SOME RESULTS ON LITTLEWOOD'S PROBLEM AND ORLICZ'S PROBLEM

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ABSTRACT. We give a concrete example to the known problem of Littlewood by applying the stationary phase (or saddle point) method. We also give a trigonometric series which is not Borel summable and not a Fourier series. The result is an affirmative answer to Orlicz's problem.

1. INTRODUCTION

In [**12**] Littlewood asked to prove the following [cf.**14**(Appendix 5)] :
There exist complex numbers $a_1, a_2, \dots, a_N$ with $|a_n| = 1$, $n = 1, 2, \dots, N$, such that (1) *holds for all real x and for all sufficiently large N,*

$$(1) \qquad A_1\sqrt{N} \le \left| \sum_{n=1}^{N} a_n e^{2\pi i n x} \right| \le A_2\sqrt{N},$$

where A_1 and A_2 are some absolute positive constants.

Kahane[**8**] proved the existence of the complex numbers $a_1, a_2, \dots, a_N$ and the real numbers $A_1 = 1 - \epsilon_N, A_2 = 1 + \epsilon_N$, and $\epsilon_N \to 0$ as $N \to \infty$ stated above. Körner[**9**] (see [**15**]) proved the problem stated above. However Körner's method of proof is not effective and cannot supply any concrete example to Orlicz's problem[**1**]. Our answer to Littlewood's problem is that we give a concrete example and give numerical values of A_1 and A_2 by applying the stationary phase method.

Hardy and Littlewood [**6**] or [**20**] announced that they had showed that

$$(2) \qquad \sum_{n=1}^{N} \exp(2\pi i(\alpha n + \beta n \log n)) \ll \sqrt{N}$$

uniformly in α and β .

It is known that

$$f_N = \sum_{n=1}^{N} \exp(i\beta n \log n) \exp(n\theta i), \quad \beta \neq 0 \ is \ real, \quad \theta \in [0, 2\pi]$$

satisfies $|f_N| \leq A(\beta)\sqrt{N}$, where $A(\beta)$ is a positive constant depending only on β (see [**10**]).

They considered in [**6**] the possibility of the convergence and Cesàro summability of the series

$$(3) \qquad\qquad \sum_{n=1}^{\infty} n^{\rho - \frac{1}{2}} e^{2\pi i(\alpha n + \beta n \log n)}, \quad \rho \geq 0.$$

But they announced in [**7**] that they did not prove the possibility of the convergence and Cesàro summability of (3) by using their method when $\rho = 0$. In the case of $\rho = 0$, we can show as a consequence of Theorem 4.2 that the series (3) is not Borel summable and not a Fourier series.

For more related results, we give references [**3, 12, 19**].

Orlicz[**13**:No.121] proposed the following :

Give an example of a trigonometric series $\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

everywhere divergent and such that $\sum_{n=1}^{\infty}(|a_n|^{2+\epsilon} + |b_n|^{2+\epsilon}) < \infty$

for every $\epsilon > 0$.

In Proposition 2.4 we estimate $\int_a^b e^{ith(x)}dx$ for some function $h(x)$. Then, in Theorem 3.1, we obtain both an upper bound and a lower bound of the exponential sum $\sum_{n=1}^{N} \exp(2\pi i(\alpha n + \beta n \log n))$. In Theorem 3.2, we give an affirmative and concrete example to the problem of Littlewood. In Theorem 4.2, we obtain a concrete trigonometric series, which is not Borel summable and not a Fourier series everywhere in **R**. The proof of Theorem 4.2 is more directly shown by using another estimation than that of Theorem 3.1. The trigonometric series also gives an example to Orlicz's problem.

2. STATIONARY PHASE METHOD

We prepare Proposition 2.1 and Proposition 2.4, by using stationary phase method, for proving Theorem 3.1 in the next section.

Proposition 2.1 ([18, Lemma 4.7]). *Let $f(x)$ be a real function on $[a, b]$ with a continuous steadily decreasing f'. Let $H_1 = f'(b)$ and $H_2 = f'(a)$. Set $H = H_2 - H_1 + 2$. Then we have*

$$\sum_{a < n \le b} e^{2\pi i f(n)} = \sum_{H_1 - \epsilon < m < H_2 + \epsilon} \int_a^b e^{2\pi i (f(x) - mx)} dx + O(\log H),$$

where ϵ is a positive constant less than 1 and the constant implied by the O is absolute.

Lemma 2.2 ([4, Chap. IV]). *Let a and $c \ne 0$ be real numbers. Then we have for all $b > 0$,*

$$\int_0^b e^{it(a+cx^2)} dx = A \frac{e^{iat}}{2(|c|t)^{1/2}} - \frac{i}{2bct} e^{i(a+cb^2)t} + O\left(\frac{1}{b^3(ct)^2}\right),$$

as $t \to \infty$, where

$$A = \int_0^\infty u^{-1/2} e^{iu \, \text{sgn}(c)} du = e^{\frac{1}{4}\pi i \, \text{sgn}(c)} \sqrt{\pi}$$

and the constant implied by the O is absolute.

