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MULTI LOCAL INVARIANTS ON REAL UNIT BALLS

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1. Introduction

Quantities depending on several points of a space have been considered from early geometry on as there is the description of action at distance, two point functions, distance functions, bitangent curves et cetera. Functions which depend on tangent vectors or on higher order tangent vectors at distinct points of the manifold although typical differential geometric objects, also appear in analysis as for example invariant kernel functions and recently also in geometric analysis of feature detection in computer and human vision [12] [10] [4]. A classical example of a differential invariant depending on distinct points is the complex Poisson kernel [11] which is invariant with respect to complex Möbius transformations and depends as well on an interior point of the hyperbolic unit ball in $\mathbb{C}^n$ as on the rotation invariant volume form on the boundary. In this paper we consider invariants for the real Möbius transformations acting on the real hyperbolic unit ball in $\mathbb{R}^n$ and construct differential invariants, which we call multi local invariants, at several distinct points in the interior and/or at the boundary. In particular we find the real Möbius invariant Poisson kernel. The proof that it satisfies the invariant Laplace equation is based upon the Lie algebra structure of invariant vector fields which depend on extra boundary points. Most attention is given to invariants which characterize motions of points in B^n or in ∂B^n depending parametrically on sets of extra points.

The general case consists of two parts. First we consider only points in the interior of B^n and secondly we treat invariants depending as well on interior points as on boundary points. Dimension two and three are special and hence are treated as particular cases. For dimension two, which is the classical well known example [1], the calculations are carried out in full detail.

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2. Basic definitions and properties

2.1. **Generating sets of invariants.** Let G be a Lie group acting on a manifold by diffeomorphisms. A function is called invariant if it is invariant under the induced action of the diffeomorphisms. A general question in geometry is to construct all invariant functions. In order to tackle this vast problem we consider sheafs of germs of real functions which are defined on the manifold or on an appropriate jet bundles over it, thus giving invariants of a certain weight.

The germ of functions at $x \in M$ is the equivalence class of functions agreeing upon open neighborhoods of x. The set of all germs of functions at a given point p is called the stack of germs at x and the union of all stacks is the sheaf of germs of functions on M. The main purpose of this approach is to avoid discussions on the specification of domains of functions and to allow a local analysis. Because we are interested in sheafs of invariant functions we like to be able to produce a finite set of germs with the property that the set contains enough information to construct all invariant functions on open dense subsets. We call a set of functions on an open neighborhood U a generating set if every germ in the invariant sheaf over U can be reconstructed from the germs of the given functions by means of functional composition. A key notion here is functional independence of the germs, which means that the exterior product of the differentials of the germs is the germ of a non zero n-form [10]. We consider C^∞-functions. Because we often take norms of germs of vector fields as generators as well as scalar products or inverses of functions, a set of generating invariants will be a set of invariant germs of maximal functional rank at an open dense subset which we call a regular subset. Extension of the germs of the generators onto a larger subset then often requires taking inverses or square roots. In order to keep the formulation transparent such extensions will be taken without mentioning.

2.2. **The tangent bundle and higher order jet bundles.** We assume some familiarity with the geometry of tangent bundles to manifolds [14]. Let M be a manifold. The tangent bundle, TM, is a bundle with projection map $\pi : TM \to M$. Let (x^i) be local coordinates on $U \subset M$, a saturated neighborhood $\pi^{-1}U$ carries the coordinates $((x^i), (\dot{x}^i))$. A one form ω on M is a function on TM in a natural way which is denoted by $\widehat{\omega}$. The following lifts are defined as natural operations on TM. Let f be a function then $f^c = \widehat{df}$ is the complete lift and $f^v = f o \pi$ is the vertical lift. Let X be a vector field on M, the complete, or first order lift $X^c \equiv X^{(1)}$, is defined by $X^c(f^c) = (X(f))^c$. Lifts of one forms then are defined by $\omega^c(X^c) = (\omega(X))^c$.

The k^{th}-order jet bundle of curves in M is defined as follows [13][3]. Let $J^k(\mathbb{R}, M)$ be the set of k-jets of maps of $\mathbb{R}$ into M. This space is equipped with the source map α which sends the k-jet of a map $\gamma(t)$ into its source point. We call the space $J^k(M) = \alpha^{-1}(0)$ the k^{th}-order jet bundle over M. The space $J^1(M)$ identifies in a natural manner with the tangent space TM. More generally we have a natural embedding $\jmath : J^{k+1}(M) \to TJ^k(M)$ for each k. The bundle of infinite jets $J(M) = \varprojlim J^k(M)$ possesses the total derivative operator, T, which is the total derivative with respect to t, as natural operator on functions on $J^k(M)$. Functions on $J^k(M)$ are included into the set of functions on $J(M)$ for this matter. We will always identify $J^1(M)$ with TM and hence use the operator T for functions, one forms and covariant tensor fields on M, which results in entities on TM. It is easily verified that for any differential one form ω on M we have $\mathcal{L}_T\omega = \omega^c$, where $\mathcal{L}_T$ stands for the Lie derivative with respect to T.

Moreover let g be a Riemannian metric on M, then $g^c = \mathcal{L}_T g$, is the complete lift of the metric g to TM. Let g^c be given in local coordinates as $g_{ij}dx^i dx^j$, the complete lift g^c is given by $2g_{ij}d\dot{x}^i dx^j + 2\Gamma^k_{ij} g_{kl}\dot{x}^i dx^l dx^j$, where Γ^k_{ij} are the connection coefficients defined by g. Let G be a transformation group acting on M, the action extends to the space TM and to any $J^k(M)$, which is called prolongation of the group action [8]. If the metric g is invariant under the induced action of G, the complete lift g^c will be invariant under the prolonged action on TM. This result is immediate because the operator T is by construction equivariant for the induced action of the group on the ring of functions. The generalization to higher order is immediate. The metric $\jmath^*(\mathcal{L}_T)^k g = g^{(k)}$, which is the complete lift of g onto $J^k(M)$, is also invariant under the induced action of the group G. We will use the notation g^c for $g^{(1)}$.

On TM one has some extra structure. The vector field $I = \dot{x}^i\partial_{\dot{x}^i}$ is the canonical vertical vector field satisfying $I(f^c) = f^c$ for any function f on M. When g is a pseudo Riemannian metric on M then $\theta = g^c(I)$ is the canonical one form associated with g. In local coordinates one has $\theta = g_{ij}\dot{x}^i dx^j$. The spray Γ is by definition the vector field on TM such that for each geodesic $\gamma(t)$ one has $\Gamma(\gamma^c(t)) = \dot{\gamma^c}(\gamma^c(t))$ citekn:Yan. The vector field Γ satisfies $\Gamma = {g^c}^{-1}(d\widehat{g})$, which in local coordinates yields $\Gamma = \dot{x}^i\partial_{x^i} - \Gamma^i_{kl}\dot{x}^k\dot{x}^l\partial_{\dot{x}^i}$.

Following properties are easily verified.

Properties 2.1. *i)* $\Gamma(\widehat{g}) = 0$, *ii)* $g^c(\Gamma, \Gamma) = 0$, *iii)* $g^c(I, \Gamma) = \widehat{g}$.

Properties 2.2. *Let $\gamma(t)$ be a curve in M, then*

$$g^c(\dot{\gamma}^c(t), \dot{\gamma}^c(t))(\gamma(t), \dot{\gamma}(t)) = g(\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t), \dot{\gamma}(t)).$$

2.3. **Invariant vector fields and motions.** Let G be any Lie group acting to the right on the manifold M by $\phi : G \times M \to M$, $\widetilde{x} = \phi(g,x) \equiv x.g$. It is well known that for given coordinates, (g^i), at a neighborhood of the identity in G, the vector fields $X_{g^i} = D_{g^i}\phi(g^j = 0, x)$ form a complete set of generating vector fields of the action [6]. Now if G acts simply transitively on M there exists, with respect to a given fixed point $x^o \in M$, a local diffeomorphism at the identity in G with a neighborhood of x^o in M. This local diffeomorphism is given by $\phi_o(g) = x^o.g$ on a neighborhood. Let (x^i) be a set of coordinates covering a neighborhood of x^o in M such that $D\phi_o(e) = Id$ and consider the vector fields

$$Y_{x^i} = \phi_o^{-1\,*} D_{x^i}\phi(g, x^o). \tag{1}$$

Let $\mathcal{L}(M)$ be the Lie algebra of the fundamental vector fields (X_{g^i}) and $\mathcal{R}(M)$ the Lie algebra of the vector fields (Y_{x^i}). Next proposition is a classical result in Lie transformation groups [6].

Proposition 2.3.
(1) The vector fields Y_{x^i} are invariant for the right action of G on M.
(2) There exists an anti isomorphism of the Lie algebra $\mathcal{R}(M)$ with the Lie algebra $\mathcal{L}(M)$.

Let M_1 and M_2 be two G-manifolds, with G acting on both manifolds at the right and $\xi = (\xi_1, \xi_2)$ coordinates on $M_1 \times M_2$. For $g \in G$ we denote the product action as $\xi.g = (\xi_1.g, \xi_2.g)$.

Definition 2.4.
(1) Let $\gamma(t)$ be a regular curve in the G-manifold M, with $\xi^o = \gamma(t = 0)$. The curve $\gamma(t)$ is called a motion (or G-motion) if $\exists g_t$, one parameter subgroup of G, such that $\gamma(t) = \xi^o.g_t$.
(2) Let $\gamma(t)$ be an M_1-regular curve in $M_1 \times M_2$, both G-manifolds, with $\gamma(t = 0) = (\xi_1^o, \xi_2^o)$. Then $\gamma(t)$ is a motion (G-motion) depending parametrically on M_2 if $\exists\ g_t$ one parameter subgroup of G and $\exists\ f(t)$ a curve in $\mathcal{I}_{\xi_1^o}$, isotropy group at ξ_1^o, such that $\gamma(t) = (\xi_1^o.g_t,\ \xi_2^o.f(t).g_t)$.

Next proposition provides a criterion for a curve to be a motion in the above sense. The proof is a direct consequence of the isomorphism of the vector space of generators over an orbit with the vector space of left invariant vector fields on the Lie group.

Proposition 2.5.
(1) Let X_{g^k} be a complete set of generators of the right action of G on M. Then a curve $\gamma(t)$ in M is a motion iff $\dot{\gamma}(t) = \sum a^i X_{g^i}(\gamma(t))$, with a^i constants.

(2) Let X_{g^k} be a complete set of generators of the right product action on $M_1 \times M_2$. Then a M_1-regular curve in $M_1 \times M_2$ is a motion depending parametrically on M_2 iff $\pi_{1*}\dot{\gamma}(t) = \sum a^i \pi_{1*} X_{g^i}(\gamma(t))$, π_1 denotes the projection on M_1.

