A famous result of B.Osofsky says that a ring $R$ is semisimple artinian if and only if every cyclic left $R$-module is injective. The crucial point of her proof was to show that such a ring has finite uniform dimension. In [7], B.Osofsky and P.F.Smith proved more generally that a cyclic module $M$ has finite uniform dimension if every cyclic subfactor of $M$ is an extending module. Extending modules have been studied extensively in recent years and many generalizations have been considered by many authors (see, for examples, [1-4, 6, 8, 9]). Lopez-Permouth, Oshiro and Tariq Rizvi in [6] introduced the concepts of extending modules and (quasi-)continuous modules relative a given left $R$-module $X$. Let $S$ be the class of all semisimple left $R$-modules and all singular left $R$-modules. We say a left $R$-module $N$ is $S$-extending if $N$ is $X$-extending for any $X \in S$. Every extending left $R$-module is $S$-extending but the converse is not true. Exploiting the techniques of [7] we prove the following result: Let $M$ be a cyclic left $R$-module. Assume that all cyclic subfactors of $M$ are $S$-extending. Then $M$ satisfies ACC on direct summands. As a corollary we show that if cyclic left $R$-module $M$ is extending and all cyclic subfactors of $M$ are $S$-extending, then $M$ has finite uniform dimension.

Throughout this note we write $A \leq_e B$ ($A|B$) to denote that $A$ is an essential submodule (a direct summand) of $B$.

A left $R$-module $M$ is called singular if, for every $m \in M$, the annihilator $l(m)$ of $m$ is an essential left ideal of $R$.

**Lemma 1 ([4, 4.6]).** The following are equivalent for a left $R$-module $M$.

1. $M$ is singular.
2. $M \cong L/K$ for a left $R$-module $L$ and $K \leq_e L$.

Let $M, X$ be left $R$-modules. Define the family

$$\mathcal{A}(X, M) = \{A \subseteq M| \exists Y \subseteq X, \exists f \in \text{Hom}(Y, M), f(Y) \leq_e A\}.$$ 

Consider the properties

$\mathcal{A}(X, M)$-$(C_1)$: For all $A \in \mathcal{A}(X, M)$, $\exists A^*|M$, such that $A \leq_e A^*$.

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\(A(X, M)-(C_2)\): For all \(A \in A(X, M)\), if \(B|M\) is such that \(A \cong B\), then \(A|M\).

\(A(X, M)-(C_3)\): For all \(A \in A(X, M)\) and \(B|M\), if \(A|M\) and \(A \cap B = 0\) then \(A \oplus B|M\).

According to [6], \(M\) is said to be \(X\)-extending, \(X\)-quasi-continuous or \(X\)-continuous, respectively, if \(M\) satisfies \(A(X, M)-(C_1)\), \(A(X, M)-(C_1)\) and \(A(X, M)-(C_3)\), \(A(X, M)-(C_1)\) and \(A(X, M)-(C_2)\).

According to [8, 1, 2], a left \(R\)-module \(M\) is called a CESS-module if every complement with essential socle is a direct summand, equivalently, every submodule with essential socle is essential in a direct summand of \(M\). Now the following result is clear.

**Proposition 2.** A left \(R\)-module \(M\) is a CESS-module if and only if \(M\) is \(X\)-extending for any semisimple left \(R\)-module \(X\).

**Definition 3.** Let \(S\) be the class of all semisimple left \(R\)-modules and all singular left \(R\)-modules. A left \(R\)-module \(M\) is called \(S\)-extending if \(M\) is \(X\)-extending for any \(X \in S\).

Note that every extending left \(R\)-module is clearly \(S\)-extending. But the following example shows that the converse is not true.

**Example 4.** Let \(M\) be a free \(\mathbb{Z}\)-module of infinite rank. Since \(M\) is non-singular and has no socle, \(M\) is clearly \(S\)-extending. But \(M\) is not extending by [5, Theorem 5].

Let \(S_1\) and \(S_2\) be the classes of all semisimple left \(R\)-modules, of all singular left \(R\)-modules, respectively. Then \(S_1 \oplus S_2\) is defined to be the class of left \(R\)-modules \(M\) such that \(M = A \oplus B\) is a direct sum of \(A \in S_1\) and \(B \in S_2\).

**Proposition 5.** A left \(R\)-module \(M\) is \(S\)-extending if and only if it is \(X\)-extending for any \(X \in S_1 \oplus S_2\).