Proof. The proof runs along the same lines as [**4**, Chap. IV], except that we have to pay attentions to the dependency of parameters in the O terms. The constants implied by the O's are absolute in this proof. Without loss of generality, we may assume $c > 0$. If we put $u = ctx^2$ for $t > 0$, then we have

$$\int_0^b e^{it(a+cx^2)} dx = \frac{e^{iat}}{2(ct)^{1/2}} \int_0^{cb^2 t} u^{-1/2} e^{iu} du.$$

We prove that the integral $\int_0^\infty u^{-\frac{1}{2}} e^{iu} du$ converges. Integrating by parts, we have

$$\int_N^\infty u^{-\frac{1}{2}} e^{iu} du = -\frac{1}{i} N^{-\frac{1}{2}} e^{iN} + \frac{1}{2i} \int_N^\infty u^{-\frac{3}{2}} e^{iu} du,$$

where $N = cb^2 t$. The function $u^{-\frac{3}{2}}$ is monotone decreasing. Applying the second mean value theorem to the real and imaginary parts of the second integral, we obtain

$$\int_N^\infty u^{-\frac{1}{2}} e^{iu} du = -\frac{1}{i} N^{-\frac{1}{2}} e^{iN} + O(N^{-\frac{3}{2}}),$$

as $N \to \infty$. Then we obtain

$$\frac{e^{iat}}{2(ct)^{1/2}} \int_N^\infty u^{-1/2} e^{iu} du = \frac{e^{iat}}{2(ct)^{1/2}} \frac{-1}{i} (cb^2 t)^{-\frac{1}{2}} e^{iN} + O\left(\frac{1}{b^3(ct)^2}\right)$$

$$= \frac{-1}{2bcti}e^{iat}e^{iN} + O\left(\frac{1}{b^3(ct)^2}\right).$$

Therefore

$$\int_0^b e^{it(a+cx^2)}dx = \frac{e^{iat}}{2(ct)^{1/2}}\int_0^{cb^2t}u^{-1/2}e^{iu}du$$

$$= \frac{e^{iat}}{2(ct)^{1/2}}\left\{\int_0^\infty u^{-1/2}e^{iu}du - \int_N^\infty u^{-1/2}e^{iu}du\right\}$$

$$= A\frac{e^{iat}}{2(ct)^{1/2}} - \frac{i}{2bct}e^{i(a+cb^2)t} + O\left(\frac{1}{b^3(ct)^2}\right),$$

which completes the proof. □

Lemma 2.3 ([4, Chap. IV]). *Suppose that a real function $h(x)$ on $[0,b]$ satisfies the following conditions:*

    (i) $h(x)$ is of class C^3 , $h'(0) = 0$, $\quad h''(0) \neq 0$, $\quad$ and $\quad h'(x) \neq 0$ on $(0,b]$.

    (ii) $h''(x) > 0$, or $h''(x) < 0$, throughout $(0,b]$.

Under these conditions, for any real number δ with $0 < \delta < b$, we have

$$\int_0^b e^{ith(x)}dx = \frac{\sqrt{\pi}e^{ih(0)t+\frac{1}{4}\pi i\ \mathrm{sgn}(h''(0))}}{\sqrt{2t|h''(0)|}}$$

$$+ O\left(\frac{1}{t^2|h(\delta)-h(0)|^{3/2}\sqrt{|h''(0)|}}\right) + O\left(\frac{1}{t}\right)$$

$$+ O\left(\frac{1}{t|h(\delta)-h(0)|^{1/2}\sqrt{|h''(0)|}}\right) + O\left(\left(\left|\frac{1}{h'(b-)}\right| + \left|\frac{1}{h'(\delta)}\right|\right)\frac{1}{t}\right),$$

as $t \to \infty$, where the constants implied by the O's are absolute.

    If we choose $\delta \geq b$, then

$$\int_0^b e^{ith(x)}dx = \frac{\sqrt{\pi}e^{ih(0)t+\frac{1}{4}\pi i\ \mathrm{sgn}(h''(0))}}{\sqrt{2t|h''(0)|}} + O\left(\frac{1}{t^2|h(b)-h(0)|^{3/2}\sqrt{|h''(0)|}}\right)$$

$$+ O\left(\frac{1}{t|h(b)-h(0)|^{1/2}\sqrt{|h''(0)|}}\right),$$

as $t \to \infty$, where the constants implied by the O's are absolute.

Proof. The proof runs along the same lines as [**4**, Chap. IV]. Since $h(x)$ is of class C^3, we have

$$h(x) = a + dx^2 + o(x^2),$$

by Taylor's theorem, where $a = h(0), \quad d = h''(0)/2$, and $o's$ are Landau's small o.

Without loss of generality, we may suppose that $h''(x) > 0$ on $(0, b]$. Consider the function $\varphi(x) = \sqrt{h(x) - a}$, which is differentiable on $[0, b]$. The function $h(x)$ is strictly increasing. There exists $\Psi(u)$ the inverse function of $\varphi(x)$ on $(0, b]$. Then $\Psi(u)$ is three times continuously differentiable and strictly increasing on the interval $[0, \varphi(\delta)]$ with $\Psi(0) = 0$ and $\Psi'(0) = d^{1/2}$. We choose any $\delta \in [0, b)$.