Corollary 2.6. *The above proposition remains true if the set X_{g^k} is replaced by a complete set of invariant vector fields.*

Corollary 2.7. *Let G be a semi simple transformation group acting on the manifold M such that the dimensions of the orbits equals the dimension of the group. Let $\gamma(t)$ be a regular curve in M, then $\gamma(t)$ is a motion in M iff (a) $\gamma(t)$ is lying in an orbit of G, (b) a generating set of first order invariants is constant along $\gamma(t)$.*

Proof. Since G is semi simple the Killing metric has maximal rank and each orbit, which being locally diffeomorphic to G, carries an invariant metric which we denote by K. Let $\gamma(t)$ be lying in an orbit. Because the first order invariants along the orbit are generated by the invariant one forms $K(X_i)$, with X_i a complete set of invariant vector fields tangent to the orbit, constancy of the first order invariants along $\gamma(t)$ implies that along γ the tangent vector $\dot{\gamma}(t)$ is a constant combination of the right invariant vector fields. Conditions are sufficient because the curve is supposed to be regular. Necessity is immediate. □

3. Möbius transformations on B^n

3.1. **Structure and generators.** In this section we review the structure of the Möbius transformations more closely and derive some properties needed to achieve our main theorems. We utilize here a construction and some results of Beardon [2].

Let $\widehat{\mathbb{R}}^n$ be the compactified $\mathbb{R}^n$ space. A Möbius transformation on $\widehat{\mathbb{R}}^n$ is defined as a finite composition of reflections in spheres or planes. The general Möbius group, which is the transformation group generated by all reflections, is denoted by $GM(n)$ and has the proper subset $M(n)$ of orientation preserving transformations. Furthermore one has the Lie group isomorphism : $GM(n) \cong O^+(1, n+1)$ [2]. Notice that $\dim GM(n) = \frac{1}{2}(n+2)(n+1)$.

On $\mathbb{R}^n$ we will make use of the Euclidean metric $< . , . >$ and corresponding norm function $\|.\|$. The Möbius transformations which preserve B^n then are compositions of rotations with center 0 and reflections with respect to the spheres or planes which are orthogonal to the boundary of B^n [2]. This group is $GM(n-1)$ which can be deduced from the Poincaré extension and the Cayley map [2]. Reflection of B^n with respect to the sphere $S(a, 1 - \|a\|^2)$, with $\|a\| > 1$ is given by :

$$\phi_a(x) = a - (1 - \|a\|^2)\frac{x-a}{\|x-a\|^2}\,. \tag{2}$$

We define $\xi_0 \in B^n$ as $\xi_0 = \frac{a}{\|a\|^2}$. Then with $\xi \in B_n$

$$\phi_{\xi_0}(\xi) = \frac{1}{\Omega}\left[\frac{\|\xi_0\|^2(\|\xi\|^2+1) - 2 < \xi_0, \xi >}{\|\xi_0\|^2}\xi_0 - (\|\xi_0\|^2 - 1)\xi\right], \tag{3}$$

where $\Omega = \|\xi_0\|^2\|\xi\|^2 + 1 - 2 < \xi_0, \xi >$.

The mapping ϕ_{ξ_0} is a diffeomorphism of B^n and maps the boundary, S^{n-1}, diffeomorphically into itself. One also has $\phi_{\xi_0}(0) = \xi_0$, $\phi_{\xi_0}(\xi_0) = 0$ and $\phi_{\xi_0}(\phi_{\xi_0}(\xi)) = \xi$, which means that ϕ_{ξ_0} is an involution.

The Lie algebra of generating vector fields of the action is found by taking the generators of the mappings :

$$\psi_{t\,u} = \phi_u o\, \phi_{t\,u} : B^n \to B^n, \tag{4}$$

for u unit vector in $\mathbb{R}^n$. The flow of this map is given by

$$\psi_{t\,u}(\xi) = \frac{1}{\Omega}\left[(\|\xi\|^2+1)t\,u - 2 < \xi, u > u - (t^2-1)\xi\right], \tag{5}$$

with

$$\Omega = t^2\|\xi\|^2 + 1 - 2t < \xi, u > . \tag{6}$$

We then find $\psi(t=0) = Id$ and

$$D_t(\psi)(\xi)(t=0) = \left[2 < \xi, u > -(\|\xi\|^2+1)\right]\partial_\xi. \tag{7}$$

We define the generating vector field determined by u, unit vector in $\mathbb{R}^n$, as $X_u = -\frac{1}{2}D_t(\psi)(\xi)(t=0)$ or

$$X_u = \left[\frac{1}{2}(\|\xi\|^2+1)u - < \xi, u > \xi\right]\partial_\xi. \tag{8}$$

Remark that the only zeros of the vector field X_u are u and $-u$ in ∂B^n. The point u is attractive while $-u$ is repulsive.

Let u, v be two orthonormal vectors, then

$$[X_u,\, X_v] = (< \xi, u > v - < \xi, v > u)\,\partial_\xi. \tag{9}$$

Let (u_i) be an orthonormal base we find that (X_{u_i}) generates the Lie algebra $\mathfrak{so}(1,n)$. Verification follows from the calculation of the brackets.

Let $X_{uv} = [X_u, X_v]$, we take $\{Z_{u_i},\, Z_{u_iu_j}\}$ as base. Additional to (9) we find

$$[Z_{u_iu_j},\, Z_{u_ku_l}] = \delta_{jk}Z_{u_iu_l} - \delta_{ik}Z_{u_ju_l} - \delta_{jl}Z_{u_iu_k} + \delta_{il}Z_{u_ju_k} \tag{10}$$

and

$$[Z_{u_i}, Z_{u_j,u_k}] = \delta_{ij}Z_{u_k} - \delta_{ik}Z_{u_j}. \tag{11}$$

As a corollary from the above construction we find that invariance under the set X_{u_i} implies infinitesimal invariance.

Proposition 3.1. *The Killing form $\mathcal{B}$ with respect to this basis is given by*

$$\mathcal{B}(Z_{u^i}, Z_{u^j}) = 2(n-1)\,\delta_{ij},\ \mathcal{B}(Z_{u^i}, Z_{u^ju^k}) = 0,$$

$$\mathcal{B}(Z_{u^iu^j}, Z_{u^ku^l}) = -2(n-1)\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,. \tag{12}$$

3.2. **The fundamental invariants.** For any $\{x, y, u, v\}$, distinct points in $\mathbb{R}^n$, we define their cross ratio as $\mathcal{Q}(x, y; u, v) = \{\|x-u\|.\|y-v\|\}/\{\|x-v\|.\|y-u\|\}$, which is invariant under the action of $GM(n)$. It is a classical result that the cross-ratio can be extended for points in $\widehat{\mathbb{R}}^n$. We recall the theorem of Beardon asserting that a map, $\phi : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$, is a Möbius transformation if and only if it preserves cross-ratios. In the real case the cross ratio is the basic invariant from which all other ones are derived.

Let $(\xi_1,\, \xi_2)$ be two points in B^n. Then under the action of ϕ_{ξ_1} the points are send into $(0, \phi_{\xi_1}(\xi_2))$. Taking the line through both images gives two intersection points $(\varrho,\, -\varrho)$ with the boundary S^{n-1} of B^n. Then $\varrho = \phi_{\xi_1}(\xi_2)/\|\phi_{\xi_1}(\xi_2)\|$. Hence the function

$$\mathcal{Q}(\xi_1, \xi_2) = \mathcal{Q}(\xi_1, \phi_{\xi_1}(\varrho); \xi_2, \phi_{\xi_1}(-\varrho)) \tag{13}$$

is invariant. Then

$$\mathcal{Q}(\xi_1, \xi_2) = \frac{\|\xi_1 - \xi_2\|\,\|\phi_{\xi_1}(\varrho) - \phi_{\xi_1}(-\varrho)\|}{\|\xi_1 - \phi_{\xi_1}(-\varrho)\|\,\|\phi_{\xi_1}(\varrho) - \xi_2\|}.$$

Using invariance this becomes

$$= \frac{\|\phi_{\xi_1}(\xi_2)\|\,\|2\varrho\|}{\|\varrho\|\,\|\varrho - \phi_{\xi_1}(\xi_2)\|},$$

which by simplification gives

$$= 2\frac{\|\phi_{\xi_1}(\xi_2)\|}{1 - \|\phi_{\xi_1}(\xi_2)\|}. \tag{14}$$

This function is clearly symmetric in ξ_i and ξ_j. We remark that $\|\phi_{\xi_i}(\xi_j)\|$ measures the invariant distance, $\tilde{d}$, between the two points in B^n, because

$$\tilde{d}(\xi_i, \xi_j) = \|\phi_{\xi_i}(\xi_i) - \phi_{\xi_i}(\xi_j)\| = \|\phi_{\xi_i}(\xi_j)\|. \tag{15}$$

Proposition 3.2. [2]

$$1 - \|\phi_{\xi_i}(\xi_j)\|^2 = \frac{1}{1 + \frac{\|\xi_i - \xi_j\|^2}{(\|\xi_i\|^2 - 1)(\|\xi_j\|^2 - 1)}}. \tag{16}$$

Hence following function is invariant.

Definition 3.3.

$$\mathcal{Q}_{ij} = \frac{\|\xi_i - \xi_j\|^2}{(\|\xi_i\|^2 - 1)(\|\xi_j\|^2 - 1)}. \tag{17}$$

The most important geometric invariant derived from $\mathcal{Q}_{ij}$ is the hyperbolic Riemannian metric. Consider $B^n \times B^n$ equipped with the coordinates (ξ_1, ξ_2). The diagonal $\triangle = \{\xi_1 = \xi_2\}$ of $B^n \times B^n$ is an invariant subspace. The function $\mathcal{Q}_{12}$ is zero on $\triangle$ with zero first order total derivatives with respect to ξ_1 and ξ_2 at $\triangle$. Hence the second order total derivatives of $\mathcal{Q}_{12}$ at $\triangle$ determine an invariant quadratic form. Set $\xi = \xi_1 = \xi_2$ at $\triangle$, then

$$T_{\xi_1} T_{\xi_1} \mathcal{Q}_{12}|_{\triangle} = 2\frac{< \dot{\xi}, \dot{\xi} >}{(1 - \|\xi\|^2)^2}. \tag{18}$$

which determine the invariant metric. The normalized metric with curvature $K = -1$ equals

$$g_o : ds^2 = 4\frac{d\xi d\xi}{(1 - \|\xi\|^2)^2}. \tag{19}$$

4. Multi local invariants on B^n

4.1. **Invariants depending on points in the interior.** In this section we examine invariants depending exclusively on points in B^n. Therefore consider the n-fold product of the unit ball with itself, $\mathcal{E}_n = \Pi_{i=0}^{n-1} B_i$, n is called the rank of the space.

The rank of $\mathcal{E}_n$ is taken equal to n because generic orbits are of maximal dimension, for smaller rank dimension of the generic orbits drops. We denote the i^{th} factor by B_i and call this the i^{th} layer for $i = 0, 1, 2, \cdots, , n-1$. It is the zeroth layer which will be of most importance to us.

4.1.1. *At zero order.*
(a) Invariant functions Let $(\xi_i), i = 0, \cdots, n-1$, be the natural coordinates on $\mathcal{E}_n$ and $\mathcal{W}_0$ the set of all n-tuples of points in B^n of rank $n-1$, considered as subset of $\mathcal{E}_n$.

Theorem 4.1. *The set*

$$\mathcal{I}_0 = \{\mathcal{Q}_{ij},\ i < j\} \tag{20}$$

is a set of generating invariant functions on $\mathcal{W}_0$.

