds for \(P = \text{(integral) domain, coprimary (a ring } A \text{ is called coprimary if } \#\text{Ass}(A) = 1), (R_0), (S_1), \text{ reduced, and normal. Moreover, as corollaries of these results, we easily see that the following proposition is true for } P = \text{Cohen-Macaulay ([3], [4]), Gorenstein ([4]), domain, coprimary, } (R_0), (S_1), \text{ and reduced.}\)

Let \(P\) be a property for which \((NC)\) holds. Then, for a ring \(A\) satisfying \(P\), the \(P\)-locus of a homomorphic image of \(A\) is open.

*The present paper contains part of the bachelor thesis of the author at Faculty of Integrated Human Studies, Kyoto University.
It is known that the properties “regular”, “Cohen-Macaulay”, “reduced”, and “normal” are described by using \((R_n)\) and \((S_n)\). Since \((NC)\) holds for each of these properties, we naturally expect that \((NC)\) may hold for \((R_n)\) and \((S_n)\) for every \(n \geq 0\). This is in fact true, and the main purpose of this paper is to give its complete proof.

Acknowledgement: The author should thank Professor Yuji Yoshino who gave him a lot of valuable advices.

2. (NC) for \((S_n)\)-condition

The following lemma should be referred to [3] §22.

Lemma 2.1. Let \(A\) be a domain, \(B\) an \(A\)-algebra of finite type, and \(M\) a finite \(B\)-module. Then there exists \(f(\neq 0) \in A\) such that \(M_f\) is \(A_f\)-free (where \(A_f\) is the localization of \(A\) with respect to the multiplicatively closed set \(\{1, f, f^2, \ldots\}\)).

Now we can prove the main result of this section.

Theorem 2.2. (NC) holds for \(P = (S_n)\).

Proof. We prove the theorem by induction on \(n\). It is easy to see that (NC) holds for \(P = (S_0)\) and \((S_1)\) respectively, hence we assume \(n \geq 2\) in the rest. Suppose that a ring \(A\) satisfies the assumption in (NC). We want to prove that the locus \(S_n(A)\) is open in \(\text{Spec}(A)\). Since \((S_n)\) implies \((S_{n-1})\), the locus \(S_{n-1}(A)\) is open in \(\text{Spec}(A)\) by induction hypothesis. Therefore we can write \(S_{n-1}(A) = \bigcup_{i=1}^{s} D(f_i)\) with \(f_i \in A\), hence \(S_n(A) = \bigcup_{i=1}^{s}(S_n(A) \cap D(f_i)) = \bigcup_{i=1}^{s} S_n(A_{f_i})\). Since \(S_{n-1}(A_{f_i}) = S_{n-1}(A) \cap D(f_i) = D(f_i) = \text{Spec}(A_{f_i})\), the condition \((S_{n-1})\) holds for \(A_{f_i}\). Thus, replacing \(A\) by \(A_{f_i}\), to prove the openness of \(S_n(A)\) we may assume that

\((\ast)\quad \text{the condition } (S_{n-1}) \text{ holds for } A.\)

Put \(I = \{I \mid I \text{ is an ideal of } A \text{ and } S_n(A)^c \subseteq V(I)\}\), where \(S_n(A)^c\) is the complement set \(\text{Spec}(A) - S_n(A)\). We have \(I \neq \emptyset\) because \((0) \in I\). Since \(A\) is noetherian, \(I\) has maximal elements. Let \(I\) be one of them. If \(I = A\) then \(S_n(A) = \text{Spec}(A)\) which is open in \(\text{Spec}(A)\). Therefore we assume that \(I \nsubseteq A\). It is easy to see from the maximality that \(\sqrt{I} = I\) and that \(S_n(A)^c = V(I)\). It follows from this that \(I\) has a primary decomposition of the form \(I = p_1 \cap \cdots \cap p_t\), where each \(p_i\) is a prime ideal, and we may assume that there are no inclusion relations between the \(p_i\)'s and that \(\text{ht}(p_1) \leq \text{ht}(p_i)\) for all \(i\).