Divide the interval $[0, b]$ into $[0, \delta]$ and $[\delta, b]$. In the first interval, by changing the variable $u = \varphi(x)$, we have

$$\int_0^\delta e^{ith(x)}dx = \int_0^{\varphi(\delta)} e^{ith(\Psi(u))}\Psi'(u)du$$

$$= d^{-1/2} \int_0^{\varphi(\delta)} e^{it(a+u^2)}du + \int_0^{\varphi(\delta)} \left(\Psi'(u) - \Psi'(0)\right) e^{ith_1(u)}du,$$

where $h_1(u) = a + u^2$. We have as $u \to +0$

(4) $$\frac{\Psi'(u) - \Psi'(0)}{h_1'(u)} \to \frac{1}{2}\Psi''(0).$$

Thus

(5) $$\int_0^b e^{ith(x)}dx = \int_0^\delta e^{ith(x)}dx + \int_\delta^b e^{ith(x)}dx$$

$$= d^{-1/2} \int_0^{\varphi(\delta)} e^{it(a+u^2)}du + \int_0^{\varphi(\delta)} (\Psi'(u) - \Psi'(0))e^{ith_1(u)}du + \int_\delta^b e^{ith(x)}dx.$$

We consider the above three integrals, respectively. By Lemma 2.2, we have

(6) $$\int_0^{\varphi(\delta)} e^{it(a+u^2)}du = \frac{e^{iat}}{2t^{1/2}}A - \frac{i}{2\varphi(\delta)t}e^{i(a+\varphi(\delta)^2)t} + O\left(\frac{1}{\varphi(\delta)^3t^2}\right)$$

$$= \frac{e^{iat}}{2t^{1/2}}A + O\left(\frac{1}{\varphi(\delta)^3}\frac{1}{t^2}\right) + O\left(\frac{1}{\varphi(\delta)}\frac{1}{t}\right).$$

Since $h'(x) \neq 0$ on $[\delta, b]$ and $h''(x)$ has a constant sign, we have, as $t \to \infty$,

$$
\int_\delta^{b-} e^{ith(x)} dx = \frac{1}{it} \int_\delta^{b-} \frac{1}{h'(x)} \frac{d}{dx} e^{ith(x)} dx
$$

$$
= \frac{1}{it} \left[\frac{1}{h'(b-)} e^{ith(b-)} - \frac{1}{h'(\delta)} e^{ith(\delta)} \right] - \frac{1}{it} \int_\delta^{b-} e^{ith(x)} \frac{d}{dx} \left(\frac{1}{h'(x)} \right) dx
$$

$$
= O\left(\left(\frac{1}{|h'(b-)|} + \frac{1}{|h'(\delta)|} \right) \frac{1}{t} \right).
$$

By virtue of (4), we have, for sufficiently small $\epsilon > 0$,

$$
\left| \int_\epsilon^{\varphi(\delta)} \left(\Psi'(u) - \Psi'(0) \right) e^{ith_1(u)} du \right| = \left| \frac{1}{it} \int_\epsilon^{\varphi(\delta)} \frac{\Psi'(u) - \Psi'(0)}{h_1'(u)} \frac{d}{du} e^{ith_1(u)} du \right|
$$

$$
= \left| \frac{1}{it} \left[\frac{\Psi'(u) - \Psi'(0)}{h_1'(u)} e^{ith_1(u)} \right]_\epsilon^{\varphi(\delta)} - \frac{1}{it} \int_\epsilon^{\varphi(\delta)} e^{ith_1(x)} \frac{d}{dx} \left(\frac{\Psi'(x) - \Psi'(0)}{h_1'(x)} \right) dx \right|
$$

$$
= \frac{1}{t} \left\{ O(1) + \int_\epsilon^{\varphi(\delta)} \left| \frac{d}{dx} \left(\frac{\Psi'(x) - \Psi'(0)}{2x} \right) \right| dx \right\} = O\left(\frac{1}{t} \right),
$$

since $\Psi(u)$ is of class C^3 and $\Psi'''(0)$ exists. Thus we have, as $t \to \infty$,

$$
\int_0^{\varphi(\delta)} (\Psi'(u) - \Psi'(0)) e^{ith_1(u)} du = O\left(\frac{1}{t} \right).
$$

Therefore

$$
\int_0^b e^{ith(x)} dx
$$

$$
= d^{-1/2} \int_0^{\varphi(\delta)} e^{it(a+u^2)} du + O\left(\frac{1}{t} \right) + O\left(\left(\left| \frac{1}{h'(b-)} \right| + \left| \frac{1}{h'(\delta)} \right| \right) \frac{1}{t} \right)
$$

$$
= d^{-1/2} \left\{ \frac{e^{iat}}{2\sqrt{t}} A + O\left(\frac{1}{|h(\delta) - h(0)|^{3/2}} \frac{1}{t^2} \right) + O\left(\frac{1}{|h(\delta) - h(0)|} \frac{1}{t} \right) \right\}
$$

$$
+ O\left(\frac{1}{t} \right) + O\left(\left(\left| \frac{1}{h'(b-)} \right| + \left| \frac{1}{h'(\delta)} \right| \right) \frac{1}{t} \right).
$$

Thus we have, as $t \to \infty$,

$$\int_0^b e^{ith(x)} dx$$

$$= A \frac{e^{iat}}{2\sqrt{dt}} + O\left(\frac{1}{|h(\delta) - h(0)|^{3/2}} \frac{1}{t^2\sqrt{d}}\right) + O\left(\frac{1}{|h(\delta) - h(0)|} \frac{1}{t\sqrt{d}}\right)$$

$$+ O\left(\frac{1}{t}\right) + O\left(\left(\frac{1}{|h'(b-)|} + \frac{1}{|h'(\delta)|}\right)\frac{1}{t}\right).$$