Proof. The set $\mathcal{I}_0$ consists of $n(n-1)/2$ functions. The differentials of their logarithm equal

(21)

$$d \log \mathcal{Q}_{ij} = 2\left[\frac{\xi_i - \xi_j}{\|\xi_i - \xi_j\|^2} - \frac{\xi_i}{\|\xi_i\|^2 - 1}\right] d\xi_i - 2\left[\frac{\xi_i - \xi_j}{\|\xi_i - \xi_j\|^2} + \frac{\xi_j}{\|\xi_j\|^2 - 1}\right] d\xi_j.$$

The exterior product is a form of maximal rank on $\mathcal{W}_0$ as can be seen from examining the form at the points where $\xi_0 = 0$. This is sufficient because $GM(n-1)$ acts transitively on the zeroth-layer. Call $\mathcal{V}_o$ the set $\xi_0 = 0$. The differentials $d \log \mathcal{Q}_{0i}$ at $\mathcal{V}_o$ restricted to the zeroth layer equal $(-2/(1-\|\xi_i\|^2))\xi_i d\xi_0$, which are $n-1$ independent differentials. The differentials restricted to the first layer of $\mathcal{Q}_{1i}$ are given by a linear combination of $\xi_1 d\xi_1$ and $\xi_i d\xi_1$ at each point of the set $\xi_0 = 0$ in $\mathcal{W}_0$ and yield a set of $n-2$ linear independent differentials. These differentials are clearly independent of the former ones because they belong to a different vector space. This construction repeats for each i which gives $n-i-1$ independent one forms independent of the former ones. But this proves the theorem, the rank equals $n(n-1)/2$. □

(b) Invariant vector fields We construct a set of invariant vector fields which lie in the first layer of $\mathcal{E}_n$. Let g_o denote the invariant metric on the first layer we then define the invariant vector fields $Y_i = g_o^{-1} d \log \mathcal{Q}_{0i}$, $i = 1, \cdots, n-1$. We find

$$Y_i = \frac{1}{2}\left[\frac{(1 - \|\xi_0\|^2)^2}{\|\xi_0 - \xi_i\|^2}(\xi_0 - \xi_i) + (1 - \|\xi_0\|^2)\xi_0\right] \partial_{\xi_0}. \tag{22}$$

Next define an extra invariant vector field Y_n by selecting an extra point ϱ in B^n such that $\mathcal{Q}_0(\xi_\alpha, \varrho) = 1$ for each ξ_α, $\alpha = 0, \cdots, n-1$. Then ϱ belongs to B^n because (1) $\mathcal{Q}_0(\xi_0, \varrho) = 1$ implies $\|\varrho\|^2 = 1/2$ at $\xi_0 = 0$ and

(2) $\mathcal{Q}_0(\xi_i, \varrho) = 1$ implies $\varrho_i = 3/4\|\xi_i\|^2\xi_i$. The set (ξ_i) being a set of rank $n-1$ for $\xi_0 = 0$. Hence the vector field

$$Y_n = \frac{1}{2}\left[\frac{(1-\|\xi_0\|^2)^2}{\|\xi_0 - \varrho\|^2}(\xi_0 - \varrho) + (1-\|\xi_0\|^2)\xi_0\right]\partial_{\xi_0} \tag{23}$$

is invariant. Let $Y_\alpha = (Y_i, Y_n)$ denote this set of vector fields with $\alpha = 1\cdots, n$.

Proposition 4.2. *The set Y_α has rank n at each point in $\mathcal{W}_0$.*

Proof. In order to show linear independence of the set Y_i it is sufficient to consider the vectors at $\mathcal{V}_0$. The vectors

$$Y_\alpha(\xi_0 = 0) = -\frac{1}{2}\frac{\xi_\alpha}{\|\xi_\alpha\|^2}\partial_{\xi_0}, \tag{24}$$

with $\xi_n = \varrho$, are linear independent at all points $(\xi_0 = 0, \xi_1, \cdots, \xi_{n-1})$ in $\mathcal{W}_0$ and hence on $\mathcal{W}_0$.

□

4.1.2. *At first order.*

(a) Invariant functions The first order invariants we are interested in are those which depend on derivatives of curves in the zeroth-layer, B_0, of $\mathcal{E}_n$. We therefore consider the space $\mathcal{J}^{1,0,\cdots,0}\mathcal{E}_n = J^1B_0 \times B_1 \times \cdots \times B_{n-1}$ on which we define the following coordinates $(\xi_0, \dot{\xi}_0, \xi_1, \cdots, \xi_{n-1})$.

The n invariant vector fields Y_α determine an independent set of invariant one forms $\mu_\alpha = g_o(Y_\alpha)$. These forms lift to invariant functions on the space $\mathcal{J}^{1,0,\cdots,0}\mathcal{E}_n$. Let $\mathcal{W}_0^{(1)}$ be the subset $\pi_o^{-1}\mathcal{W}_0 \setminus \{\dot{\xi}_0 = 0\}$, with $\pi_o : \mathcal{J}^{1,0,\cdots,0}\mathcal{E}_n \to \mathcal{E}_n$.

Theorem 4.3. *The set*

$$\mathcal{I}_0^{(1)} = \pi_o^*\mathcal{I}_0 \cup \{\widetilde{\mu}_\alpha\} \tag{25}$$

is a generating set of invariants on $\mathcal{W}_0^{(1)}$.

Applying corollary (2.7) we find :

Theorem 4.4. *Let $\gamma(t)$ be a curve in $\mathcal{E}_n$. Then $\gamma(t)$ is a motion in B_0 depending parametrically on ξ_i, $i = 1, \cdots, n-1$ iff (1) all $\mathcal{Q}_{ij}$, for $i, j = 0, 1, \cdots n-1$ are constant along $\gamma(t)$ and (2) the one form μ_n is constant along $\gamma(t)$.*

Corollary 4.5. *In the above theorem constancy of the one form μ_n can be replaced by constancy of the square of the length $g_o(\dot{\gamma}(t), \dot{\gamma}(t))$ of the tangent vector to the curve.*

Next consider the space

$$\mathcal{J}^{1,0,\cdots,0}\mathcal{E}_{n-1} = J^1 B_0 \times \Pi_{i=1}^{n-2} B_i \tag{26}$$

which is equipped with the coordinates $(\xi_0, \dot{\xi}_0, \xi_1, \cdots, \xi_{n-2})$. Consider the subset $\mathcal{W}_1$ which is defined by the conditions : (1) $\dot{\xi}_0 \neq 0$ and (2) $rk\,\{\xi_0, \dot{\xi}_0, \xi_1, \cdots, \xi_{n-2}\} = n-1$. The set of fundamental vector fields has maximal rank on $\mathcal{W}_1$ because : (1) the rotation vector fields lift linearly onto the tangent bundle to B_0 and (2) the vector fields X_u lift to $[\frac{1}{2}(<\xi, \xi> +1)u> - <\xi, u> \xi])\partial_\xi + [<\dot{\xi}, \xi> u - <\dot{\xi}, u> \xi - <\xi, u> \dot{\xi}]\partial_{\dot{\xi}}$.

Theorem 4.6. *The following set is a generating set of invariants on* $\mathcal{W}_1$ *:*

$$\mathcal{I}_1 = \{\mathcal{Q}_{0i},\, \mathcal{Q}_{ij},\, T_0\mathcal{Q}_{0,i},\, \widetilde{g_o}\} \tag{27}$$

$i, j = 1, \cdots, n-2$.

Proof. $rk\,\{\mathcal{Q}_{0i}, \mathcal{Q}_{ij}\} = (n-1)(n-2)/2$ by the same arguments as used in theorem (4.3), while the remaining subset of functions are first order in the variables $\dot{\xi}_0$ and has rank $n-1$. Hence the rank of $\mathcal{I}_1$ on $\mathcal{W}_1$ equals $\frac{n(n-1)}{2}$, while the space has dimension n^2. It follows that the level surfaces have same dimension as the group $GM(n-1)$.

□

The invariant sheaf $\mathcal{I}_0^{(1)}$ on $\mathcal{W}_0^{(1)}$ extends uniquely onto the invariant diagonal $\xi_{n-1} = \xi_{n-2}$. We use the same symbol for the extension. Consider the embedding

$$\triangle : \mathcal{J}^{1,0,\cdots,0}\mathcal{E}_{n-1} \to \mathcal{J}^{1,0,\cdots,0}\mathcal{E}_n \tag{28}$$

which takes $\mathcal{J}^{1,0,\cdots,0}\mathcal{E}_{n-1}$ into the diagonal $\xi_{n-1} = \xi_{n-2}$. This embedding is equivariant and hence we find equivalence of two sets of invariant generators, namely $\triangle^*\mathcal{I}_0^{(1)}$ and $\mathcal{I}_1$. Remark that taking the embedding $\triangle$ means taking the limit for $\xi_n \to \xi_{n-1}$, which results into restriction to the $n-1$ first points in B.

(b) Invariant vector fields We now construct invariant vector fields tangent to the first layer of $\mathcal{J}^{1,0,\cdots,0}\mathcal{E}_{n-1}$. Let $L_\alpha = Y_\alpha^{(1)}$, where α runs from 1 to $n-1$, be the complete lifts of the $n-1$ vector fields Y_α onto $\mathcal{J}^{1,0,\cdots,0}\mathcal{E}_{n-2}$. They constitute a linear independent set. Define $N_\alpha = g_o^{c\;-1}\mu_\alpha$ and let further Γ be the geodesic spray and Z the vector field on J^1B_0 orthonormal to all the former ones with respect to the metric g_o^c.

Proposition 4.7. *The set*

$$Z_\alpha = \{L_\alpha,\, N_\alpha,\, \Gamma,\, Z\} \tag{29}$$

is a linear independent set of $2n$ *invariant vector fields at each point of* $\mathcal{W}_1$.

Proof. To show linear independence of the vector fields it is sufficient to look at points with $\xi_0 = 0$. We find with $\alpha = 1, \cdots, n-1$:

$$Y_\alpha^{(1)}(\xi_0 = 0) = -\frac{1}{2}\frac{\xi_\alpha}{\|\xi_\alpha\|^2}\partial_{\xi_0} + \left[\frac{1}{2}(\frac{1}{\|\xi_\alpha\|^2}+1)\dot{\xi}_0 - \frac{<\dot{\xi}_0, \xi_\alpha>}{\|\xi_\alpha\|^4}\xi_\alpha\right]\partial_{\dot{\xi}_0} \tag{30}$$

$$N_\alpha = g_o^{c\,-1}\mu_\alpha(\xi_0 = 0) = -\frac{1}{2}\frac{\xi_\alpha}{\|\xi_\alpha\|^2}\partial_{\dot{\xi}_0}$$

$$\Gamma(\xi_0 = 0) = \dot{\xi}_0\partial_{\xi_0}$$

from which linear independence is easily determined. Invariance is obvious. □

Let $\gamma(t)$ be a curve in the product $\Pi_{i=0}^{n-2}B_i$ then $\gamma_0^{(1)}(t)$ denotes the prolongation of this curve in $J^1B_0 \times B_1 \times \cdots \times B_{n-2}$. Using theorem (2.7) we find :

Theorem 4.8. *Let* $\gamma(t)$ *be a curve in* $B_0 \times B_1 \times \cdots, \times B_{n-2}$ *which is regular in the zeroth layer. Then* $\gamma(t)$ *is a motion depending parametrically on* $B_1 \times \cdots \times B_{n-2}$ *iff (1)* $\gamma_0^{(1)}(t)$ *is lying in an orbit and (2)* $\dot{\gamma}_0^{(1)}$ *is a constant combination of the set* Z_α.

