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PURITY OF THE IDEAL OF CONTINUOUS FUNCTIONS WITH COMPACT SUPPORT(1)

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Abstract. Let C(X) be the ring of all continuous real valued functions defined on a completely regular T_1-space. Let $C_K(X)$ be the ideal of functions with compact support.

Purity of $C_K(X)$ is studied and characterized through the subspace X_L, the set of all points in X with compact neighborhoods (nbhd).

It is proved that $C_K(X)$ is pure if and only if $X_L = \bigcup_{f \in C_K} \operatorname{supp} f$. if $C_K(X)$ and $C_K(Y)$ are pure ideals, then $C_K(X)$ is isomorphic to $C_K(Y)$ if and only if X_L is homeomorphic to Y_L. It is proved that $C_K(X)$ is pure and X_L is basically disconnected if and only if for every $f \in C_K(X)$, the ideal $(f\ )$ is a projective C(X)-module. Finally it is proved that if $C_K(X)$ is pure, then X_L is an F'-space if and only if every principal ideal of $C_K(X)$ is a flat C(X)-module.

Concrete examples exemplifying the concepts studied are given.

1. Introduction

Let X be a completely regular T_1-space, βX the Stone-Cech compactification of X and let C(X) be the ring of all continuous real valued functions defined on X.

For each $f \in C(X)$, Let $Z(f)=\{x \in X: f(x)=0\}$, $\operatorname{coz} f = X - Z(f)$, $\operatorname{supp} f = \overline{\operatorname{coz} f}$.

$$f^*(x) = \begin{cases} 1 & f(x) > 1 \\ f(x) & -1 \leq f(x) \leq 1 \\ -1 & f(x) < -1 \end{cases}$$

and $\bar{f} = (f^*)^\beta$, the continuous extension of f^* to βX. If I is an ideal of C(X), let coz $I = \bigcup_{f \in I} \operatorname{coz} f$. For each $K \subset \beta X$, let $O^K = \{f \in C(X) : K \subseteq \operatorname{Int}_{\beta X} \operatorname{cl}_{\beta X} Z(f)\}$ and $M^K = \{f = C(X) : K \subset \operatorname{cl}_{\beta X} Z(f)\}$. Let $C_K(X)$ denotes the ideal of functions with compact support.

Recall that an ideal I of C(X) is called pure if for each $f \in I$ there exists $g \in I$ such that $f = fg$ and in this case $g = 1$ on supp f. I is called

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a projective ideal if it is a projective C(X)-module, it is called a flat ideal if it is a flat C(X)-module. A ring is called a PP-ring if every principal ideal of it is projective, it is called a PF-ring if every principal ideal of it is flat. A space X is called paracompact if every open cover of X has a locally finite open refinement, it is called basically disconnected if supp f is open for each $f \in \mathrm{C(X)}$. X is called an F-space if any two disjoint cozero sets in X are completely separated, it is called an F′-space if any two disjoint cozero sets have disjoint closures. For an F′-space which is not an F-space, see [7]. For all notations and undefined terms in this paper the reader may consult [8].

Bkouche in [3] proved that if X is locally compact, then $\mathrm{C_K(X)}$ is pure. in this paper without assuming local compactness, we characterize purity of the ideal $\mathrm{C_K(X)}$ using the subspace $\mathrm{X_L}$. And for this we prove that $\mathrm{X_L}$ is an open subspace of βX. Bkouche also proved in [3] that if X is locally compact, then $\mathrm{C_K(X)}$ is projective if and only if X is paracompact. We extend this to arbitrary space X and prove that $\mathrm{C_K(X)}$ is projective if and only if $\mathrm{X_L}$ is paracompact and $\mathrm{C_K(X)}$ is pure.

Vechtomov in [12] proved that if X is locally compact, then every principal ideal of $\mathrm{C_K(X)}$ is projective if and only if X is basically disconnected. We extend this result and prove that if $\mathrm{C_K(X)}$ is pure, then every principal ideal of $\mathrm{C_K(X)}$ is projective if and only if $\mathrm{X_L}$ is basically disconnected.