On the other hand, by (5) and (6),

$$\int_0^b e^{ith(x)} dx = d^{-1/2} \int_0^{\varphi(b)} e^{it(a+u^2)} du + \int_0^{\varphi(b)} (\Psi'(u) - \Psi'(0)) e^{ith_1(u)} du$$

$$= \frac{\sqrt{\pi} e^{ith(0) + \frac{1}{4}\pi i \operatorname{sign}(h''(0))}}{\sqrt{2t|h''(0)|}} + O\left(\frac{1}{\varphi(b)^3} \frac{1}{t^2} \frac{1}{\sqrt{|h''(0)|}}\right) + O\left(\frac{1}{\varphi(b)} \frac{1}{t} \frac{1}{\sqrt{|h''(0)|}}\right),$$

which completes the proof.   □

The following Proposition 2.4 is an immediate consequence of Lemma 2.3.

Proposition 2.4. *Suppose that a real function $h(x)$ is of class C^3 on $[a,b]$, $h'(c) = 0$ at just one point c with $a < c < b$, and $h''(c) \neq 0$. Moreover, $h''(x) > 0$, or $h''(x) < 0$, throughout $[a,b]$. Then, for any real number δ,*

$$\int_a^b e^{ith(x)} dx = \sqrt{\frac{2\pi}{t|h''(c)|}} \exp\left(ith(c) + \frac{1}{4}i\pi \operatorname{sgn}(h''(c))\right)$$

$$+ O\left(\left(\frac{1}{t^2|h(\min(c+\delta,b)) - h(c)|^{3/2}} + \frac{1}{t|h(\min(b,c+\delta)) - h(c)|^{1/2}}\right)\frac{1}{\sqrt{|h''(c)|}}\right)$$

$$+ O\left(\chi(b - (c+\delta))\frac{1}{t}\left(1 + \frac{1}{|h'(b-)|} + \frac{1}{|h'(\min(c+\delta,b))|}\right)\right)$$

$$+ O\left(\left(\frac{1}{t^2|h(\max(a,c-\delta)) - h(c)|^{3/2}} + \frac{1}{t|h(\max(a,c-\delta)) - h(c)|^{1/2}}\right)\frac{1}{\sqrt{|h''(c)|}}\right)$$

$$+ O\left(\chi((c-\delta) - a)\frac{1}{t}\left(1 + \frac{1}{|h'(a+)|} + \frac{1}{|h'(\max(a,c-\delta))|}\right)\right),$$

as $t \to \infty$, where $\chi(x)$ is the charactor function with $\chi(x) = 0$ if $x \geq 0$, otherwise $\chi(x) = 1$.

3. Littlewood's problem

Theorem 3.1. *(i) Let $\beta > 0$. We have, for any $0 < \epsilon < 1$,*

$$\left|\sum_{n=1}^{N} e^{2\pi i(\alpha n + \beta n \log n)}\right| \leq \frac{1}{\sqrt{\beta}} \frac{e^{\frac{\epsilon+1-\{\alpha+\beta(\log N+1)+\epsilon\}}{2\beta}}}{e^{\frac{1}{2\beta}}-1}\sqrt{N}$$

$$+ O\left(\frac{e^{(1+\epsilon)/2\beta}}{\sqrt{\beta}(e^{1/2\beta}-1)}\right) + O(\log N),$$

where the constants implied by the O's are absolute and $\{x\}$ is the fractional part of x.

(ii) Let $0 < \beta \leq 1/(2\log 2) = 0.7213\cdots$. For any $0 < \epsilon < 1$, we have

$$\left|\sum_{n=1}^{N} e^{2\pi i(\alpha n + \beta n \log n)}\right|$$

$$\geq \frac{1}{\sqrt{\beta}} e^{-\frac{1+\{\alpha+\beta(\log N+1)+\epsilon\}}{2\beta}}\sqrt{N} \cdot \left\{ e^{\frac{1}{2\beta}} - 1 - \frac{1}{e^{\frac{1}{2\beta}}-1} \right\}$$

$$+ O\left(\frac{1}{\epsilon^{3/2}\sqrt{\beta}}\right) + O(\log N)$$

$$\geq \frac{1}{\sqrt{\beta}} e^{-\frac{1}{\beta}}\sqrt{N} \cdot \left\{ e^{\frac{1}{2\beta}} - 1 - \frac{1}{e^{\frac{1}{2\beta}}-1} \right\} + O\left(\frac{1}{\sqrt{\beta}}\right) + O(\log N),$$

where the constants implied by the O's are absolute.

Remark 3.1. $e^{\frac{1}{2\beta}} - 1 - \frac{1}{e^{\frac{1}{2\beta}}-1} > 0$ *holds for $0 < \beta < 1/2\log 2$.*

Proof. The constants implied by the O's are absolute in this proof. Without loss of generality, we may assume $0 \leq \alpha < 1$. We write $e(x) = e^{2\pi ix}$. We set $f(x) = \alpha x + \beta x \log x$.

