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PSEUDO-RIEMANNIAN SUBMANIFOLDS WITH POINTWISE PLANAR NORMAL SECTIONS

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0. Introduction. A normal section of a surface in a Euclidean space is naturally defined. B.-Y. Chen [3], [4], defined a normal section of submanifolds in a Euclidean space and studied some geometric properties. We can extend this definition to that of pseudo-Riemannian submanifolds in a pseudo-Euclidean space and we will give the definition in § 1.

In the present paper, we study some properties of pseudo-Riemannian submanifolds with pointwise planar normal sections in a pseudo-Euclidean space.

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1. Preliminaries. Let $M^n_{r,s}$ be an n-dimensional smooth manifold with a scalar product $\langle\ ,\ \rangle$ whose canonical form is

$$\begin{pmatrix} I_{n-r-s} & & \\ & -I_r & \\ & & O_s \end{pmatrix}$$

where I_r is the $r\times r$-identity matrix and O_s the $s\times s$-0 matrix. The scalar product $\langle\ ,\ \rangle$ is nondegenarate if and only if $s=0$. In particular, $M^n_{r,o}$ will be denoted by M^n_r which is said to be an n-dimensional pseudo-Riemannian manifold of signature $(r, n-r)$.

Let M^n_r be an n-dimensional pseudo-Riemannian submanifold of signature $(r, n-r)$ in an m-dimensional pseudo-Euclidean space E^m_s of signature $(s, m-s)$. For any point p in M^n_r and any non-zero vector t at p tangent to M^n_r, the vector t and the normal space $T^{\perp}_pM^n_r$ determine an $(m-n+1)$-dimensional affine space $E(p, t)$ in E^m_s. The intersection of $E(p, t)$ and M^n_r gives rise to a curve $\gamma(s)$ in a neighborhood of p which is called the normal section of M^n_r at p in the direction t. In general, the normal section γ is a twisted space curve in $E(p, t)$. A pseudo-Riemannian submanifold is said to have planar normal sections if its normal sections are planar curves, that is, $\gamma'\wedge\gamma''\wedge\gamma'''=0$ for each normal section γ. A pseudo-Riemannian submanifold M^n_r is said to have pointwise planar normal sections if each normal section γ at p satisfies $(\gamma'\wedge\gamma''$

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$\wedge\gamma''')(p) = 0$ for every point p in M_r^n.

Let ∇ and $\tilde{\nabla}$ be Levi-Civita connections of M_r^n and E_s^m, respectively.

For any tangent vector fields X and Y to M_r^n, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.1}$$

where h is the second fundamental form.

For any normal vector field ξ to M_r^n, we write

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{1.2}$$

where $-A_\xi X$ and $D_X\xi$ denote the tangential and normal components of $\tilde{\nabla}_X \xi$, respectively. Then we have

$$\langle A_\xi X, Y\rangle = -\langle h(X, Y), \xi\rangle, \tag{1.3}$$

where $\langle\ ,\ \rangle$ denotes the scalar product defined in E_s^m. For the second fundamental form h, we define the covariant derivative, denoted by $\bar{\nabla}_X h$, to be

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{1.4}$$

We then have the equation of Codazzi;

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z) = (\bar{\nabla}_Z h)(Y, X). \tag{1.5}$$

Let us introduce some typical pseudo-Riemannian manifolds:

(1) $S_r^n(c) = \{x \in E_r^{n+1} \mid \langle x-a, x-a\rangle = 1/c\}, \quad c > 0.$

(2) $H_r^n(c) = \{x \in E_{r+1}^{n+1} \mid \langle x-a, x-a\rangle = 1/c\}, \quad c < 0.$

(1) is called a pseudo-Riemannian sphere with radius $1/\sqrt{c}$ and (2) is called a pseudo-hyperbolic space with radius $1/\sqrt{-c}$. Both spaces have planar geodesics.

2. Pointwise planar normal sections. Let M_r^n be an n-dimensional pseudo-Riemannian submanifold of signature $(r, n-r)$ of an m-dimensional pseudo-Euclidean space E_s^m of signature $(s, m-s)$.