The construction of the invariants by means of invariant vector fields is not the most suitable one at higher order, we give a different construction based upon the Killing form.

Consider again the space $\mathcal{J}^{1,0,\cdots,0}\mathcal{E}_{n-1}$. Each layer has an invariant metric : g_o^c on the first layer and g_i on the i^{th}-layer, with $i = 1, \cdots, n-2$. Let $g = g_o^c + \sum_{i=1}^{n-2} g_i$ be the metric $\mathcal{J}^{1,0,\cdots,0}\mathcal{E}_{n-1}$.

By means of the invariant set $\mathcal{I}_1$ we construct following set of invariant vector fields which are transversal to the orbits nl., $N_i = g^{-1}(d\mathcal{Q}_{0i})$, $L_i = g^{-1}(dT_0\mathcal{Q}_{0i})$, $M_{ij} = g^{-1}(d\mathcal{Q}_{ij})$, $i < j$, $I = g_o^{c\,-1}(\theta)$.

Because the set $\mathcal{I}_1$ is linear independent at each point the above set of vector fields is a linearly independent set at each point of $\mathcal{W}_1$. Moreover the set spans an invariant normal space to all orbits of the group action. Taking the Killing metric in the tangent space to each orbit, which is well defined because the generic orbits have maximal dimension, and taking the above vector fields as orthonormal vector fields normal to the orbits one obtains an invariant metric on $\mathcal{W}_1$, which we denote by K_2.

We will have to show that the differential of K_2 is independent of the differentials of the other generators. Let $p \in J^1B_0 \times B_1 \times \cdots \times B_{n-2}$ be such that $\xi_0 = 0$ and the $n-1$ points $\{\dot{\xi}_0, \xi_1, \cdots, \xi_{n-2}\}$ are linear independent. Choose two unit vectors $u, v \in \mathbb{R}^n$ such that u is orthogonal to the set $\{\dot{\xi}_0, \xi_1, \cdots, \xi_{n-2}\}$ and v orthogonal to $\{u, \xi_1, \cdots, \xi_{n-2}\}$. Then

$$X_{uv}(p) = - < \dot{\xi}_0, v > u\partial_{\dot{\xi}_0}. \tag{31}$$

But from the definition of the Killing form we have

$$K_2(< \xi_0, v > u\partial_{\dot{\xi}_0}, < \xi_0, v > u\partial_{\dot{\xi}_0})(p) = -2(n-1). \tag{32}$$

Then

$$K_2(u\partial_{\dot{\xi}_0}, u\partial_{\dot{\xi}_0}) = -\frac{2(n-1)}{(< \dot{\xi}_0, v >)^2}, \tag{33}$$

which proves independence with respect to the set of functions $\{\mathcal{Q}_{oi}, T_0\mathcal{Q}_{oi}\mathcal{Q}_{ij}, \widetilde{g}_o\}$ at each point of the regular subset.

Together with theorem (2.7) we now have following theorem.

Theorem 4.9. *Let $\gamma(t)$ be a B_0-regular curve in $\Pi_{i=0}^{n-2}B_i$. Then $\gamma(t)$ is a motion depending parametrically on a point in $\Pi_{i=1}^{n-2}B_i$ iff the set $\{\mathcal{Q}_{0i},\ \mathcal{Q}_{ij},\ g_o,\ K_2|^o\}$ is constant along $\gamma(t)$.*

Remark that because $\Gamma(\widetilde{g}_o) = 0$ and $g^c(\Gamma, \Gamma) = 0$ the vector field Γ is tangent to invariant null surfaces in J^1B_0 with respect to the metric g_o^c. The null surfaces are given by $\widetilde{g}_o = constant$ for $\dot{\xi}_0 \neq 0$ and hence are orbits which have signature $(0, n-1, n-1)$. Now if $\gamma(t)$ is constant in the metric g_o the tangent vector field $\dot{\gamma}_0^c(t)$ is a null vector such that $g_o^c(\Gamma, \dot{\gamma}^c(t))(\gamma(t), \dot{\gamma}(t)) = 0$ (2.2). Hence $\dot{\gamma}_0^c(t)$ is lying in an n dimensional invariant totally isotropic null space containing Γ. If $\gamma(t)$ is a motion then $\dot{\gamma}_0^c(t)$ is a fixed point in this space determined by $T_0T_0\mathcal{Q}_{0i}, T_0\widetilde{g}_o$ and $\widetilde{K}_2|^o$.

4.1.3. *Second order invariants.* The construction given in former paragraph yields the derivation of generating second order invariants. Consider $\mathcal{J}^{2,0,\cdots,0}\mathcal{E}_{n-1} = J^2B_0 \times \Pi_{i=1}^{n-2}B_i$.

Theorem 4.10. *The following sets are equivalent generating sets of invariants*

$$\mathcal{I}_2 =< \mathcal{Q}_{ij},\ \mathcal{Q}_{0i},\ T_0\mathcal{Q}_{0i},\ T_0T_0\mathcal{Q}_{0i},\ \widetilde{g}_o,\ T_0\widetilde{g}_o,\ \widetilde{K}_2 > \tag{34}$$

$$\mathcal{J}_2 =< \mathcal{Q}_{ij},\ \mathcal{Q}_{0i},\ T_0\mathcal{Q}_{0i},\ \widetilde{g}_o,\ \widetilde{g_o^cL_\alpha},\ \widetilde{g_o^cZ} > \tag{35}$$

where $L_\alpha = Y_\alpha^{(1)}$ and $Y_\alpha = (Y_i, Y_n)$ and Z the vector field defined in (29).
Construct following embeddings :

$$J^2B_0 \times \Pi_{i=1}^{n-3} B_i \xrightarrow{\triangle} J^2B_0 \times \Pi_{i=1}^{n-2} B_i \xrightarrow{\jmath} TJ^1B_0 \times \Pi_{i=1}^{n-2} B_i, \tag{36}$$

where $\triangle$ is the diagonal embedding $\xi_{n-2} = \xi_{n-3}$ and $\jmath$ the canonical embedding of the jet bundle into the tangent space. Define the subset $\mathcal{W}_2$ of $J^2B_0 \times \Pi_{i=1}^{n-3} B_i$ as the set of points $\{\xi_0, \dot{\xi}_0, \ddot{\xi}_0, \xi_1, \cdots, \xi_{n-3}\}$ of rank $n-1$ with $\dot{\xi}_0 \neq 0, \ddot{\xi}_0 \neq 0$.

Theorem 4.11. *The set*

$$\mathcal{I}_{\mathcal{W}_2} = \{\mathcal{Q}_{oi}, T_0\mathcal{Q}_{0i}, \mathcal{Q}_{ij}, \widetilde{g}_o, T_0T_0\mathcal{Q}_{oi}, \widetilde{g}_o^c, \widetilde{K|^o}\}, \tag{37}$$

where all functions are pulled back via the above defined mapping $\jmath \circ \triangle$, is a generating set of invariants on $\mathcal{W}_2$.

4.1.4. *Higher order invariants.* At higher order we summarize the results. Verification follows from the same considerations as the ones at second order. Consider the space $\mathcal{J}^{(k,0,\cdots,0)}\mathcal{E}_{n-k} = J^kB_0 \times \Pi_{i=1}^{n-(k+1)} B_i$.

Theorem 4.12. *The set*

$$\mathcal{I}_{n-k} = \left\{ \begin{array}{cccccc} \mathcal{Q}_{ij} & \mathcal{Q}_{oi} & & & & \\ & T_0\mathcal{Q}_{oi} & \widetilde{g}_o & & & \\ & T_0^2\mathcal{Q}_{oi} & T_0\widetilde{g}_o & \widetilde{K_2} & & \\ & \vdots & \vdots & \vdots & \vdots & \\ & T_0^k\mathcal{Q}_{oi} & T_0^{k-1}\widetilde{g}_o & T_0^{k-2}\widetilde{K}_2 & \cdots & \widetilde{K}_k \end{array} \right\} \tag{38}$$

is a generating set of invariants at order k.

Given a generating set at order k, a set at order $k+1$ is found by prolongation of the former set together with $\widetilde{K}_{k+1}$, which is constructed in the same way as K_2. The embedding

$$\triangle : J^{n-1}B_0 \to J^{n-1}B_0 \times B_1 \tag{39}$$

then allows the construction of a generating set at order $n-1$, nl.

$$\mathcal{I}_{n-1} = \left\{ \begin{array}{cccc} \widetilde{g}_o & & & \\ T_0\widetilde{g}_o & \widetilde{K_2} & & \\ \vdots & \vdots & & \\ T_0^{n-2}\widetilde{g}_o & T_0^{n-3}\widetilde{K}_2 & \cdots & \widetilde{K}_{n-1} \end{array} \right\} \tag{40}$$

Then at order n we have

Theorem 4.13. *The set*

$$\mathcal{I}_{n-1} \cup \{T_0^{n-1}\widetilde{g}_o,\, T_0^{n-2}\widetilde{K}_2,\, \cdots,\, T_0\widetilde{K}_{n-1},\, \widetilde{K}_n\} \tag{41}$$

is a generating set of invariants of order n.

Motions in B^n then are characterized by following theorem.

Theorem 4.14. *Let $\gamma(t)$ be a regular curve in J^nB^n, lift of a curve in B^n, then $\gamma(t)$ is a motion iff the functions $\{\widetilde{g}_o,\, \widetilde{K}_2,\, \cdots,\, \widetilde{K}_n\}$ are constant along $\gamma^c(t) = (\gamma(t), \dot{\gamma}(t), \cdots, \gamma^{(n+1)}(t))$.*

4.2. **Invariants depending on boundary and interior points.** Consider the space

$$\mathcal{E}_n = B_0 \times \underbrace{S^{n-1} \times \cdots S^{n-1}}_{n-1\ times}. \tag{42}$$

On the first factor B_0 we use the standard coordinates (ξ_0) from $\mathbb{R}^n$. Occasionally we use on each of the spheres, S^{n-1}, spherical coordinates induced from the standard coordinates on $\mathbb{R}^n$, but mostly when not mentionned we use the external coordinates which are denoted by (ξ_i), $i = 1, \cdots, n-1$, i stand for the layer. The generating vector fields of the action in this case are easily written down as

$$X_u = \sum_{j=0}^{n-1} \left(- < \xi_j, u > \xi_j + \frac{1}{2}(< \xi_j, \xi_j > +1)u \right) \partial_{\xi_j} \tag{43}$$

$$X_{(u,v)} = \sum_{j=0}^{n-1} (- < \xi_j, v > u\partial_{\xi_j} + < \xi_j, u > v\partial_{\xi_j}),$$

where it is understood that each ξ_i, for $i \neq 0$, is a point in the boundary S^{n-1} of B^n.