Al-Ezeh, Natsheh and Hussein in [1] proved that C(X) is a PF-ring if and only if X is an F-space. We prove a theorem analogous to this and prove that if $\mathrm{C_K(X)}$ is pure, then every principal ideal of $\mathrm{C_K(X)}$ is flat if and only if $\mathrm{X_L}$ is an F′-space.

2. The Subspace $\mathrm{X_L}$

Most of the results about the ideal $\mathrm{C_K(X)}$ were proved under the assumption that X is locally compact space, although there are some non-locally compact spaces with the ideal $\mathrm{C_K(X)}$ having some nice properties.

In trying to characterize the properties of $\mathrm{C_K(X)}$, we found that we don't need local compactness but only the properties of the subspace of all points with compact nbhds which we will denote by $\mathrm{X_L}$. In fact, X is locally compact if and only if $\mathrm{X} = \mathrm{X_L}$. X is nowhere locally compact if and only if $\mathrm{X_L} = \phi$.

The following lemma establishes the relationship between $\mathrm{C_K(X)}$ and $\mathrm{X_L}$ and its proof follows immediately using complete regularity of the space X.

Lemma 2.1. *For each space X, $X_L = coz(C_K(X))$.*

Nowe, we prove the following theorem that generalizes a result for realcompact spaces proved by Harris in [9].

Theorem 2.2. *For each space X, $X_L = Int_{\beta X}X$.*

Proof. Let $x \in Int_{\beta X}X$, then there exists an open set U of βX such that $x \in U \subseteq X$. Regularity of βX implies that there exists an open set V of βX such that $x \in V \subseteq cl_{\beta X}V \subseteq U$. Hence, $cl_{\beta X}V$ is a compact nbhd of x in X and so $x \in X_L$.

If $x \in X_L$, then by lemma 2.1, there exists $f \in C_K(X)$ such that $f(x) \neq 0$. But $\beta X\text{-}X \subseteq Int_{\beta X}Z(\bar{f}) \subseteq Z(\bar{f})$, since $C_K(X) = O^{\beta X\text{-}X}$. So $Z(\bar{f}) = Z(f) \bigcup (\beta X\text{-}X)$. Hence $\beta X\text{-}Z(\bar{f}) = (\beta X\text{-}Z(f)) \bigcap X = X - Z(f) = coz\, f$. Therefore coz f is an open set in βX, and so $x \in coz\, f \subseteq Int_{\beta X}X$. □

The above theorem have the following important consequences.

Corollary 2.3. *X_L is an open subspace of both X and βX.*

Corollary 2.4. *X_L is a locally compact subspace of X.*

Corollary 2.5. *For each $f \in C_K(X)$, coz f is an open subset of βX.*

Corollary 2.6. *$X_L = \phi$ if an only if βX-X is dense in X if and only if $C_K(X) = \{0\}$.*

Proof. The result follows since $\beta X\text{-}X_L = cl_{\beta X}(\beta X\text{-}X)$. □

3. Purity of $C_K(X)$

Bkouche in [3] proved that if X is locally compact, then $C_K(X)$ is a pure ideal. Brookshear in [4] and Natsheh and Al-Ezeh in [10] gave simpler proofs of Bkouche's result. We found that there are non-locally compact spaces X such that $C_K(X)$ is a non-trivial pure ideal. An example is given at the end of this section.

Here we characterize purity of $C_K(X)$ using the subspace X_L for arbitrary space X.