By Proposition 2.1, we have

$$I_N = \sum_{n=1}^{N} e^{2\pi i(\alpha n + \beta n \log n)} = \sum_{H_1-\epsilon < h < H_2+\epsilon} \int_{1}^{N} e(f(x) - hx)dx + O(\log H),$$

where $H_1 = \alpha+\beta$, $H_2 = \alpha+\beta(\log N+1)$, and $H = H_2-H_1+2 = \beta \log N+2$ for an arbitrary positive number $\epsilon < 1$.

Set $h(x) = \alpha x+\beta x \log Nx-hx$ as in Proposition 2.4, $M = \lceil \alpha+\beta-\epsilon \rceil$, and $K = [\alpha + \beta(\log N + 1) + \epsilon]$, where $\lceil x \rceil$ denotes the smallest integer $\geq x$, and $[x]$ denotes the largest integer $\leq x$.

The real number c_h satisfies the equation $h'(c_h) = 0,$ i.e., $\alpha + \beta(\log Nc_h + 1) - h = 0$, $\log Nc_h = \dfrac{h - \alpha}{\beta} - 1$. For $c = c_h - \delta$ or $c_h + \delta$, we have

$$\left| \frac{1}{h'(c)} \right| = \left| \frac{1}{\alpha + \beta \log Nc + \beta - h} \right| = \left| \frac{1}{\beta(\log c - \log c_h)} \right| \leq \frac{c_h + \delta}{\beta \delta},$$

by the mean value theorem.

Thus

$$I_N = \sum_{h=M}^{K} \int_0^N e(f(x) - hx)dx + O(\log N)$$

$$= N \sum_{h=M}^{K} \int_0^1 e^{2\pi i N(\alpha x + \beta x \log(Nx) - hx)} dx + O(\log N).$$

Therefore, since c_h satisfies $h'(c_h) = 0, h''(x) > 0$ for $x > 0$, we obtain

$$I_N = N \sum_{h=M}^{K} \left\{ 2\sqrt{\frac{\pi c_h}{2 \cdot 2\pi N \cdot \beta}} e\left(Nh(c_h) + \frac{1}{8} \right) + O\left(\frac{1}{\delta^{3/2}} \sqrt{\frac{c_h}{\beta}} \cdot \frac{1}{N^2} \right) \right.$$

$$(7) \qquad \left. + O\left(\frac{1}{\delta^{1/2}} \sqrt{\frac{c_h}{\beta}} \cdot \frac{1}{N} \right) + O\left(\frac{e^{\frac{h-\alpha}{\beta} - 1}}{N^2 \beta \delta} \right) \right\} + O(\log N)$$

We have

$$|h(0) - h(c_h)| = |h(c_h)| = c_h \beta,$$

$$|h(1) - h(c_h)| = \alpha + \beta \log N - h - c_h \beta \to \infty \quad \text{as} \quad N \to \infty.$$

For any small $\delta > 0$, if $c_h - \delta \leq 0$, or $c_h + \delta \geq 1$, then the term of

$$J_h = O\left(\frac{1}{\delta^{3/2}} \sqrt{\frac{c_h}{\beta}} \cdot \frac{1}{N^2} \right) + O\left(\frac{1}{\delta^{1/2}} \sqrt{\frac{c_h}{\beta}} \cdot \frac{1}{N} \right)$$

is replaced by

$$O\left(\frac{1}{(c_h \beta)^{3/2}} \sqrt{\frac{c_h}{\beta}} \cdot \frac{1}{N^2} \right) + O\left(\frac{1}{(c_h \beta)^{1/2}} \sqrt{\frac{c_h}{\beta}} \cdot \frac{1}{N} \right),$$

by Proposition 2.4. Thus $\sum_{h=M}^{K} J_h = O(\frac{1}{\beta} \log N)$. Therefore, for any small $\delta > 0$,

$$
\begin{aligned}
I_N = & \sqrt{N} \sum_{h=M}^{K} \left\{ \sqrt{\frac{c_h}{\beta}} e(N(\alpha c_h + \beta c_h \log N c_h - h c_h) + \frac{1}{8}) \right. \\
& \left. + O\left(\frac{1}{\delta^{3/2} N^2 \sqrt{N}} \sqrt{\frac{c_h}{\beta}} \right) + O\left(\frac{1}{\delta^{1/2} N \sqrt{N}} \sqrt{\frac{c_h}{\beta}} \right) + O\left(\frac{e^{\frac{h-\alpha}{\beta}} - 1}{N^2 \beta} \right) \right\} \\
& + O(\log N).
\end{aligned}
$$