Now we claim that

\((1)\quad \text{ht}(I) \geq n,\)
(2) $p_i \in S_n(A)^c$ for all $i$,
(3) $S_n(A)^c = V(I)$.

It follows from (3) that $S_n(A) = D(I)$, which shows that $S_n(A)$ is open in Spec($A$), proving the theorem. We prove these in turn.

(1) It suffices to prove that $ht(p_1) \geq n$. To prove this by contradiction, suppose that $l := ht(p_1) \leq n - 1$. By (*) we get $\text{depth}(A_{p_1}) \geq \inf (n - 1, \ ht(p_1)) = ht(p_1) = l$, hence there exist $c_1, \ldots, c_l$ in $p_1$ and $f \in A - p_1$ such that $c_1, \ldots, c_l$ is an $A_f$-sequence in $p_1 A_f$ and that $(c_1, \ldots, c_l) A_f$ is $p_1 A_f$-primary. Now we can take $g \in \bigcap_{i=2}^l p_i - p_1$ such that $IA_g = p_1 A_g$ because $p_i \not\subset p_1$ for all $i \geq 2$. Moreover, by the assumption in (NC), there exists $h \in A - p_1$ such that $D(h) \cap V(p_1) \subseteq S_n(A/p_1)$, hence the condition $(S_n)$ holds for $A_h/p_1 A_h$. Put $x = fgh \in A - p_1$. Replacing $A$ by $A_x$, we may assume that

\[\begin{align*}
&c_1, \ldots, c_l \text{ is an } A\text{-sequence in } p_1, \\
&(c_1, \ldots, c_l) \text{ is } p_1\text{-primary (hence } p_1^r \subseteq (c) \text{ for some } r \in \mathbb{N}), \\
&I = p_1 \text{ (hence } S_n(A)^c = V(p_1)), \\
&(S_n) \text{ holds for } A/p_1.
\end{align*}\]

Moreover, by Lemma 2.1, replacing $A$ by $A_y$ with some $y \in A - p_1$, we may assume that

$$p_1^i/p_1^{i+1} + (c) \cap p_1^i \text{ is } A/p_1\text{-free } (1 \leq i < r).$$

Now note that $S_n(A)^c \neq \emptyset$. In fact, if $S_n(A)^c = \emptyset$ then $V(p_1) = S_n(A)^c = \emptyset$ hence $p_1 = A$, a contradiction. Therefore we have $S_n(A)^c \neq \emptyset$. We would like to prove that $A_p$ satisfies the condition $(S_n)$ for any $p_1 \in S_n(A)^c$. If this is true, then we have a contradiction since $p_1 \not\in S_n(A)$. Therefore, we will have $ht(p_1) \geq n$ as desired. To prove that $(S_n)$ holds for $A_p$, take $p' \in Spec(A)$ with $p' \subseteq p_1$ and $p'' \in V(p' + p_1)$ such that $ht(p' + p_1/p_1) = ht(p''/p_1)$. (Since $p', p_1 \subseteq p$, we have $V(p' + p_1) \neq \emptyset$.) We should divide the proof into two cases.

i) The case when $ht(p' + p_1/p_1) \leq n$:

Since $ht(p''/p_1) \leq n$, $A_{p''/p_1} A_{p''} = (A/p_1)_{p''/p_1}$ is CM. Replacing $A$ by $A/(c)$, we may assume that $p_1^r = (0)$ and that $p_1^i/p_1^{i+1}$ is $A/p_1$-free. Therefore, $\text{depth}(A_{p''}) = \text{depth}(A_{p''/p_1} A_{p''}) = \text{depth}(A_{p''/p_1} A_{p''}) = \text{ht}(p''/p_1) = \text{ht}(p'')$, hence $A_{p''}$ is CM. It follows that $A_{p'} = (A_{p''})_{p' A_{p''}}$ is CM.

ii) The case when $ht(p' + p_1/p_1) \geq n$:

Let $q/p_1 \in V(p' + p_1/p_1)$. Then $ht(q/p_1) \geq n$, hence $\text{depth}((A/p_1)_q/p_1) \geq n$.