Lemma 3.1. *If I is a pure ideal of C(X), then coz I = $\bigcup_{f \in I} supp\, f$.*

Proof. It is clear that coz I = $\bigcup_{f \in I}$ coz $f \subseteq \bigcup_{f \in I}$ supp f. Now, if $f \in I$, then there exists $g \in I$ such that $f = fg$. So g=1 on supp f. Thus supp$f \subseteq$coz g. Hence coz I = $\bigcup_{f \in I}$ supp f. □

Theorem 3.2. *Let I be a z-ideal contained in $C_K(X)$. Then the following statements are equivalent :*

(1) I is a pure ideal.

(2) I = $O^{\beta X\text{-}coz\ I}$.

(3) coz I = $\bigcup\limits_{f\in I}$ supp f.

Proof. (1) $\Rightarrow$ (2) : Suppose I is a pure ideal. Then I = O^{A}, where A = $\bigcap\limits_{f\in I} Z(\bar{f})$, see [10]. So βX - A = $\bigcup\limits_{f\in I} \beta X - Z(\bar{f})$.

βX - A = $\bigcup\limits_{f\in I} \beta X - (Z(f) \bigcup (\beta X - X)) = \bigcup\limits_{f\in I} (\beta X - Z(f)) \bigcap X = \bigcup\limits_{f\in I}$ coz f, since $\beta X - X \subseteq Z(\bar{f})$.

(2) $\Rightarrow$ (3) : It follows by corollary 2.5 that βX - coz I is a closed subset of βX. Hence I is a pure ideal, see [10]. So it follows by 3.1 that coz I = $\bigcup\limits_{f\in I}$ supp f.

(3) $\Rightarrow$ (1) : Let $g \in$ I, then supp $g \subseteq \bigcup\limits_{f\in I}$ supp f = coz I = $\bigcup\limits_{f\in I}$ coz f.

So, supp $g \subseteq \bigcup\limits_{i=1}^{n}$ coz f_i , for f_1, f_2, ... , $f_n \in$ I, since supp g is compact.

Let $h = \sum\limits_{i=1}^{n} f_i^2$, then $h \in$ I and coz $h = \bigcup\limits_{i=1}^{n}$ coz f_i. Let $k \in$ C(X) such that k(supp g) = 1 and $k(Z(h)) = 0$. Hence $g = gk$ and $Z(h) \subset Z(k)$. Therefore, $k \in$ I, since I is a z-ideal. Thus I is a pure ideal. $\square$

Here we give some important examples of pure and non-pure ideals, that might be of interest to the reader.

Example 3.1. *Let R, N and W be the set of reals,natural numbers and the set of all ordinals less than the first uncountable ordinal number, respectively. Then $C_K(R)$, $C_K(N)$ and $C_K(W)$ are pure ideals, since all these spaces are locally compact.*

Example 3.2. *Let Q be the set of rational numbers and S the real numbers with the Sorgenfrey line topology. Then $C_K(Q)$, $C_K(S)$ and $C_K(S\times S)$ are pure, since all these spaces are nowhere locally compat.*

Example 3.3. *Let X = [-1,1] with all points are isolated except for x=0 has its usual nbhds. Then X_L = X-{0}.*

Let $f(x) = \begin{cases} x & x = \frac{1}{n}\ n \in Z^* \\ 0 & otherwise \end{cases}$

Then supp f $=\{\frac{1}{n} : n \in Z^*\} \bigcup \{0\}$ *. So $f \in C_K(X)$ and supp f is not contained in X_L. So $C_K(X)$ is not a pure ideal. Now, let J = { $f \in C(X)$:*

f = 0 except on a finite set }. Then J is a z-ideal contained in $C_K(X)$ and supp g= coz $g \subseteq$ coz $J = X_L$ for each $g \in J$. Hence, J is a pure ideal.