Thus

$$
|I_N| \le \frac{1}{\sqrt{\beta}} \sum_{h=M}^{K} e^{\frac{1}{2}(\frac{h-\alpha}{\beta} - 1)} + O\left(\frac{1}{\sqrt{N}\sqrt{\beta}} \sum_{h=M}^{K} e^{\frac{1}{2}(\frac{h-\alpha}{\beta} - 1)} \right) + O(\log N)
$$

$$
\begin{aligned}
(8) \\
\le \frac{1}{\sqrt{\beta}} \frac{A(\alpha,\beta) e^{\frac{1}{2\beta}(K-M+1)}}{\exp(\frac{1}{2\beta}) - 1} + O\left(\frac{1}{\sqrt{N}\sqrt{\beta}} \frac{A(\alpha,\beta) e^{\frac{1}{2\beta}(K-M+1)}}{e^{\frac{1}{2\beta}} - 1} \right) + O(\log N),
\end{aligned}
$$

where $A(\alpha, \beta) = e^{\frac{1}{2\beta}(M-\alpha-\beta)}$. Therefore

$$
|I_N| \le \frac{\sqrt{N}}{\sqrt{\beta}} \frac{e^{\frac{\epsilon+1-\{\alpha+\beta(\log N+1)+\epsilon\}}{2\beta}}}{e^{\frac{1}{2\beta}} - 1} + O\left(\frac{e^{\frac{\epsilon+1-\{\alpha+\beta(\log N+1)+\epsilon\}}{2\beta}}}{\sqrt{\beta}(e^{\frac{1}{2\beta}} - 1)} \right) + O(\log N),
$$

which completes the proof of (i).

Next we show (ii). From (7) and (8) with $K := K - 2$, we have

$$
\begin{aligned}
|I_N| \ge & \left| \sqrt{N} \sum_{h=M}^{K} \left\{ \sqrt{\frac{c_h}{\beta}} e^{-2\pi i N \beta c_h + \frac{i}{4}\pi} + O\left(\frac{1}{\sqrt{N}} \sqrt{\frac{c_h}{\beta}} \right) \right\} \right| + O(\log N) \\
= & \left| \sqrt{N} \sum_{h=K-1}^{K} \sqrt{\frac{c_h}{\beta}} e^{-2\pi i N \beta c_h + \frac{i}{4}\pi} + \sqrt{N} \sum_{h=M}^{K-2} \sqrt{\frac{c_h}{\beta}} e^{-2\pi i N \beta c_h + \frac{i}{4}\pi} \right| \\
& + O\left(\sqrt{\frac{1}{\beta}} \right) + O(\log N) \\
(9) \quad \ge & \frac{\sqrt{N}}{\sqrt{\beta}} \left| \sqrt{c_K} e(-N\beta c_K + \frac{1}{8}) + \sqrt{c_{K-1}} e(-N\beta c_{K-1} + \frac{1}{8}) \right| \\
& - \left| \frac{A(\alpha,\beta) e^{\frac{1}{2\beta}(K-2-M+1)}}{\sqrt{\beta}(e^{\frac{1}{2\beta}} - 1)} \right| + O\left(\sqrt{\frac{1}{\beta}} \right) + O(\log N).
\end{aligned}
$$

Using $\left|re^{i\theta} + r'e^{i\theta'}\right| = r'\left|\frac{r}{r'}e^{i(\theta-\theta')} + 1\right|$, we obtain

$$|I_N| \geq \frac{1}{\sqrt{\beta}}e^{\frac{1}{2\beta}(K-1-\alpha)-\frac{1}{2}}\left\{\left|\sqrt{e^{\frac{1}{\beta}}}e\left(-\beta e^{-1-\frac{\alpha}{\beta}}e^{\frac{1}{\beta}(K-1)}(e^{\frac{1}{\beta}}-1)\right)+1\right|\right.$$

$$(10) \qquad \left. - \frac{1}{e^{\frac{1}{2\beta}}-1}\right\} + O\left(\sqrt{\frac{1}{\beta}}\right) + O(\log N)$$

$$\geq \frac{1}{\sqrt{\beta}}e^{-\frac{1}{2\beta}}e^{-\frac{1}{2\beta}\{\alpha+\beta(\log N+1)+\epsilon\}}\sqrt{N}\cdot\left\{e^{\frac{1}{2\beta}}-1-\frac{1}{e^{\frac{1}{2\beta}}-1}\right\}$$

$$+ O\left(\sqrt{\frac{1}{\beta}}\right) + O(\log N)$$

$$\geq \frac{1}{\sqrt{\beta}}e^{-\frac{1}{\beta}}\sqrt{N}\cdot\left\{e^{\frac{1}{2\beta}}-1-\frac{1}{e^{\frac{1}{2\beta}}-1}\right\} + O\left(\sqrt{\frac{1}{\beta}}\right) + O(\log N).$$

Thus we have

$$(11) \qquad \left|\sum_{n=2}^{N}\exp(2\pi i(\alpha n + \beta n\log n))\right| = |I_N|$$

$$\geq \frac{1}{\sqrt{\beta}}e^{-\frac{1}{\beta}}\sqrt{N}\cdot\left\{e^{\frac{1}{2\beta}}-1-\frac{1}{e^{\frac{1}{2\beta}}-1}\right\} + O\left(\sqrt{\frac{1}{\beta}}\right) + O(\log N) > 0,$$

which completes the proof of Theorem 3.1. $\qquad\square$

Now, as an immediate consequence of Theorem 3.1, we can give an answer to Littlewood's problem.

Theorem 3.2. *Let $a_n = \exp(2\pi i\beta n\log n)$, $\quad n = 1, 2, \ldots$, where β is any constant with $0 < \beta \leq 1/(2\log 2)$. Then (1) holds for all real x and all sufficiently large N, where both constants A_1 and A_2 depend only on β.*

4. ORLICZ'S PROBLEM

In this section, the proof of Orlicz's problem is directly shown by using another estimation than that of Theorem 3.1, which does not need the condition such that the coefficient in main term of (11) is positive.