**Proof.** It follows from the fact that if \(0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0\) is an exact sequence then \(M\) is \(X\)-extending if and only if it is both \(X'\)-extending and \(X''\)-extending by [6, Proposition 2.7]. \(\Box\)

**Proposition 6.** Let \(M\) be a cyclic left \(R\)-module. Assume that all cyclic subfactors of \(M\) are \(S\)-extending. Then \(M\) satisfies ACC on direct summands.

**Proof.** We prove this by adapting the proof of [7, Theorem 1 and 4, 7.12]. Suppose that \(M\) does not satisfy ACC on direct summands and that \(A_1 \subset A_2 \subset A_3 \subset \ldots \) is an infinite ascending chain of direct summands \(A_i(i \geq 1)\) of \(M\). Then there exists a submodule \(B_1\) of \(M\) such
that $M = A_1 \oplus B_1$. Thus $A_2 = A_2 \cap (A_1 \oplus B_1) = A_1 \oplus (A_2 \cap B_1)$ so that $A_2 \cap B_1$ is a direct summand of $B_1$. Let $B_2$ be a submodule of $B_1$ such that $B_1 = (A_2 \cap B_1) \oplus B_2$. Then $M = A_2 \oplus B_2$. Repeating this argument we can produce an infinite descending chain

$$B_1 \supset B_2 \supset B_3 \supset \ldots .$$

of direct summands $B_i$ of $M$ such that $M = A_i \oplus B_i$. For each $i \geq 1$, there exists a nonzero submodule $C_{i+1}$ of $M$ such that $B_i = B_{i+1} \oplus C_{i+1}$. Put $C_1 = A_1$. Then

$$M = C_1 \oplus C_2 \oplus \cdots \oplus C_n \oplus B_n$$

and $\oplus_{i=n+1}^\infty C_i \subset B_n$ for all $n \geq 1$. Clearly $C_i$ is cyclic since $M$ is cyclic, and so $C_i$ contains a maximal submodule $W_i$. Put

$$P = M/(\oplus_{i=1}^\infty W_i), \quad Q = (\oplus_{i=1}^\infty C_i)/(\oplus_{i=1}^\infty W_i).$$

Then clearly $P$ is a cyclic subfactor of $M$ and $Q$ is a semisimple submodule of $P$. By the hypothesis, $P$ is $X$-extending for any $X \in S$. Particularly $P$ is $Q$-extending. It is easy to see that $Q \in \mathcal{A}(Q, P)$, and so there exists a direct summand $Q^*$ of $P$ such that $Q \leq_e Q^*$.

Note that $Q = \oplus_{i=1}^\infty S_i$ is an infinite direct sum of simple left $R$-modules $S_i$ ($i \geq 1$). Let $\{1, 2, \ldots \}$ be a disjoint union of countable sets $\{I_j|j = 1, 2, \ldots \}$. Set $Q_j = \oplus_{i \in I_j} S_i$, $j = 1, 2, \ldots$. Then $Q_j$ is a non-finitely generated semisimple left $R$-module. Clearly $Q^*$ is a cyclic subfactor of $M$. By the hypothesis, $Q^*$ is $X$-extending for any $X \in S$. Particularly $Q^*$ is $Q_j$-extending. It is easy to see that $Q_j \in \mathcal{A}(Q_j, Q^*)$, and so there exists a direct summand $Q^*_j$ of $Q^*$ such that $Q_j \leq_e Q^*_j$. Clearly $Q^*_j$ is finitely generated, and thus $Q_j \neq Q^*_j$.

Let $D_j = (Q^*_j + Q)/Q$. Since $Q^*_j \cap (\oplus_{k \neq j} Q_k) = 0$ and $Q_j \neq Q^*_j$, it is easy to see that $D_j \neq 0$. Also $Q_j \leq Q \cap Q^*_j \leq Q^*_j$, so $Q \cap Q^*_j \leq_e Q^*_j$. This implies that $D_j \simeq Q^*_j/(Q^*_j \cap Q)$ is singular by Lemma 1. Hence

$$D = \sum_{j=1}^\infty D_j = \oplus_{j=1}^\infty D_j$$

is a singular submodule of $Q^*/Q$. Since $Q^*/Q$ is a cyclic subfactor of $M$, it follows that $Q^*/Q$ is $X$-extending for any $X \in S$. Particularly $Q^*/Q$ is $D$-extending. It is easy to see that $D \in \mathcal{A}(D, Q^*/Q)$, and so there exists a direct summand $D^*$ of $Q^*/Q$ such that $D \leq_e D^*$.