Example 3.4. *Let $X = Q$ with all points having their usual nbhds except for $x = 0$ is isolated. Then $X_L = \{0\}$ and $C_K(X) = \{ f \in C(X)$: $f = 0$ except for $x = 0$ } is a pure ideal.*

Example 3.5. *Let $X = \{r \in R$: $r \in Q$ or $-1 \leq r \leq 1\}$ with the subspace topology. Then $X_L = (-1 , 1)$ and $C_K(X) = \{f \in C(X)$: coz $f \subseteq (-1 , 1)\}$.*

Let $g(x) = \begin{cases} 0 & x < -1 \\ 1+x & -1 \leq x \leq 0 \\ 1-x & 0 < x \leq 1 \\ 0 & x > 1 \end{cases}$

Then $g \in C_K(X)$, but supp g =$[-1 , 1]$ is not contained in X_L. So $C_K(X)$ is not a pure ideal.

Example 3.6. *Let $X = R$ with the integers are isolated and any other point has the Sorgenfrey line topology nbhd. Then $X_L = Z$ -the set of all integers- is discrete and clopen and $C_K(X) = \{ f \in C(X)$: $f = 0$ except on a finite set of integers } is pure.*

4. Some Applications

In this section we prove some properties of C_K(X) when it is pure, using the condition that C_K(X) is pure if and only if $X_L = \bigcup_{f \in C_K}$ supp f .

The following theorem generalizes the result of Vechtomov in [12], which he proved for locally compact spaces.

Theorem 4.1. *Suppose that $C_K(X)$ is a pure ideal. Then for each proper ideal I of $C_K(X)$, coz I is contained properly in X_L.*

Proof. Suppose I is an ideal of C_K(X) such that coz I = X_L.
Let $f \in C_K$(X). Since C_K(X) is a pure ideal, then supp $f \subseteq X_L$ = coz I. Hence supp $f \subseteq \bigcup_{i=1}^{n}$ coz f_i , where $f_i \in$I, for each i. Let $g = \sum_{i=1}^{n} f_i^2$, then $g \in$ I, and coz g = $\bigcup_{i=1}^{n}$ coz f_i.

Define $h(x) = \begin{cases} (\frac{f}{g})(x) & x \in \text{coz } g \\ 0 & \text{otherwise} \end{cases}$

Then $h \in$C(X), since supp $f \subseteq$ coz g. Moreover $f = gh \in$I. Hence I = C_K(X). □

Remark 4.1. *If $C_K(X)$ is not pure, then the above theorem need not be true, since for the function g defined in example 3.7 $g \notin I = gC_K(X)$, but coz $I = X_L$.*

The following theorem generalizes the result of Bkouche in [3] which he proved for locally compact spaces.

Theorem 4.2. *Let I be a z-ideal contained in $C_K(X)$. Then I is a projective C(X)-module if and only if coz I is paracompact and I is pure.*

Proof. See theorem 3.2 and [4]. □

Remark 4.2. *The two conditions in 4.2 are both necessary. Let W be the set of all ordinal numbers less than ω_1 the first uncountable ordinal, then $C_K(W) = M^{\omega_1}$. Then $C_K(W)$ is pure, but W is not paracompact and $C_K(W)$ is not projective since a prime projective C(X)-module is of the form M_x for some isolated point $x \in X$, see [5]. On the other hand, for the space defined in Example 3.7, X_L is paracompact, while $C_K(X)$ is not projective since it is not pure. For the spaces defined in examples 3.6 and 3.8, $C_K(X)$ is a projective ideal.*

The following theorem generalizes the result of Vechtomov in [12] which he proved for locally compact spaces.

Theorem 4.3. *Let $C_K(X)$ and $C_K(Y)$ be pure ideals. Then X_L is homeomorphic to Y_L if and only if $C_K(X)$ is ring isomorphic to $C_K(Y)$.*

Proof. If $C_K(X)$ is isomorphic to $C_K(Y)$, then X_L is homeomorphic to Y_L, since the maximal ideals of $C_K(X)$ are precisely the sets $M^x \bigcap C_K(X)$ for each $x \in X$, see [11].