Lemma 4.1 ([cf.**18**, 4.12, 4.9]). *Let $f(x)$ be a real function on $[a, b]$ with a continuous steadily decreasing f'. Let $\alpha = f'(b)$ and $\beta = f'(a)$. Let $g(x)$ be a real positive monotone function with a continuous derivative, and*

let $|g'(x)|$ be steadily decreasing. Then

$$\sum_{a<n\le b} g(n)e^{2\pi i f(n)} = e^{-\pi i/4} \sum_{\alpha<k\le\beta} \frac{e^{2\pi i(f(x_k)-kx_k)}}{\sqrt{|f''(x_k)|}} g(x_k)$$
$$+ O\left(\max(g(a), g(b))\log(|\beta-\alpha|+2)\right) + O\left(|g'(a)|\right),$$

where x_k is the number satisfying $f'(x_k) = k$.

Proof. It comes from directly from [**18**].   □

Next, we give a concrete trigonometric series which is not a Fourier series. Moreover it is proved that the series is nowhere Borel summable.

Theorem 4.2. *Let $\beta \ne 0$ and*

$$\begin{cases} a_n = n^{-1/2}\cos(2\pi\beta n\log n) \\ b_n = -n^{-1/2}\sin(2\pi\beta n\log n). \end{cases}$$

Then though

$$\sum_{n=1}^{\infty}\left(|a_n|^{2+\epsilon} + |b_n|^{2+\epsilon}\right) < \infty \quad \text{holds for every } \epsilon > 0,$$

the trigonometric series $\displaystyle\sum_{n=1}^{\infty}(a_n\cos 2\pi nx + b_n\sin 2\pi nx)$ *is not a Fourier series .*

Moreover the series is nowhere Borel summable.

Remark 4.1. *By [**16**,pp.54-55], the trigonometric series given in Theorem 4.2 is nowhere Euler-Knopp summable, Taylor summable, and Meyer-König's summable.*

Theorem 4.2 shows that the trigonometric series is an example to Orlicz's problem. Moreover, since $\sum_{n=1}^{\infty}(a_n\cos 2\pi nx + b_n\sin 2\pi nx) = \text{Re}\left(\sum_{n=1}^{\infty} n^{-\frac{1}{2}}e^{2\pi i(nx+\beta n\log n)}\right)$, the series $\sum_{n=1}^{\infty} n^{-\frac{1}{2}}e^{2\pi i(nx+\beta n\log n)}$ is not Borel summable and not a Fourier series (cf.the comment on the series (3) in the section 1).

Proof. Without loss of generality, we may assume $\beta > 0$. We show that the series is not a Fourier series. To prove this it will suffice to show that the series is not Cesàro summable on a set of positive measure of x [**15**,p.89]. In fact we prove that the series is nowhere Cesàro summable.

We set

$$c_k = \frac{1}{\sqrt{k}}\cos 2\pi(\beta k\log k + kx), \quad s_n = \sum_{k=0}^{n} c_k, \quad \text{and} \quad t_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k.$$

Then

$$t_n = s_n - \frac{1}{n+1} \sum_{k=0}^{n} k c_k.$$

We set $f(n) = \beta n \log n + nx$. For an arbitrary positive integer $M(\geq [\beta \log(\beta+1)+\beta+x+1])$, we can choose n with $M = [\beta \log n + \beta + x]$.

By Lemma 4.1, we have

$$t_n = \text{Re} \left[\sum_{k=1}^{M} \frac{1}{\sqrt{k}} e^{2\pi i f(k)} + \sum_{k=M+1}^{n} \frac{1}{\sqrt{k}} e^{2\pi i f(k)} - \frac{1}{n+1} \sum_{k=1}^{n} \sqrt{k} e^{2\pi i f(k)} \right]$$

$$= \text{Re} \left[\sum_{k=1}^{M} \frac{1}{\sqrt{k}} e^{2\pi i f(k)} + \frac{e^{-\pi i/4}}{\sqrt{\beta}} \sum_{\beta \log M + \beta + x < k \leq M} e^{-2\pi i \beta x_k} \right.$$

$$+ O\left(\frac{1}{\sqrt{M}} \log(\beta \log n + 2) \right) - \frac{e^{-\pi i/4}}{\sqrt{\beta}} \frac{1}{n+1} \sum_{\beta + x < k \leq M} x_k e^{-2\pi i \beta x_k}$$

$$\left. + O\left(\frac{1}{\sqrt{n}} \log(\beta \log n + 2) \right) \right].$$

The number $x_k = \exp\left(\frac{k - \beta - x}{\beta} \right)$ is the only root of the equation $\beta \log x_k + \beta + x - k = 0$.