Thus, $\text{depth}_{p'+p_1/p_1}(A/p_1) \geq n$. Therefore there exist $c_1', \ldots, c_n'$ such that

$$c_1', \ldots, c_n' \text{ is an } A/p_1\text{-sequence in } p' + p_1/p_1.$$
Since $p_1^i/p_1^{i+1} + (c) \cap p_1^i$ is $A/p_1$-free, one can show that
\[ c_1', \ldots, c_n' \] is an $A/(c)$-sequence in $p' + p_1/p_1$.
Hence $c_1, \ldots, c_l, c_1', \ldots, c_n'$ is an $A$-sequence in $p''$, so an $A_{p''}$-sequence in $p''A_{p''}$. Therefore,
\[ c_1', \ldots, c_n', c_1, \ldots, c_l \] is an $A_{p''}$-sequence in $p''A_{p''}$.
Hence $c_1', \ldots, c_n'$ is an $A_{p''}$-sequence in $p'A_{p''}$, so an $A_{p''} = (A_{p''})_{p'A_{p''}}$-sequence in $p'A_{p''} = p'(A_{p''})_{p'A_{p''}}$. It follows that depth$(A_{p''}) \geq n$.

As we have remarked above, it follows from $i), ii)$ that \( \text{ht}(p_1) \geq n \).

(2) To prove it by contradiction, suppose that $p_k \in S_n(A)$ for some $k$. Since $I \subseteq p_k$, we have $\text{ht}(p_k) \geq n$, hence depth$(A_{p_k}) \geq \inf (n, \text{ht}(p_k)) = n$. Therefore, there exist $c_i \in p_k$ and $f \in A - p_k$ such that $c_1, \ldots, c_n$ is an $A_f$-sequence in $p_kA_f$ and that $IA_f = p_kA_f$. Since $p_k \in V(I) = S_n(A)^c$, we have $D(f) \cap S_n(A)^c \neq \emptyset$. Let $p$ be a minimal element of this set. Since $p \in S_n(A)^c \subseteq V(I)$, we have $I \subseteq p$, hence $pA_f \supseteq IA_f = p_kA_f$. Therefore $c_1, \ldots, c_n$ is an $A_f$-sequence in $pA_f$, hence is an $A_p = (A_f)_{pA_f}$-sequence in $pA_p$. It follows that depth$(A_p) \geq n = \inf (n, \text{ht}(p))$. On the other hand, if $p' \in \text{Spec}(A)$ such that $p' \not\subseteq p$, then we have $p' \notin D(f) \cap S_n(A)^c$ by the minimality of $p$. Since $p \in D(f)$, we have $p' \notin D(f)$. Therefore we have $p' \notin S_n(A)^c$, hence $(S_n)$ holds for $A_{p'}$. Thus, we see that $(S_n)$ holds for $A_p$, contrary to the choice of $p$.

(3) We have $S_n(A)^c \subseteq S_n(A)^c = V(I)$. Suppose that $S_n(A)^c \not\subseteq V(I)$. Then there exists $p \in V(I)$ such that $p \notin S_n(A)^c$. Hence we have $p_k \subseteq p$ for some $k$ and $p \in S_n(A)$. Therefore $(S_n)$ holds for $(A_p)_{p_kA_p} = A_{p_k}$. It follows that $p_k \in S_n(A)$, contrary to (2).

\[ \square \]

3. (NC) For $(R_n)$-Condition

Consider the following condition. Let $n$ be an integer and let $A$ be a local ring.

\((R'_n)\) : If $p \in \text{Spec}(A)$ and codim$(p) \leq n$, then $A_p$ is regular.