4.2.1. *At zero order.* Let ξ_0 be a point in the interior of B_0 and ξ_i a point at the boundary S^{n-1}. Through ξ_0 and ξ_i construct the circle Γ_i, which is orthogonal to the boundary. This circle is unique. Let ζ_i be the second intersection point of the circle Γ_i and the boundary, which is given by $\zeta = \phi_{\xi_0}(-\phi_{\xi_0}(\xi_i))$. Because circles orthogonal to the boundary are geodesics of the invariant metric, the construction is invariant.
Let then ξ_j be a point at the boundary such that $\xi_j \neq \xi_i$ and $\xi_j \neq \zeta_i$, the function

$$\mathcal{Q}_{0ij} = [\xi_i, \xi_j; \xi_0, \zeta_i] = \frac{\|\xi_0 - \xi_j\| \, \|\xi_i - \zeta_i\|}{\|\xi_0 - \zeta_i\| \, \|\xi_i - \xi_j\|} \tag{44}$$

is invariant under the action of Möbius transformations on the closed unit ball. Using the invariance we find

$$\mathcal{Q}_{0ij} = \frac{2}{\|\phi_{\xi_0}(\xi_i) - \phi_{\xi_0}(\xi_j)\|} = \frac{1}{1 - < \phi_{\xi_0}(\xi_i), \phi_{\xi_0}(\xi_j) >}. \tag{45}$$

Remark that $\mathcal{Q}_{0ij}(\xi_0 = 0) = \frac{2}{\|\xi_i - \xi_j\|}$ is an invariant for the action of $SO(n)$ and hence translating this function over B_0 by means of ϕ_{ξ_0} yields the invariant $\mathcal{Q}_{0ij}$.

The Möbius transformation (3) for ξ in the boundary S^{n-1} is

$$\phi_{\xi_0}(\xi) = \frac{2(1 - \frac{<\xi, \xi_0>}{<\xi_0, \xi_0>})\xi_0 - (\|\xi_0\|^2 - 1)\xi}{\|\xi_0 - \xi\|}. \tag{46}$$

Let $u = \xi_0 / \|\xi_0\|$ and set $\|\xi\| = t$ we then find

$$\phi_{t\,u}(\xi) = \frac{2(t - < \xi, u >)\, u - (t^2 - 1)\xi}{\|t\, u - \xi\|^2} \tag{47}$$

and

$$D_t \phi_{t\,u}(\xi)|_{t=0} = 2[(1 - 2(< \xi, u >)^2)u + < \xi, u > \xi]. \tag{48}$$

From the above expressions we find the total derivatives of the invariant functions at the points $\xi_0 = 0$.

Proposition 4.15.

$$T_0 \mathcal{Q}_{0ij}(\xi_0 = 0) = -4 \frac{< \xi_i + \xi_j, \dot{\xi}_0 >}{\|\xi_i - \xi_j\|^2} \tag{49}$$

$$T_j \mathcal{Q}_{0ij}(\xi_0 = 0) = 2 \frac{< \xi_j - \xi_i, \dot{\xi}_j >}{\|\xi_i - \xi_j\|^3} \tag{50}$$

The singular subsets in $\mathcal{E}_n$ are given by $\Sigma_0 \;:\; \xi_i = \xi_j,\ i < j, i, j = 1, \cdots, n-1$ and $\Sigma_1 \;:\; -\phi_{\xi_0}(\xi_i) = \phi_{\xi_0}(\xi_j),\ i < j,\ i, j = 1, \cdots, n-1$. The subset $\mathcal{W}_0 = \mathcal{E}_n \setminus (\Sigma_0 \cup \Sigma_1)$ is a regular subset. It is sufficient to verify that the set of fundamental vector fields has maximal rank at points in $\mathcal{W}_0$ with $\xi_0 = 0$.

For u, v orthonormal vectors in $\mathbb{R}^n$ we have

$$(51)\qquad X_u|_{\xi_0=0} = -\frac{1}{2}u\partial_{\xi_0} + \sum_{i=1}^{n-1}[- < \xi_i, u > \xi_i + \frac{1}{2}(< \xi_i, \xi_i > +1)u]\partial_{\xi_i} \quad (1)$$

$$X_{(u,v)}|_{\xi_0=0} = \sum_{i=1}^{n-1} < \xi_i, u > v\partial_{\xi_i} - < \xi_i, v > u\partial_{\xi_i} \quad (2)$$

Set (1) has rank n and is independent of the set (2), which are the rotations on the S^{n-1} spheres. Outside the singular subsets the set (2) is linear independent and of rank $n(n-1)/2$.

Theorem 4.16. *The set of functions $\mathcal{Q}_{0ij}$, with $i < j$ is a generating set of invariants on $\mathcal{W}_0$.*

Proof. The dimension of $\mathcal{E}_n$ equals $n^2 - n + 1$ while the dimension of the group $GM(n-1)$ equals $\frac{1}{2}n(n+1)$. On the other hand the set $\{\mathcal{Q}_{0ij}\}$ contains $\frac{1}{2}(n-1)(n-2)$ invariant functions on $\mathcal{W}_0$. To prove that this set has maximal rank it is sufficient to look at the particular points with $\xi_0 = 0$ because the group $GM(n-1)$ acts transitively on B^n. The set of differentials with respect to the j^{th} factor, taken at $\xi_0 = 0$, which is given by the set (50) with $i < j$, has maximal rank for the points in $\mathcal{W}_0$ with $\xi_0 = 0$. Hence the set has maximal rank on the whole of $\mathcal{W}_0$. □

4.2.2. *At first order.*

(1) First order at interior points of B^n Let

$$\mathcal{J}^{(1,0,\cdots,0)}\mathcal{E}_n = J^1B_0 \times \underbrace{S^{n-1} \times \cdots \times S^{n-1}}_{n-1\ times}.$$

The invariant sheaf $\mathcal{A}_0$ is in a natural way an invariant sheaf on $\pi_0^{-1}(\mathcal{W}_0)$, where

$$\pi_o : J^1B_0 \times \Pi_{l=1}^{n-1} S_l^{n-1} \to B_0 \times \Pi_{l=1}^{n-1} S_l^{n-1}$$

is the projection. We will use the same notation for both. The rank of $\mathcal{A}_0$ equals $\frac{1}{2}(n-1)(n-2)$. The sheaf $\mathcal{A}_0$ contains a subsheaf of invariant functions which are independent of ξ_0 and which has dimension $\frac{1}{2}(n-1)(n-4)$ [5]. More generally we have following lemma.

Lemma 4.17. *Consider the space $\mathcal{J}^{(1,0,\cdots,0)}\mathcal{E}_{n-1} = J^1B^n \times \underbrace{S^{n-1} \times \cdots \times S^{n-1}}_{k\ times}$. Let $n \geq 4$ and $k \geq 2$, then outside the singular subset the number of functional independent, in the variables ξ_0, invariant functions contained in the set $\{\mathcal{Q}_{0ij};\ i < j = 1, \cdots, k\}$, equals $k-1$.*

Proof. Let $2 \leq k$. The differentials in the first layer of the set $\mathcal{Q}_{0ij}$, $i, j = 1, \cdots, k$ at $\xi_0 = 0$ are given by (49) and form a linear independent set of rank $k-1$ outside the singular subsets of $\mathcal{E}_n$.

□

On the space $\mathcal{J}^{(1,0,\cdots,0)}\mathcal{E}_n$ we find $n-1$ functional independent, in ξ_0, invariants. Hence in order to have a generating set of invariants we need one invariant function more. The invariant metric (19) has maximal rank on B_0 and hence is a good candidate. We have following theorem.

Theorem 4.18. *The invariant sheaf $\mathcal{A}_1$ on $\pi^{-1\,*}\mathcal{W}_0 \setminus \{\dot{\xi}_0 = 0\}$ is generated by the first prolongation $\mathcal{A}_0^{(1)}$ together with $\widetilde{g}_o$*

Applying theorem (2.7) we have :

Theorem 4.19. *Let $\gamma(t)$ be a B_0-regular curve in $\mathcal{E}_n$. Then $\gamma(t)$ is a motion in B^n parametrized by $n-1$ points at the boundary S^{n-1} of B^n iff the curve is lying in an orbit of $GM(n-1)$ and $\widetilde{g}_0$ is constant along the curve.*

Using theorem (4.18) one finds a set of invariant generators on $\mathcal{J}^{(1,0,\cdots,0)}\mathcal{E}_{n-1}$.

Theorem 4.20. *Let $\mathcal{W}_1$ be the regular subset of $\mathcal{J}^{(1,0,\cdots,0)}\mathcal{E}_{n-1}$. On $\mathcal{W}_1$ the following set is a set of invariant generators.*

$$\mathcal{I}_1 = \{\mathcal{Q}_{0ij},\, T_0\mathcal{Q}_{0ij},\, \widetilde{g}_o \,:\, i, j = 1, \cdots, n-2\}. \tag{52}$$

The proof follows from the diagonal embedding

$$\jmath : \mathcal{J}^{(1,0,\cdots 0)}\mathcal{E}_{n-1} \to \mathcal{J}^{1,0,\cdots 0}\mathcal{E}_n \tag{53}$$

where the image is in the diagonal $\xi_n = \xi_{n-1}$.

let $\mathcal{O} \subset \mathcal{W}_0$ be an orbit and π_o the projection $\mathcal{E}_n \to B_0$. Then π_o is surjective on $\mathcal{O}$. From

$$d\mathcal{Q}_{0ij}(\xi_0 = 0) = \frac{-1}{\|\xi_i - \xi_j\|}\left((\xi_i + \xi_j)d\xi_0 + \frac{< \xi_i - \xi_j, d\xi_i - d\xi_j >}{\|\xi_i - \xi_j\|^2}\right), \tag{54}$$

we find as a consequence of the implicit function theorem that (ξ_0) are coordinate functions on the orbits and $\pi_o^{-1}(p) \cap \mathcal{O}$, with $p \in B^n$, is isomorphic to $SO(n)$.

(2) First order at boundary points Let ξ_1 and ξ_2 be two points in the boundary. Then $\mathcal{Q}_{012}$ is invariant and the inverse function $\Psi = 1/(\mathcal{Q}_{012})^2$ is zero at the diagonal $\triangle : \xi_1 = \xi_2$ of $\mathcal{E}_3 = B_0 \times S^{n-1} \times S^{n-1}$ with zero first order total derivatives with respect to ξ_1 and ξ_2. We set $\xi = \xi_1 = \xi_2$.

Theorem 4.21.

$$T_{\xi_1} T_{\xi_2} \mathcal{Q}_{012}|_{\triangle} = \frac{(\|\xi_0\|^2 - 1)^2}{\|\xi_0 - \xi\|^4} < \dot{\xi}, \dot{\xi} > \tag{55}$$

Proof. From $< \phi_{\xi_0}(\xi_i), \phi_{\xi_0}(\xi_i) >= 1$ we find

$$< T_{\xi_i} \phi_{\xi_0}(\xi_i), \phi_{\xi_0}(\xi_i) >= 0 \tag{56}$$

With the use of (46) we then find

$$T_\xi \phi_{\xi_0}(\xi) = -\frac{2 < \xi_0, \dot{\xi} > \xi_0 - (\|\xi_0\|^2 - 1)\dot{\xi}}{\|\xi_0 - \xi\|^2} + 2\frac{< \xi_0, \dot{\xi} > \phi_{\xi_0}(\xi)}{\|\xi_0 - \xi\|^2}. \tag{57}$$

Then $< T_\xi \phi_{\xi_0}(\xi), T_\xi \phi_{\xi_0}(\xi) >$ yields relation (55).