Since $D^*$ is a cyclic submodule of $Q^*/Q$, there exists a cyclic submodule $H$ of $Q^*$ such that $D^* = (H + Q)/Q$. It is easy to see that $Q^*_j \cap H \neq 0$. Thus $Q_j \cap H = (Q^*_j \cap H) \cap Q_j \neq 0$. Hence there exists a non-zero simple submodule $V_j$ of $Q_j \cap H$. Let $V = \oplus_{j=1}^\infty V_j$. Then $V \leq H$. Since $H$ is a cyclic subfactor of $M$, it follows that $H$ is $X$-extending for any $X \in S$. 

Particularly $H$ is $V$-extending. Clearly $V \in \mathcal{A}(V, H)$, and so there exists a direct summand $V^*$ of $H$ such that $V \leq_e V^*$. It is easy to see that $V \neq V^*$ since $V^*$ is cyclic. If $(V^* + Q)/Q = 0$, then $V^* \leq Q$, and thus $V^*$ is semisimple. Hence $V$ is a direct summand of $V^*$. But $V \leq_e V^*$, it follows that $V = V^*$, a contradiction. Thus $(V^* + Q)/Q \neq 0$.

For any $n \geq 1$, we have $(V^* \cap \oplus_{j=1}^n Q^*_j) \cap Q = V^* \cap (\oplus_{j=1}^n Q^*_j \cap Q) = V^* \cap (\oplus_{j=1}^n Q_j) \geq (\oplus_{j=1}^\infty V_j) \cap (\oplus_{j=1}^n Q_j) = \oplus_{j=1}^n V_j$. Since $V^* \cap (\oplus_{j=1}^n Q_j)$ is semisimple, it follows that

$$
(V^* \cap \oplus_{j=1}^n Q^*_j) \cap Q = \oplus_{j=1}^n V_j.
$$

Clearly, $\oplus_{j=1}^n V_j$ is a finitely generated submodule of $Q$. Thus there exists a finitely generated submodule $N$ of $\oplus_{i=1}^\infty C_i$ such that $(N + \oplus_{i=1}^\infty W_i)/(\oplus_{i=1}^\infty W_i) = \oplus_{j=1}^n V_j$. Suppose that $N \leq \oplus_{i=1}^m C_i$. It is easy to see that

$$
L = (\oplus_{i=1}^m C_i + \oplus_{i=1}^\infty W_i)/(\oplus_{i=1}^\infty W_i)
$$

is semisimple. Thus $\oplus_{j=1}^n V_j$ is a direct summand of $L$. It is easy to see that $L$ is a direct summand of $P$. Thus $\oplus_{j=1}^n V_j$ is a direct summand of $P$. Let $P = (\oplus_{j=1}^n V_j) \oplus P_1$. By modularity, $V^* \cap (\oplus_{j=1}^n Q^*_j) = (V^* \cap (\oplus_{j=1}^n Q^*_j) \cap Q) \oplus (V^* \cap (\oplus_{j=1}^\infty Q^*_j) \cap P_1)$. But it is easy to see that $(V^* \cap (\oplus_{j=1}^n Q^*_j)) \cap Q \leq_e V^* \cap (\oplus_{j=1}^n Q^*_j)$. Thus $(V^* \cap (\oplus_{j=1}^n Q^*_j)) \cap Q = (V^* \cap (\oplus_{j=1}^n Q^*_j)) \cap Q = (\oplus_{j=1}^\infty Q^*_j)$. This holds for each $n \geq 1$, hence it follows that $V^* \cap (\oplus_{j=1}^\infty Q^*_j) \leq Q$. But $Q \leq \oplus_{j=1}^\infty Q^*_j$, it follows that

$$
(\oplus_{j=1}^\infty (Q^*_j + Q)/Q) \cap ((V^* + Q)/Q) = 0.
$$

Now it follows that $(V^* + Q)/Q = 0$, which is a contradiction, because $D \leq_e D^*$. This completes the proof of the proposition. \hfill \Box

Now we have the main result of this paper, which generalizes Osofsky-Smith theorem ([7, Theorem 1]).

**Theorem 7.** Let $M$ be a cyclic extending left $R$-module. Assume that all cyclic subfactors of $M$ are $S$-extending. Then $M$ has finite uniform dimension.

**Proof.** By Proposition 6, $M$ is a finite direct sum of indecomposable submodules. Since every direct summand of an extending module is extending, the result follows by the fact that each indecomposable extending module is uniform. \hfill \Box

**References**


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