Conversely, Suppose $\varphi : X_L \to Y_L$ is a homeomorphism. Let $f \in C_K(Y)$, then $f_1 \circ \varphi \in C(X_L)$, where $f_1 = f_{|Y_L}$. But coz $f = \varphi$ (coz $f_1 \circ \varphi$) , which implies that $\varphi^{-1}(\text{coz } f) = \text{coz } f_1 \circ \varphi$.

Therefore $\text{cl}_{X_L} \text{coz } f_1 \circ \varphi = \text{cl}_{X_L} \varphi^{-1}(\text{coz } f) = \varphi^{-1} \text{supp } f$, since supp f is contained in Y_L.(here we used purity of $C_K(Y)$). Now, for each $f \in C_K(X)$, define

$$g_f : X \to R \text{ by } g_f(x) = \begin{cases} f_1 \circ \varphi(x) & x \in X_L \\ 0 & x \in X\text{-}\varphi^{-1}(\text{supp } f) \end{cases}$$

Then, $g_f \in C_K(X)$, since supp $g_f = \text{cl}_{X_L}\text{coz } f_1 \circ \varphi$ is compact.

Define $\bar{\varphi} : C_K(Y) \to C_K(X)$ by $\bar{\varphi}(f) = g_f$. Then $\bar{\varphi}$ is a ring homomorphism.

Now, suppose $\bar{\varphi}(f) = 0$, then $f_1 \circ \varphi(x) = 0$ for every $x \in X_L$. But coz $f_1 \circ \varphi = \varphi^{-1}(\text{coz } f)$.

So $\phi = \varphi^{-1}(\text{coz } f)$. Thus $f = 0$. Let $f \in C_K(X)$. Define $g : Y \to R$ by

$$g(\mathrm{y}) = \begin{cases} f \circ \varphi^{-1}(y) & y \in \mathrm{Y}_L \\ 0 & y \in \mathrm{Y}\text{-}\varphi(\mathrm{supp}\ f) \end{cases}$$

Then $g \in$C(Y), since φ(supp f) is compact. (We used here purity of C_K(X), since we assumed that supp $f \subseteq \mathrm{X}_L$). Moreover, if g(y)$\neq 0$, then φ^{-1}(y) $\in$ coz f. So coz $g \subseteq \varphi$(coz f).

So, cl_{Y_L}coz $g \subseteq \mathrm{cl}_{Y_L}\varphi$(coz f) $=\varphi(\mathrm{cl}_X$ coz f) $= \varphi$(supp f). Hence supp $g = \mathrm{cl}_{Y_L}$coz g is compact.

Thus $g \in \mathrm{C}_K$(Y).

Now, $\bar{\varphi}\ (g)(\mathrm{x}) = \begin{cases} g_1 \circ \varphi(\mathrm{x}) & x \in \mathrm{X}_L \\ 0 & x \in \mathrm{X}\text{-}\varphi^{-1}(\mathrm{supp}\ g) \end{cases}$

$= \begin{cases} f \circ \varphi^{-1} \circ \varphi(\mathrm{x}) & x \in \mathrm{X}_L \\ 0 & x \in \mathrm{X}\text{-}\varphi^{-1}(\mathrm{supp}\ g) \end{cases}$

$= \begin{cases} f(\mathrm{x}) & x \in \mathrm{X}_L \\ 0 & \text{otherwise} \end{cases}$

$= f(\mathrm{x})$.

So $\bar{\varphi}\ (g) = f$. Hence C_K(X) is ring isomorphic to C_K(Y). □

Corollary 4.4. *If C_K(X) is a pure ideal, then C_K(X) is isomorphic to $C_K(X_L)$.*

Proof. Just take Y $= \mathrm{X}_L$ in theorem 4.3. □

The condition that C_K(X) is pure in 4.3 and 4.4 can not be removed all the way, since if C_K(X) is not pure , then take Y$= \mathrm{X}_L$. Then $\mathrm{X}_L = \mathrm{Y}_L =$ Y, but C_K(X) is not isomorphic to C_K(Y), since the latter is pure because Y is locally compact.