Here the codimension of an ideal $I$ of $A$ is defined as follows.
\[ \text{codim}(I) = \dim(A) - \dim(A/I). \]

**Lemma 3.1.** Let $A$ be a local ring. Then $(R_n)$ holds for $A$ if and only if $(R'_n)$ holds for $A_p$ for every $p \in \text{Spec}(A)$.
Proof. Suppose that \((R_n)\) holds for \(A\). Let \(p \in \text{Spec}(A)\), and let \(p' \in \text{Spec}(A)\) such that \(p' \subseteq p\) and that \(\text{codim}(p'A_p) \leq n\). Then we have \(\text{ht}(p'A_p) \leq \text{codim}(p'A_p) \leq n\), hence \((A_p)p'A_p\) is regular since \((R_n)\) holds for \(A_p\). It follows that \((R_n')\) holds for \(A_p\). Conversely, suppose that \((R_n')\) holds for \(A_p\) for every \(p \in \text{Spec}(A)\). Let \(p \in \text{Spec}(A)\), and let \(p' \in \text{Spec}(A)\) such that \(p' \subseteq p\) and that \(\text{ht}(p'A_p) \leq n\). Then we have \(\text{codim}(p'A_p') = \text{ht}(p') = \text{ht}(p'A_p) \leq n\), hence \((A_p)p'A_p = A_{p'} = (A_{p'})p'A_{p'}\) is regular. It follows that \((R_n)\) holds for \(A_p\). Therefore, \((R_n)\) holds for \(A\). \(\square\)

The following theorem is the main result of this section.

**Theorem 3.2.** \((\text{NC})\) holds for \(\mathcal{P} = (R_n)\).

**Proof.** We prove this theorem by induction on \(n\). It is easy to see that \((\text{NC})\) holds for \(\mathcal{P} = (R_0)\), hence we assume \(n \geq 1\) in the rest. We discuss in the same way as the proof of Theorem 2.2. Suppose that a ring \(A\) satisfies the assumption in \((\text{NC})\). Let \(I\) be one of the maximal elements of the set \(\{I \mid I\text{ is an ideal of }A\text{ and }R_n(A)^c \subseteq V(I)\}\). We may assume that

\[
\begin{align*}
(R_{n-1}) & \text{ holds for } A \cdots (*), \\
I & \subseteq A, \\
\sqrt{I} & = I, \\
R_n(A)^c & = V(I), \\
I & = p_1 \cap \cdots \cap p_t \text{ (with some } p_i \in \text{Spec}(A)),
\end{align*}
\]

there are no inclusion relations between the \(p_i\)'s,

\[\text{ht}(p_1) \leq \text{ht}(p_i) \text{ for all } i.\]

Now we prove that \(\text{ht}(p_1) \geq n\). To prove this by contradiction, suppose that \(l := \text{ht}(p_1) \leq n - 1\). By (*) we see that \(A_{p_1}\) is regular. Hence replacing \(A\) by \(A_x\) for some \(x \in A - p_1\), we may assume that

\[
\begin{align*}
(c_1, \cdots, c_l) & \text{ is an } A\text{-sequence in } p_1 \text{ (with some } c_i \in p_1), \\
(c_1, \cdots, c_l) & = p_1, \\
I & = p_1 \text{ (hence } R_n(A)^c = V(p_1)), \\
(R_n) & \text{ holds for } A/p_1 \cdots (**).\n\end{align*}
\]