□

Let

$$f(\xi_0, \xi) = \frac{(1- < \xi_0, \xi_0 >)}{\|\xi_0 - \xi\|^2} \tag{58}$$

then $g_S(\xi_0, \xi) = f^2(\xi_0, \xi)\, d\sigma^2$, with $d\sigma^2$ the standard spherical metric on S^{n-1}, defines an invariant Riemannian metric on the boundary S^{n-1} depending parametrically on ξ_0. The conformal function $f(\xi_0, \xi)$ is nowhere zero on $B_0 \times S^{n-1}$. Define

$$\eta(\xi_0, \xi) = f^{n-1}(\xi_0, \xi)\, \nu \tag{59}$$

with ν the spherical volume form normalized by ($\int_{S^{n-1}} \nu = 1$). Remark that $\int_{S^{n-1}} \eta$ is constant, because the integral is an invariant function on B_0. Moreover the form $\eta(\xi_0, \xi)$ has to be a constant multiple of the associated volume form of g_S because invariant functions on $B_0 \times S^{n-1}$ are constant. The choice of the normalization constant is of no importance in future calculations.

It is now sufficient to verify that $f(\xi_0 = 0) = 1$ which implies that $\int_{S^{n-1}} \eta = 1$. This was the reason for choosing the multiplicative constant of f as above. The value of the differential with respect to B_0 of $\log f$ at $\xi_0 = 0$ equals

Proposition 4.22.

$$\frac{d_o f}{f}(\xi_0 = 0) = 2 < \xi, \dot{\xi}_0 > . \tag{60}$$

let $p = (\xi_0, \xi_1, \cdots, \xi_{n-1}) \in \mathcal{E}_n$ such that $\xi_0 = 0$ and $(\xi_i), i = 1, \cdots, n-1$, is an orthonormal set. Let $\mathcal{O}_p$ be the orbit through p and choose ξ_n a unit vector orthogonal to the set $(\xi_i), i = 1, \cdots, n-1$, such that the set is positively oriented.

With each ξ_a, $a = 1, \cdots, n$, corresponds an invariant one form

$$\mu_a = d_0 \log f_a = -2\left[\frac{\xi_0}{1 - \|\xi_0\|^2} + \frac{\xi_0 - \xi_a}{\|\xi_0 - \xi_a\|^2}\right] d\xi_0 \tag{61}$$

which is well defined on $\mathcal{E}_n$ and is a basic one form for the projection $\pi_o : \mathcal{O}_p \to B_0$. The functions $g_o^{-1}(\mu_a, \mu_b)$ are invariant and hence are constant on the orbit $\mathcal{O}_p$. The vector fields

$$Z_a = g_o^{-1}\mu_a = -\frac{(1 - \|\xi_0\|^2)^2}{2}\left[\frac{\xi_0}{1 - \|\xi_0\|^2} + \frac{\xi_0 - \xi_a}{\|\xi_0 - \xi_a\|^2}\right] \partial_{\xi_0} \tag{62}$$

are invariant vector fields and are tangent to the first layer B^n of $\mathcal{E}_n$.

We restrict the vectorfields Z_a to the orbit $\mathcal{O}_p$. Because they are invariant and all invariant functions are constant on $\mathcal{O}_p$ it is sufficient to calculate their brackets at p. We use following expressions.

$$(63)\qquad \begin{aligned} &(a) & Z_a|_p &= \frac{1}{2}\xi_a\partial_{\xi_0} \\ &(b) & Z_a(Z_b)|p &= -\frac{1}{2}\xi_a\partial_{\xi_0} \\ &(c) & Z_1(g_o(Z_a, Z_b))|p &= -\frac{1}{2}(g_o(\xi_1, \frac{1}{2}\xi_b) + g_o(\frac{1}{2}\xi_a, \xi_1) \\ &&&= - < \xi_1, \xi_a + \xi_b > \end{aligned}$$

We then find on $\mathcal{O}_p$

$$[Z_a,\, Z_b] = Z_b - Z_a. \tag{64}$$

Lemma 4.23. *The Lie algebra spanned by the vector fields Z_a is the nilpotent Lie subalgebra complementing the subalgebra $K \equiv \mathfrak{o}(n)$, isotropy algebra at $\xi_0 = 0$.*

Proof. It is sufficient to verify $[Z_1, [Z_1, Z_a]] = [Z_1, Z_a]$ and $[[Z_1, Z_b], [Z_1, Z_b]] = 0$. □

It is important to notice that the invariant subbundle of tangent vectors to B_0 over $\mathcal{O}_p$ not a subbundle is of $T\mathcal{O}_p$. Hence the vector fields Z_a

are not tangent to the orbits. Define their orthonormalization with respect to g_o. Let $\bar{Z}_1 = Z_1/\|Z_1\|$ and

$$\bar{Z}_a = \frac{Z_a - g_o(\bar{Z}_{a-1}, Z_a)\bar{Z}_{a-1} - \cdots - g_o(\bar{Z}_1, Z_a)\bar{Z}_1}{\|Z_a - g_o(\bar{Z}_{a-1}, Z_a)\bar{Z}_{a-1} - \cdots - g_o(\bar{Z}_1, Z_a)\bar{Z}_1\|}. \tag{65}$$

for $a > 1$. The norm is take with respect to g_o.

Proposition 4.24. *For any $a \neq 1$ we have :*

$$[\bar{Z}_1,\, \bar{Z}_a] = \bar{Z}_a. \tag{66}$$

Proof. It is now sufficient to verify the bracket at p. Using (63) and $Z_1(N_a)(p) = 0$, with N_a the denominator in (65) we find

$$\begin{aligned}[\bar{Z}_1, \bar{Z}_a]|_p &= [Z_1, Z_a]|_p - Z_1(g_o(Z_a, Z_1))Z_1|_p \\ &= Z_a|_p,\end{aligned}$$

which proves the proposition. □

We clearly have that μ_1 is a closed one form which also follows from the brackets of the vector fields (66).
For arbitrary a and $b = 1$ we have

$$d_{B^n}\mu_1(\bar{Z}_a, \bar{Z}_1) = \bar{Z}_a(\mu_1(\bar{Z}_1)) - \bar{Z}_1(\mu_1(\bar{Z}_a)) - \mu_1([\bar{Z}_a, \bar{Z}_1])$$

$$= \bar{Z}_a(g_o(\bar{Z}_1, \bar{Z}_1)) - \bar{Z}_1(g_o(\bar{Z}_1, \bar{Z}_a)) - g_o(\bar{Z}_1, [\bar{Z}_a, \bar{Z}_1]) = 0.$$

Proposition 4.25. *Let $\lambda = \mu_2 \wedge \cdots \wedge \mu_n$ and $\omega_o = \mu_1 \wedge \lambda$ the associated volume form of g_o. Then*

$$d\lambda = -(n-1)\omega_o \tag{67}$$

and

$$d\eta = (n-1)\mu_1 \wedge \eta \tag{68}$$

Proof. The differential of λ is given by

$$d\lambda(\bar{Z}_1, \cdots, \bar{Z}_n) = \frac{1}{n}\sum_{i=1,\cdots,n}(-1)^{i-1}\bar{Z}_i(\lambda(\bar{Z}_1, \cdots, \check{\bar{Z}}_i, \cdots, \bar{Z}_n)$$

$$-\frac{1}{n}\sum_{1\leq i<j\leq n}(-1)^{i+j}\lambda([\bar{Z}_i, \bar{Z}_j], \bar{Z}_1, \cdots, \check{\bar{Z}}_i, \cdots \check{\bar{Z}}_j, \cdots, \bar{Z}_n)$$

which gives with the use of the definition of λ and the brackets of the vector fields

$$d\lambda(\bar{Z}_1, \cdots, \bar{Z}_n) = -\frac{n-1}{n}\frac{1}{(n-1)!}. \tag{69}$$

But then because $\omega_o(\bar{Z}_1, \cdots, \bar{Z}_n) = \frac{1}{n!}$, we find (1).

Part (2) follows from

$$d\eta = (n-1)f^{n-1}df \wedge \nu = (n-1)\frac{d_{B^n}f}{f} \wedge \eta \tag{70}$$

together with (61). In the proof we have been using the brackets determined over the orbit $\mathcal{O}_p$, but the result clearly is independent of the choosen vector fields and hence is valied in general.

□

Proposition 4.26. *The function f^{n-1} is harmonic with respect to the Laplace-Beltrami operator determined by the invariant metric g_o.*

Proof. From

$$d_o(f^k\lambda) = kf^{k-1}d_{B^n}f \wedge \lambda - \frac{n-1}{2}f^k\omega_o \tag{71}$$

follows

$$d_o(f^k\lambda) = \frac{1}{2}kf^k\mu_1 \wedge \lambda - \frac{n-1}{2}f^k\omega_o. \tag{72}$$

Then by taking $k = n-1$ we find

$$d_o(f^{n-1}\lambda) = 0. \tag{73}$$

Now $\bar{Z}_1 = g_o^{-1}\mu_1$ and $d_{B^n}f = \frac{1}{2}f\mu_1$, we find $grad_o\ (f^{n-1}) = \frac{1}{2}f^{n-1}Z_1$. $grad_o$ stands for the gradient on B_o with respect to g_o. Hence

$$L_{grad\ f^{n-1}}\omega_o = d(\imath_{(grad_o\ f^{n-1})}\omega_o) = \frac{1}{2}\,d(f^{n-1}\lambda) = 0. \tag{74}$$

This proves the proposition. □

Proposition 4.27. *The vector field $\bar{Z}_1$ is geodesic for each value of $\xi_1, \cdots, \xi_{n-1}$.*

Proof. The statement follows from

$$\begin{aligned} g_o(\nabla_{\bar{Z}_1}\bar{Z}_1, \bar{Z}_a) &= -g_o(\bar{Z}_1, \nabla_{\bar{Z}_1}\bar{Z}_a) \\ &= -2\,g_o(\bar{Z}_1, \bar{Z}_a) = 0 \end{aligned}$$

for all $a \neq 1$. $g_o(\nabla_{\bar{Z}_1}\bar{Z}_1, \bar{Z}_1) = 0$ because $\bar{Z}_1$ is of constant length for g_o.

(3) Volume forms and metrics Consider the space $B_0 \times S^{n-1}$. One has the invariant metric g_o and volume form ω_o on the first factor B_0. On the second factor the invariant metric g_S, which depends parametrically on B_0. Indicate this metric as g_1 and call the associated invariant volume form η_1. We now have $g_o + g_1$ as invariant metric on $B_0 \times S^{n-1}$ and $\omega_o \wedge \eta_1$ as invariant volume form. This procedure extends to all factors in the $n-1$ product $S^{n-1} \times \cdots \times S^{n-1}$. The metric then becomes

$$g = g_o + \sum_{i=1}^{n-1} g_i \tag{75}$$

with $g_i = f_i^2 d\sigma_i$, $f_i = (1 - \|\xi_0\|^2)/(\|\xi_0 - \xi_i\| \, |1- < \xi_i, \xi_0 > |)$ and $d\sigma_i^2$ the spherical metric on S_i^{n-1}.