Brookshear in [4] proved that the principal ideal (f) is a projective C(X)-module if and only if supp f is clopen, and so C(X) is a PP-ring if and only if X is basically disconnected. Here we used the above to generalize the result of Vechtomov in [12] which he proved it for locally compact spaces.

Theorem 4.5. *Let I be a pure ideal contained in C_K(X). Then coz I is basically disconnected if and only if every principal ideal of I is a projective C(X)-module.*

Proof. Let Y $=$ coz I. Suppose that Y is basically disconnected and let $f \in$I, then supp $f \subseteq$Y since I is pure. Let $f_1 = f_{|_Y}$, then cl_Y(Y-Z(f_1)) $=$ supp f . So supp f is open in Y and therefore it is open in X.

Hence the ideal (f) is a projective C(X)-module.

Conversely, suppose that every principal ideal of I is a projective C(X)-module. We first show that for each $f \in \mathrm{C}_K$(Y) , supp f is clopen , then we use it to show that for each $f \in$C(Y), supp f is clopen .

So let $f_1 \in \mathrm{C}_K$(Y). Define f (x) $= \begin{cases} f_1(x) & x \in \mathrm{cl}_Y(\mathrm{Y}\text{-}\mathrm{Z}(f_1)) \\ 0 & x \in \mathrm{X}\text{-}(\mathrm{Y}\text{-}\mathrm{Z}(f_1)) \end{cases}$

Then f $\in$I , since it is a z-ideal and supp f is a compact set contained in Y . So (f) is a principal ideal of I , and therefore it is projective. Hence cl$_Y$(Y-Z(f_1)) = supp f is clopen.

Now, Let k $\in$C(Y) , and a $\in$cl$_Y$(Y-Z(k)) $\subseteq$Y . So there exists an open set U such that $\bar{\bar{U}}$ is compact, and a $\in$U$\subseteq$ $\bar{\bar{U}}$ $\subseteq$Y . There exists f $\in$C(X) such that f (a) = 1 and f (X-U) = 0 . Then f $\in$I since supp f is compact and contained in Y. Let f_1= f $|_Y$.

Thus a $\in$(Y-Z(f $_1$))$\bigcap$ cl$_Y$ (Y-Z(k))$\subseteq$ cl$_Y$ ((Y-Z(f $_1$)) $\bigcap$cl$_Y$ (Y-Z(k))) = cl$_Y$((Y-Z(f_1)) $\bigcap$(Y-Z(k))) = cl$_Y$ (Y-Z(f_1k)) $\subseteq$cl$_Y$ (Y-Z(k)) . But cl$_Y$ (Y-Z(f_1k)) is compact , and so is clopen since f_1k $\in$C$_K$(Y) . So, cl$_Y$ (Y-Z(k)) is clopen in Y. Thus Y = coz I is basically disconnected . □

Corollary 4.6. *X_L is basically disconnected and C_K(X) is pure if and only if for each f $\in C_K$(X), the ideal (f) is a projective C(X)-module.*

Proof. If supp f is clopen, then let g be the characteristic function of supp f . Then g $\in$C$_K$(X) and f=fg. □

The following corollary shows that for a locally compact space X, it is enough to prove that every principal ideal of C$_K$(X) is a projective C(X)-module to show that C(X) is a PP-ring.

Corollary 4.7. *Let X be a locally compact space. Then the following statements are equivalent:*

(1) C(X) is a PP-ring.

(2) X is basically disconnected.

(3) Every principal ideal of C_K(X) is a projective C(X)-module.