Since \(R_n(A)^c \neq \emptyset\), one can take \(p \in R_n(A)^c\). Then we have \(p_1 \subseteq p\). To show that \(A_p\) satisfies \((R_n')\), we take \(p' \in \text{Spec}(A)\) such that \(p' \subseteq p\) and that \(\text{codim}(p'A_p) \leq n\). There exists \(p'' \in V(p' + p_1)\) such that \(\text{codim}((p' + p_1/p_1)A_p) = \text{codim}((p''/p_1)A_p) = \text{codim}(p''A_p/p_1A_p))\). We
have
\[
\begin{align*}
\text{codim}((p' + p_1/p_1)A_p) &= \text{ht}(p/p_1) - \text{ht}(p/p' + p_1) \\
&= \text{ht}(p) - l - \text{ht}(p/p' + p_1), \\
\text{codim}((p' + p_1/p')A_p) &= \text{codim}((c_1, \ldots, c_l)(A/p')_{p/p'}) \leq l, \\
\text{codim}((p' + p_1/p')A_p) &= \text{ht}(p/p') - \text{ht}(p/p' + p_1).
\end{align*}
\]

It follows that
\[
\text{codim}(p''A_p/p_1A_p) = \text{codim}((p' + p_1/p_1)A_p) \\
\leq \text{ht}(p) - \text{ht}(p/p') \\
= \text{codim}(p'A_p) \leq n.
\]

By (**) we see that \((R_n)\) holds for \((A/p_1)_{p/p_1} = A_p/p_1A_p\). By Lemma 3.1, we see that \(A_{p'/p_1A_{p'}} = (A/p_1A_p)_{p'A_p/p_1A_p}\) is regular, which shows that \(A_{p'}\) is regular. It follows that \((A)p'_{A_p} = (A_{p'})_{p'A_{p'}}\) is regular. Therefore we see that \(A_p\) satisfies \((R'_n)\). Let \(q \in \text{Spec}(A)\) such that \(q \subseteq p\). If \(q \in R_n(A)\), then \((R_n)\) holds for \(A_q\), hence \((R'_n)\) holds for \(A_q\). If \(q \in R_n(A)^c\), then we see that \((R'_n)\) holds for \(A_q\), discussing in the same way as above.

Thus, it follows from Lemma 3.1 that \((R_n)\) holds for \(A_p\). Since \(p \in R_n(A)^c\), we have a contradiction. Thus we have shown that \(\text{ht}(p_1) \geq n\), hence \(\text{ht}(I) \geq n\).

Therefore we can arrange the order of \(p_1, \ldots, p_t\) to satisfy the following conditions.

\[
\begin{align*}
\text{ht}(p_i) \begin{cases} 
= n & (1 \leq i \leq s), \\
> n & (s < i \leq t), 
\end{cases}
\end{align*}
\]

\(A_p_i\) is \begin{cases} 
\text{non-regular} & (1 \leq i \leq r), \\
\text{regular} & (r < i \leq s).
\end{cases}
\]

Put \(J = p_1 \cap \cdots \cap p_r\). Let \(p \in R_n(A)^c\). Then there exists \(p' \in \text{Spec}(A)\) such that \(p' \subseteq p\), \(\text{ht}(p') \leq n\), and that \(A_{p'}\) is non-regular. By (*) we get \(\text{ht}(p') = n\). Replacing \(p\) by \(p'\), we may assume that \(\text{ht}(p) = n\). Since \(R_n(A)^c \subseteq V(I)\), we have \(I \subseteq p\), hence \(p_k \subseteq p\) for some \(k\). Since \(\text{ht}(p) = n\), we have \(p_k = p\) and \(1 \leq k \leq s\), and since \(A_p\) is non-regular, we have \(1 \leq k \leq r\). It follows that \(J \subseteq p_k = p\), i.e. \(p \in V(J)\). Therefore, we have \(R_n(A)^c \subseteq V(J)\). Since the opposite inclusion is obvious by the choice of \(p_i\), we have \(R_n(A)^c = V(J)\). Thus, we get \(R_n(A) = D(J)\), which shows that \(R_n(A)^c\) is open in \(\text{Spec}(A)^c\).
Added in Proof: The author was informed that the results similar to the present paper had been reported in the following paper: C. Massaza e P. Valabrega, Sull’apertura di luoghi in uno schema localmente noetheriano, Boll. U.M.I. 14 (1977), 564-574.

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