We have

$$t_n = \sum_{1 \leq k \leq \beta + x} \frac{1}{\sqrt{k}} \cos(2\pi f(k)) +$$

$$+ \sum_{\beta + x < k \leq M} \left[\frac{1}{\sqrt{k}} \cos(2\pi f(k)) + \left(1 - \frac{x_k}{n+1} \right) \frac{1}{\sqrt{\beta}} \cos\left(\frac{\pi}{4} + 2\pi \beta x_k \right) \right]$$

$$- \frac{1}{\sqrt{\beta}} \sum_{\beta + x < k \leq \beta \log M + \beta + x} \cos\left(\frac{\pi}{4} + 2\pi \beta x_k \right) + O\left(\frac{\log(\beta \log n + 2)}{\sqrt{\beta \log n}} \right).$$

Since $\log n - \log(n-1) \leq 1/(n-1)$ for all n, we can choose a subsequence n' of n with $[\beta \log(n-1) + \beta + x] = [\beta \log n + \beta + x] - 1 = M - 1$, say. We abbreviate n' to n. Then

$$t_n - t_{n-1} = \frac{1}{\sqrt{M}} \cos(2\pi f(M)) + \left(1 - \frac{x_M}{n+1} \right) \frac{1}{\sqrt{\beta}} \cos\left(\frac{\pi}{4} + 2\pi \beta x_M \right)$$

$$+ \frac{1}{\sqrt{\beta}} \sum_{\beta + x < k \leq M-1} \left(\frac{1}{n} - \frac{1}{n+1} \right) x_k \cos\left(\frac{\pi}{4} + 2\pi \beta x_k \right)$$

$$-\frac{1}{\sqrt{\beta}} \sum_{k \in A} \cos\left(\frac{\pi}{4} + 2\pi\beta x_k\right) + O\left(\frac{\log(\beta \log n + 2)}{\sqrt{\beta \log n}}\right),$$

where $A = \{k \in \mathbf{N} | \beta \log[\beta \log(n-1) + \beta + x] + \beta + x < k \leq \beta \log M + \beta + x\}$.

We have

$$\sum_{\beta+x<k\leq M-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) x_k \cos\left(\frac{\pi}{4} + 2\pi\beta x_k\right)$$

$$= O\left(\frac{1}{n^2} \sum_{0\leq k\leq M-1} x_k\right) = O\left(\frac{1}{n}\right),$$

where $x_M = ne^{-\theta/\beta}$ and θ is the fractional part of $\beta \log n + \beta + x$.

Thus we obtain, as $n \to \infty$,

$$|t_n - t_{n-1}| = \left| \frac{1}{\sqrt{M}} \cos(2\pi f(M)) + \left(1 - \frac{x_M}{n+1}\right) \frac{1}{\sqrt{\beta}} \cos\left(\frac{\pi}{4} + 2\pi\beta x_M\right) \right.$$

$$\left. -\frac{1}{\sqrt{\beta}} \sum_{k \in A} \cos\left(\frac{\pi}{4} + 2\pi\beta x_k\right) + O\left(\frac{\log(\beta \log n + 2)}{\sqrt{\beta \log n}}\right) + O\left(\frac{1}{n}\right) \right|$$

$$\geq \frac{1}{\sqrt{\beta}} \left| \left| \cos\left(\frac{\pi}{4} + 2\pi\beta x_a\right)\right| - \left| \left(1 - \frac{x_M}{n+1}\right) \cos\left(\frac{\pi}{4} + 2\pi\beta ne^{-\frac{\theta}{\beta}}\right)\right| \right| + o(1),$$

where $a = [\beta \log M + \beta + x]$. From the definition of M, we obtain

$$n - 1 = \exp\left(\frac{\beta \log(n-1)}{\beta}\right) < x_M = \exp\left(\frac{M - \beta - x}{\beta}\right)$$

$$\leq \exp\left(\frac{\beta \log n}{\beta}\right) = n.$$

Thus

$$\frac{n-1}{n+1} < \frac{x_M}{n+1} \leq \frac{n}{n+1}.$$

Therefore we obtain, for almost all x,

$$\limsup_{n\to\infty} |t_n - t_{n-1}| \geq \frac{1}{\sqrt{\beta}} \limsup_{n\to\infty} \left| \cos\left(\frac{\pi}{4} + 2\pi\beta x_a\right)\right|$$

$$= \frac{1}{\sqrt{\beta}} \limsup_{n\to\infty} \left| \cos\left(\frac{\pi}{4} + 2\pi\beta\exp\left(\frac{[\beta \log[\beta \log n + \beta + x] + \beta + x] - \beta - x}{\beta}\right)\right)\right|$$

$$= \frac{1}{\sqrt{\beta}} > 0,$$

by the facts that for every $\lambda > 1$, $(y\lambda^n)_{n=1}^{\infty}$ is u.d. mod 1 for almost all y, where $y = \beta e^{-\frac{\beta+x}{\beta}}$ and $\lambda = e^{\frac{1}{\beta}}$.

This shows that t_n does not converge for almost all x. Hence $\sum_{n=1}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx)$ is not Cesàro summable, that is, not a Fourie series.

Suppose that the series

$$\sum_{n=1}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx)$$

is Borel summable. We note that the order of magnitude of the coefficients of the series is clearly $O\left(\dfrac{1}{\sqrt{n}}\right)$. Hence the series converges by the known Tauberian theorem for Borel summability method [**5**, Theorem 156].

This contradicts that the series is not Cesàro summable for any x. Therefore the series is not Borel summable, which completes the proof. □

Remark 4.2. *It has brought to our notice that the original Orlicz problem had been actually solved by Stečhkin[cf.**2**, Chap.XII, p.278]. In fact he proved, possibly unware of the problem, a general theorem on divergent trigonometric series, from which the solution to Orlicz's problem follows immediately as a corollary.*

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