4.2.3. *Higher order invariants.* Let $n \geq 4$ and $1 \leq k \leq n-3$. Consider the space

$$J^k B^n \times \Pi_{i=1}^{n-(k+1)} S_i^{n-1}. \tag{76}$$

The invariant sheaf is generated by the set

$$\mathcal{I}_k = \left\{ \begin{array}{ccccc} \mathcal{Q}_{0ij} & & & & \\ T_0 \mathcal{Q}_{0ij} & \widetilde{g}_o & & & \\ T_0^2 \mathcal{Q}_{0ij} & T_0 \widetilde{g}_o & \widetilde{K}_2|_0 & & \\ \vdots & \vdots & \vdots & & \\ T_0^k \mathcal{Q}_{0ij} & T_0^{k-1} \widetilde{g}_o & T_0^{k-2} \widetilde{K}_2|_0 & \vdots & \widetilde{K}_k|_0 \end{array} \right\} \tag{77}$$

The space $J^k B^n \times \Pi_{i=1}^{n-(k+1)} S_i^{n-1}$ has dimension $n^2 - n + (k+1)$. The rank of the set $\{\mathcal{Q}_{0ij}\}$ equals $\frac{(n-(k+1))(n-(k+2))}{2}$. The rank of the set $\{T_0 \mathcal{Q}_{0ij}, \cdots, T_0^k \mathcal{Q}_{0ij}\}$ equals $k(n-(k+1))$ while the remaining set $\{\widetilde{g}_o, \cdots, \widetilde{K}_k^o\}$ has rank equal to $\frac{k(k+1)}{2}$. Then $\dim J^k B^n \times \Pi_{i=1}^{n-(k+1)} S_i^{n-1} -$ rank $\mathcal{I}_k = \frac{n(n+1)}{2}$, which equals the dimension of the group. But the regular subset is the subset such that the orbits have maximal dimension, which proves the statement.

5. The special dimensions

In determination the first order invariants we have been using condition $n \geq 4$ (4.2.2). For this reason we need to review dimensions three and two.

5.1. **Dimension three.** Consider $\mathcal{E}_3 = B_0 \times S^2 \times S^2$, with $B_0 \equiv B^3$. The group of Möbius transformations has dimension 6. On the regular subset $\mathcal{W}_0$ there exists one invariant generator, namely $\mathcal{Q}_{012}$. We construct as before the invariant metric

$$g = g_o + g_1 + g_2 \tag{78}$$

where g_o is the invariant metric on B^3 and g_i, $i = 1, 2$ the invariant metric on i^{th} boundary. With respect to this metric we construct the invariant vector field $Y = g^{-1} d\mathcal{Q}_{012}$. This field is transversal to the orbits which allows the construction of an invariant metric by taking (1) the Killing form in the tangent space to the orbits and (2) defining the vector field Y as an orthonormal vector field to the orbits. Denote this metric by K_1. Let K_1^o be the restriction of K_1 to the first layer, then the lift of K_1^o is an invariant function $\widetilde{K}_1^o$ on $J^1 B_0 \times S^2 \times S^2$.

Let $\mathcal{W}_0^{(1)} = \pi_o^{-1} \mathcal{W}_0 \cap \{\dot{\xi}_0 \neq 0\}$ as subset of $J^1 B_3 \times S^2 \times S^2$. We then have

Theorem 5.1. *The set*

$$\mathcal{I}_0^{(1)} = \{\mathcal{Q}_{012},\, T_0 \mathcal{Q}_{0ij}, \widetilde{g}_o,\, \widetilde{K}_1^o\} \tag{79}$$

is a generating set of invariants on $\mathcal{W}_0^{(1)}$

The diagonal embedding

$$\triangle : J^1 B_3 \times S^2 \to J^1 B_3 \times S^2 \times S^2 \tag{80}$$

pulls back the generating functions defining a generating set $\mathcal{I}_1$. One verifies that

$$\mathcal{I}_1 = \{\widetilde{g}_o, \widetilde{K}_1^o\} \tag{81}$$

is the resulting generating set of invariants.

The construction for higher order generating sets is exactly the same as in the general case. On $J^2 B_3 \times S^2$ one finds

$$I_2 = \{\widetilde{g}_o,\, T_0 \widetilde{g}_o,\, \widetilde{K}_1^o,\, T_0 \widetilde{K}_1^o,\, \widetilde{K}_2^o\} \tag{82}$$

We omit the details.

5.2. **Dimension two.**

5.2.1. *At zero order.* This is the classical case of the unit disc $B = B_2$ in $\mathbb{R}^2$. Let (x, y) be the natural coordinates on $\mathbb{R}^2$. The action of the connected component of $GM(1)$ is given in complex form by [2] [7] $\Phi : \widetilde{z} = (az + b)/(\overline{b}z + \overline{a})$ with $|a|^2 - |b|^2 = 1$, where z, a, b are complex variables. The group $GM(1)$ is isomorphic with $PGL(2)$ and has Lie algebra $\mathfrak{sl}(2)$. With $a = 2(\alpha + i\beta)$, $b = 2(\gamma + i\sigma)$ and (σ, β, γ) as local coordinates on the group, a set of fundamental vector fields of the Möbius action on the unit disc is given by

$$
\begin{aligned}
H &= \Phi_* \partial_\sigma (g = e) = -xy\partial_x + \frac{1}{2}(1 + x^2 - y^2)\partial_y \\
K &= \Phi_* \partial_\beta (g = e) = (y\partial_x - x\partial_y) \\
L &= \Phi_* \partial_\gamma (g = e) = \frac{1}{2}(1 - x^2 + y^2)\partial_x - xy\partial_y.
\end{aligned}
\tag{83}
$$

Their commutators are

$$
[K, H] = -L, \ [K, L] = H, \ [H, L] = K. \tag{84}
$$

At first order the regular subset $\mathcal{W}_1$ in $J^1 B_2$ is given by $\|(\dot{x}, \dot{y})\| \neq 0$.

Proposition 5.2. *The sheaf of invariant functions on $\mathcal{W}_1$ is generated by $\widetilde{g}_o$.*

Applying corollary (2.7) we find

Proposition 5.3. *A motion depending parametrically on an extra point in B is a curve in $\mathcal{E}_2 = B_0 \times B_1$, which is regular with respect to the first factor, such that $\mathcal{Q}_{01}$ and g_o are constant along the curve.*

To exhibit the geometric structure of $J^1 B_2$ we recall the complete lift of the hyperbolic metric :

$$
g_o^c = 8 \frac{x\dot{x} + y\dot{y}}{(1 - x^2 - y^2)^3}(dx^2 + dy^2) + 4 \frac{dx d\dot{x} + dy d\dot{y}}{(1 - x^2 - y^2)^2}. \tag{85}
$$

Then with $\Gamma = {g_o^c}^{-1} d\widetilde{g}_o$, the spray of the connection, we find

$$
\Gamma = \dot{x}\partial_x + \dot{y}\partial_y + 2\frac{2y\dot{x}\dot{y} + x(\dot{x}^2 - \dot{y}^2)}{-1 + x^2 + y^2}\partial_{\dot{x}} + 2\frac{2x\dot{x}\dot{y} + y(\dot{y}^2 - \dot{x}^2)}{-1 + x^2 + y^2}\partial_{\dot{y}}. \tag{86}
$$

The level surfaces of $\widetilde{g}_o$ in $\mathcal{W}_1 \subset J^1 B_2$ are codimension one surfaces, which are $PGL(2)$-orbits. The vector field $I = \dot{x}\partial_{\dot{x}} + \dot{y}\partial_{\dot{y}}$ commutes with the vector fields H, K, L on $\mathcal{W}_1$ because they are linear homogeneous in the fibre coordinates $(\dot{x}, \dot{y})$. Consider the set $Z = (H^{(1)}, K^{(1)}, L^{(1)}, I)$ and let

A be the matrix of this set. Then $\triangle_1 = \det A = \frac{1}{4}(1-x^2-y^2)^2(\dot{x}^2+\dot{y}^2)^2$, which is different from zero on the regular subset $\mathcal{W}_1 = J^1B_2 \setminus \{\|(\dot{x},\dot{y})\| = 0\}$. Hence Z is a linear independent set at each point of $\mathcal{W}_1$.
The canonical one form ω with values in $\mathfrak{sl}(2) \oplus \mathbb{R}$ is determined by

$$\begin{aligned}
\omega_1 &= \frac{2}{1-x^2-y^2}dy + \frac{2x\dot{y}d\dot{x} - 2x\dot{x}d\dot{y}}{(1-x^2-y^2)(\dot{x}^2+\dot{y}^2)} \\
\omega_2 &= \frac{-2ydx+2xdy}{1-x^2-y^2} + \frac{(1+x^2+y^2)(\dot{y}d\dot{x}-\dot{x}d\dot{y})}{(1-x^2-y^2)(\dot{x}^2+\dot{y}^2)} \\
\omega_3 &= \frac{2dx}{1-x^2-y^2} - \frac{2y\dot{y}d\dot{x}-2\dot{x}yd\dot{y}}{(1-x^2-y^2)(\dot{x}^2+\dot{y}^2)} \\
\omega_4 &= \frac{2xdx+2ydy}{1-x^2-y^2} + \frac{\dot{x}d\dot{x}+\dot{y}d\dot{y}}{\dot{x}^2+\dot{y}^2}
\end{aligned} \tag{87}$$

Remark that $\omega_4 = \frac{1}{2}d\log\widetilde{g}_o$. Using the Killing metric on the orbits we determine the invariant metric

$$K_2 = (w^1)^2 - (w^2)^2 + (w^3)^2 + (w^4)^2 \tag{88}$$

According to the general case, the level surfaces of $\widetilde{g}_o$ are null surfaces with signature $(0,+,-)$ for the metric g_o^c.

Proposition 5.4.

$$(1)\ K_2(\Gamma) = \theta,\ (2)\ K_2(I) = \omega^4,\ (3)\ K_2(I,\Gamma) = 0 \tag{89}$$

Recall that the canonical form associated with g_o in coordinates equals

$$\theta = \frac{4(\dot{x}dx+\dot{y}dy)}{(1-x^2-y^2)^2}. \tag{90}$$

Proof of the above proposition is then obtained by direct verification. Next proposition, which then is a consequence of the former, asserts that Γ is a constant vector field along the orbits of $PGL(2)$ with positive norm.

Proposition 5.5.

$$K_2(\Gamma,\Gamma) = 4\frac{\dot{x}^2+\dot{y}^2}{(1-x^2-y^2)^2} \tag{91}$$

Remark that the operator $P = Id - I \otimes \omega^4$ is an invariant operator projecting upon the tangent space to the orbits.

Proposition 5.6. *The complete lift of a regular curve $\gamma(t)$which is lying in a level surface of $\widetilde{g}_o$ has constant angle with the vector field Γ, with respect to the metric K_2.*

Proof. It suffices to calculate $K_2(\dot{\gamma}^c(t), \Gamma)$. This equals

$$\theta(\dot{\gamma}^c(t)) = 4\frac{\dot{x}^2 + \dot{y}^2}{(1 - x^2 - y^2)^2}(\gamma(t))$$

which is a constant if $\gamma^c(t)$ is lying in a level surface of $\mathcal{Q}_1$. □

Let $\mathcal{W}_2 = \pi^{-1}\mathcal{W}_1$. The tensor field g_o itself is also a tensor field on J^1B_2 ; we will use the same symbol for it. Any symmetric covariant tensor field T on a manifold is a function, denoted by $\widehat{T}$, on the tangent space. Let $\jmath : J^2B_2 \to TJ^1B_2$ be the canonical embedding, then $\widetilde{g_o} = \jmath^*\widehat{g}_o$, $\widetilde{g_o^c} = \jmath^*\widehat{g}_o^c$ and $\widetilde{K}_2 = \jmath^*\widehat{K}_2$ are invariant functions on J^2B_2.