Remark 4.3. *Let X be the space defined in example 3.8. Then C_K(X) is pure and X_L is discrete. So every principal ideal of C_K(X) is a projective C(X)-module. On the other hand the for the function f defined in example 3.5 supp f = $\{\frac{1}{n}$: n $\in Z^*\} \bigcup$ $\{0\}$ is not open. So the ideal (f) is not projective. This shows that if I is not pure, then theorem 4.5 may not be true.*

It is proved in [1] that C(X) is a PF-ring if and only if X is an F-space. It is well-known that the ideal (f) is a flat C(X)-module if and only if Ann(f) is pure. We now use the above to characterize when every principal ideal of C$_K$(X) is flat.

Theorem 4.8. *Let I be a pure ideal contained in C_K(X). Then coz I is an F'-space if and only if every principal ideal of I is a flat C(X)-module.*

Proof. Let Y = coz I. Suppose that Y is an F′-space, f ∈I and g ∈Ann(f). Let $f_1 = f\,|_Y$ and $g_1 = g\,|_Y$. Then (Y-Z(f_1))$\bigcap$ (Y-Z(g_1)) = ϕ. So, cl$_Y$ (Y-Z(f_1)) $\bigcap$cl$_Y$(Y-Z(g_1)) = ϕ, since Y is an F′-space. But supp f =cl$_Y$(Y-Z(f_1)), since I is pure. Thus supp f $\bigcap$supp $g = \phi$. There exists k ∈C(X) such that k(supp f) = 0 and k(supp g)= 1. So, k ∈Ann(f) and $g = gk$. Thus the ideal (f) is a flat C(X)-module since Ann(f) is pure.

Conversely, suppose every principal ideal of I is a flat C(X)-module. Letg, k ∈C(Y) such that $gk = 0$. Suppose y∈ cl$_Y$(Y-Z(g))$\bigcap$ cl$_Y$(Y-Z(k)). There exists f_1 ∈I such that f_1(y)≠ 0. Let $f = f_1\ |_Y$. Then y∈ cl$_Y$(Y-Z(fg))$\bigcap$ cl$_Y$(Y-Z(fk). Define h_1(x) =$\begin{cases} fg(x) & x \in \text{cl}_Y(\text{Y-Z}(fg)) \\ 0 & x \in \text{X-(Y-Z}(fg)) \end{cases}$

and h_2(x) = $\begin{cases} fk(x) & x \in \text{cl}_Y(\text{Y-Z}(fk)) \\ 0 & x \in \text{X-(Y-Z}(fk)) \end{cases}$

Then h_1, h_2 ∈ I, since I is a z-ideal and supp h_1 and supp h_2 are compact sets contained in Y. Moreover, $h_1h_2 = 0$. So, there exists h_1' ∈Ann(h_2) such that $h_1 = h_1h_1'$. Hence y ∈cl$_Y$(Y-Z(fg)) = supp $h_1 \subseteq$ coz h_1' . But h_1' (supp h_2) = 0, so y∉ supp h_2 = cl$_Y$(Y-Z(f k)). Contradiction. Hence cl$_Y$(Y-Z(g)) $\bigcap$cl$_Y$(Y-Z(k)) = ϕ and Y = coz I is an F′-space. □

Corollary 4.9. *Let X be a locally compact space. Then X is an F′-space if and only if every principal ideal of $C_K(X)$ is a flat C(X)-module.*

Example 4.1. *Let X =$\beta R^+ - R^+$, then X is a compact, connected F-space, see [8]. So every principal ideal of $C_K(X)$ = C(X) is a flat C(X)-module, but not every principal ideal is projective.*

Example 4.2. *Let X be the space defined in example 3.5. Then X_L is an F′-space. For the function f defined there, Ann(f) is not pure, since the function g(x) =* $\begin{cases} 0 & x = \frac{1}{n}\ n \in Z^* \\ x & \textit{otherwise} \end{cases}$

belongs to Ann(f), but supp g = X- $\{\frac{1}{n}$: n∈ $Z^*\}$is not a subset of coz Ann(f), since for each h ∈ Ann(f), h(0) = 0. So the ideal (f) is not a flat C(X)-module. This example shows that theorem 4.8 need not be true if I is not pure.

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