Theorem 5.7. *The invariant sheaf on the regular subset $\mathcal{W}_2$ is generated by the germs of the set*

$$\mathcal{I}_{\mathcal{W}_1} = \{\widetilde{g}, \widetilde{g^c}, \widetilde{K}_2\}. \tag{92}$$

Theorem 5.8. *Let $\gamma(t)$ be a regular curve in B_0. Then $\gamma(t)$ is the trajectory of a one parameter subgroup of $PGL(2)$ iff $\widetilde{g_o}$ and $\widetilde{K}_2$ are constant along $\gamma(t)$.*

Proof. From $g_o(\dot{\gamma}, \dot{\gamma}) = a,\ a \in \mathbb{R}$ we find $g_o^c(\dot{\gamma}^c, \dot{\gamma}^c) = 0$, which implies that the tangent vector to the curve is a null vector in g_o^c. But constant $g_o(\dot{\gamma}, \dot{\gamma})$ also implies, using proposition (5.6) that $\dot{\gamma}^c(t)$ has constant angle (with respect to K_2) with Γ and is lying in a null two plane formed by a null vector in g_o^c transversal to Γ and Γ itself. But requirement $K_2(\dot{\gamma}^c(t), \dot{\gamma}^c(t)) = c$, $c \in \mathbb{R}$ implies that $\dot{\gamma}^c(t)$ is constant with respect to a complete set of right invariant vector fields in the orbit. This implies the theorem. □

5.2.2. *Invariant functions on $J^1B_2 \times S^1$.* We now consider sets of two points (ξ_0, ξ_1) with $\xi_0 \in B_2$ and $\xi_1 \in S^1$, the boundary. Call this the space $\mathcal{E}_2 = B_2 \times S^1$. For this calculations we prefer the polar coordinates on B_2 and hence use (ρ, θ, φ) as coordinates on $\mathcal{E}_2$. The fundamental vector fields on $\mathcal{E}_1$ then are

$$\begin{aligned} H &= \frac{1}{2}(1 - r^2)\sin\theta\partial_r + \frac{1 + r^2}{2r}\cos\theta\partial_\theta + \cos\varphi\partial_\varphi \\ K &= -\partial_\theta - \partial_\varphi \\ L &= \frac{1}{2}(1 - r^2)\cos\theta\partial_r - \frac{1 + r^2}{2r}\sin\theta\partial_\theta - \sin\varphi\partial_\varphi \end{aligned} \tag{93}$$

Let $\Delta = 1 + r^2 - 2r\cos(\theta - \varphi)$, the basic one forms are

$$\begin{aligned}
\omega_1 &= 2\frac{1}{\Delta}\left[-\frac{(1+r^2)\sin\theta - r\sin\varphi}{(-1+r^2)}dr + r\cos\theta d\theta - r\cos\theta d\varphi\right]\\
\omega_2 &= \frac{1}{\Delta}\left[-2\frac{(1+r^2)\sin(\theta-\varphi)}{(-1+r^2)}dr + 2r\cos(\theta-\varphi)d\theta - (1+r^2)d\varphi\right]\\
\omega_3 &= 2\frac{1}{\Delta}\left[-\frac{(1+r^2)\cos\theta - r\cos\varphi}{(-1+r^2)}dr - r\sin\theta d\theta + r\sin\theta d\varphi\right]
\end{aligned} \tag{94}$$

The Killing metric $K = \omega_1^2 - \omega_2^2 + \omega_3^2$ restricted to the first factor of $\mathcal{E}_2$ equals $K|^o = \mu.\mu$, with

$$\mu = 2\frac{-2r + (1+r^2)\cos(\theta-\varphi)}{(-1+r^2)(1+r^2-2r\cos(\theta-\varphi))}dr + 2\frac{r\sin(\theta-\varphi)}{1+r^2-2r\cos(\theta-\varphi)}d\theta \tag{95}$$

Because $K|^o$ is invariant the one form μ is an invariant one form. As a consequence of the invariance of the metric g_o the norm $g_o^{-1}(\mu,\mu)$ has to be constant. We find $g_o^{-1}(\mu,\mu) = 1$. We are now able to construct a one parameter family of invariant metrics. Let

$$g(k) = g_o + k\,\mu.\mu \tag{96}$$

The determinant of this metric equals $(1+k)r^2/(1-r^2)^4$ from which it follows that the metrics $g(k)$ are Riemannian for $-1 < k$ and singular for $k = -1$. For $k = -1$ we find $g(k=0) = \nu.\nu$ with

$$\nu = -2\frac{r(-2r + (1+r^2)\cos(\theta-\varphi))}{(-1+r^2)(1+r^2-2r\cos(\theta-\varphi))}d\theta + 2\frac{\sin(\theta-\varphi)}{1+r^2-2r\cos(\theta-\varphi)}dr. \tag{97}$$

It follows that the one parameter family of invariant metrics can be written as

$$g(k) = \nu^2 + (k+1)\mu^2. \tag{98}$$

The one forms μ and ν are defined on B_2 and $d_{B_2}\mu = 0$ but $d_{B_2}\nu \neq 0$. Because μ is d_{B_2}-closed it is d_{B_2}-exact and hence there exists a function ϕ on $\mathcal{E}_2$ such that $\mu = d_{B_2}\phi$, namely

$$\phi = \log\frac{1+r^2-2r\cos(\theta-\varphi)}{1-r^2}. \tag{99}$$

Remark that the function ϕ is not an invariant function on $\mathcal{E}_2$ because the only invariant functions are constants. We now are able to describe the invariant sheaf on $J^1B_2 \times S^1$.

Theorem 5.9. *Let $\mathcal{W}_{(1,0)} = J^1B_2 \times S^1 \setminus \{\dot{x} = 0\}$. The sheaf of invariant functions $\mathcal{A}_{\mathcal{W}_{(1,0)}}$ is generated by the set*

$$\mathcal{I}_{\mathcal{W}_{(1,0)}} = \{g_o,\, K|^o\}. \tag{100}$$

The set $\{\dot{x} = 0\}$ stands for the zero section of the bundle $J^1B_2 \to B_2$. As a consequence of the decomposition of the invariant metric we may reformulate this proposition.

Theorem 5.10. *The set $\mathcal{J}_{\mathcal{W}_{1,0}} = \{\widetilde{\mu},\, \widetilde{\nu}\}$ is a generating set of invariants on $\mathcal{W}_{(1,0)}$.*

The one forms are seen as functions on the 1-jets of curves in B_2. Again we are able to formulate next proposition.

Properties 5.11. *A regular curve in $B_2 \times S$ is a motion of $Pl(2)$ in B_2 depending parametrically on S iff $\widetilde{\mu}$, $\widetilde{\nu}$ are constant along the curve.*

We make this more explicit. Let (ξ_0^o, ξ_1^o) be a given point in $\mathcal{E}_2$ and $\mathcal{I}_{\xi_0^o}$ be the isotropy group at $\xi_0^o \in B_2$. A curve $\gamma(t) = (\xi_0(t), \xi_1(t))$ is said to be a motion depending parametrically on a point in the boundary if $\gamma(t)$ is of the form $\gamma(t) = (g_t.\xi_0^o,\, g_t.f(t).\xi_1^o)$ with g_t a one parameter subgroup of $Pl(2)$ and $f(t)$ a curve in $\mathcal{I}_{\xi_0^o}$. The curve satisfies the above conditions if $K|^o(\dot{\gamma}(t), \dot{\gamma}(t))$ and $g_o(\dot{\gamma}(t),\, \dot{\gamma}(t))$ are constant.

5.2.3. *Invariant functions on $B_2 \times J^1S^1$.* Taking the restriction of the Killing form to the tangent space of the second factor in $\mathcal{E}_2 = B_2 \times S^1$ gives $K|^1 = \eta.\eta$ with

$$\eta = \frac{(1 - r^2)}{1 + r^2 - 2r\cos(\theta - \varphi)} d\varphi \tag{101}$$

The one form η defines an invariant measure on the boundary S^1 depending parametrically on a point in B_2. One easily finds that $\int_0^{2\pi} \eta = 2\pi$. The integral has to be a constant because it is an invariant and the only invariants on B_2 are constants. The invariant one form (101) is the Poisson kernel on the unit disk [1]. Accordingly to (4.26) the function $(1 - r^2)/(1 + r^2 - 2r\cos(\theta - \phi)$ is harmonic in the B_2-variables and $d\varphi$ is the rotation invariant Lebesgue measure on S^1 when the integral is taken as a Lebesgue integral.

We now have the theorem

Theorem 5.12. *Let $\mathcal{W}_{(0,1)}$ be the subset $B_2 \times J^1S^1 \setminus \{\dot{\varphi} = 0\}$. The sheaf of invariant functions $\mathcal{A}_{\mathcal{W}_{(0,1)}}$ is generated by the function $\widetilde{\eta}$.*

and the proposition

Properties 5.13. *Let $\gamma(t)$ being a regular curve in the boundary $B_2 \times S$. Then $\gamma(t)$ is a motion of $Pl(2)$ in the boundary S depending parametrically on a point in B_2 if and only if the one form η is constant along the curve.*

5.2.4. *Invariant functions on $J^1B_2 \times J^1S^1$.* We next define the invariant vector fields $X_\mu = g_o^{-1}\mu$, $X_\nu = g_o^{-1}\nu$ and X_η, where X_η is determined by the invariant condition $\eta(X_\eta) = 1$ and $\pi_{o\,*}X_\eta = 0$, π_o being the projection $\mathcal{E}_2 \to B_2$. We then have

$$X_\mu = \frac{(1-r^2)(-2r+(1+r^2)\cos(\theta-\varphi)}{2(1+r^2-2r\cos(\theta-\varphi))}\partial r + \frac{(-1+r^2)^2\sin(\theta-\varphi)}{2r(1+r^2-2r\cos(\theta-\varphi)}\partial_\theta$$

$$X_\nu = \frac{(-1+r^2)^2\sin(\theta-\varphi)}{2(1+r^2-2r\cos(\theta-\varphi))}\partial_r + \frac{r(-1+r^2)-(-1+r^4)\cos(\theta-\varphi)}{r(1+r^2-2r\cos(\theta-\varphi)}\partial_\theta$$

$$X_\eta = \frac{1+r^2-2r\cos(\theta-\varphi)}{1-r^2}\partial_\varphi.$$

Because the bracket of two invariant vector fields is an invariant vector field, the above vector fields constitute the dual Lie algebra up to automorphism. Their brackets are

$$[X_\mu, X_\nu] = -X_\nu,\ [X_\mu, X_\eta] = X_\nu + X_\eta,\ [X_\nu, X_\eta] = -X_\mu. \tag{102}$$

Theorem 5.14. *Let $\mathcal{W}_{(1,1)} = J^1B_2 \times J^1S^1 \setminus (\{\dot{x} = 0\} \cup \{\dot{\varphi} = 0\})$. The sheaf of invariant functions on $\mathcal{W}_{(1,1)}$ is generated by the set $\mathcal{I}_{\mathcal{W}_{(1,1)}} = \{\widetilde{\mu}, \widetilde{\nu}, \widetilde{\eta}\}$.*

Proposition 5.15. *Let $\gamma(t)$ be a regular curve in $\mathcal{E}_2$. Then $\gamma(t)$ is a motion of $Pl(2)$ iff the forms μ, ν and η on $\mathcal{W}_{(1,1)}$ are constant along the curve.*

We also have from the general case

Proposition 5.16. *For each value of φ the vector field X_μ is geodesic.*

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