We now prove

Theorem 2.1. *Let M_r^n be an n-dimensional pseudo-Riemannian submanifold of an m-dimensional pseudo-Euclidean space E_s^m. Then M_r^n has pointwise planar normal sections if and only if h and $\bar{\nabla} h$ satisfy*

$$(\bar{\nabla} h)(t, t, t)\wedge h(t, t) = 0 \tag{2.1}$$

for any vector t tangent to M_r^n, where $(\bar{\nabla} h)(t, t, t) = (\bar{\nabla}_t h)(t, t)$.

Proof. Let t be a nonzero tangent vector to M_r^n at p in M_r^n and let γ be the normal section of M_r^n at p in the direction t with $\gamma(0) = p$. Let T be the tangent vector field to the normal section $\gamma(s)$ such that $\gamma'(s) = T$ and $\gamma'(0) = t$. Then we obtain by the equation of Gauss (1.1) and that of Weingarten (1.2)

$$\gamma''(s) = \tilde{\nabla}_T T = \nabla_T T + h(T, T), \tag{2.2}$$

$$\begin{aligned} \gamma'''(s) &= \tilde{\nabla}_T \gamma''(s) \\ &= \nabla_T T + h(\nabla_T T, T) - A_{h(T,T)} T + D_T h(T, T). \end{aligned} \tag{2.3}$$

At $p = \gamma(0)$, the definition of the normal section gives

$$t \wedge \nabla_t T = 0 \quad \text{and} \quad t \wedge (\nabla_t \nabla_T T - A_{h(t,t)} t) = 0. \tag{2.4}$$

We now assume that M_r^n has pointwise planar normal sections. Let γ be a normal section of M_r^n at p in the direction t. Then, $\gamma'''(0)$ is a linear combination of $\gamma'(0)$ and $\gamma''(0)$. Thus (2.2), (2.3) and (2.4) give

$$(\bar{\nabla} h)(t, t, t) \wedge h(t, t) = 0.$$

Conversely, we assume that $(\bar{\nabla} h)(t, t, t) \wedge h(t, t) = 0$ for any nonzero tangent vector t of M_r^n at p. Let γ be the normal section of M_r^n at p in the direction t. By considering (2.4), we obtain

$$\gamma'(0) \wedge \gamma''(0) \wedge \gamma'''(0) = t \wedge h(t, t) \wedge (\bar{\nabla} h)(t, t, t) = 0.$$

Lemma 2.2. *Let M_r^n be an n-dimensional pseudo-Riemannian submanifold of signature $(r, n-r)$ in a pseudo-Euclidean space E_s^m of signature $(s, m-s)$. If M_r^n has pointwise planar normal sections, then for a nonnull vector $t \in T_p M_r^n$, we have*

$$\nabla_t T = 0, \tag{2.5}$$

where $T = \gamma'(s)$, γ being the normal section of M_r^n at in the direction t.

Proof. Since $\langle t, t\rangle \neq 0$, we may assume $\langle T, T\rangle = \varepsilon = \pm 1$ by the arc length parametrization. Thus, $\langle \nabla_T T, T\rangle = 0$ along γ. Since $t \wedge \nabla_t T = 0$, we have $\nabla_t T = 0$.

Definition. A normal section γ of M_r^n at p is said to be nondegenerate if $\gamma'(p), \cdots, \gamma^{(k)}(p)$ span a nondegenerate subspace of $E(p, t)$, where $\gamma'(p) = t$, $\gamma'(p) \wedge \cdots \wedge \gamma^{(k)}(p) \neq 0$ and $\gamma'(p) \wedge \cdots \wedge \gamma^{(k+1)}(p) = 0$.

Proposition 2.3. *Let M_r^n be an n-dimensional pseudo-Riemannian submanifold with pointwise planar normal sections in a pseudo-Euclidean space E_s^m. If every normal section is nondegenerate, then M_r^n is spacelike or timelike.*

Proof. It suffices to show that there are no null vectors tangent to M_r^n at every point p. Let p be a point of M_r^n and let t be a null vector tangent to M_r^n at p. Let γ be the normal section of M_r^n at p in the direction t. Without loss of generality we may assume $\gamma'(p)\wedge\gamma''(p)\neq 0$. Making use of (2.2), we see that $\langle\gamma'(p), \gamma'(p)\rangle\langle\gamma''(p), \gamma''(p)\rangle-\langle\gamma'(p), \gamma''(p)\rangle^2 = 0$. Thus, a plane spanned by $\gamma'(p)$ and $\gamma''(p)$ is degenerate, which implies that $\gamma'(p), \gamma''(p), \cdots, \gamma^{(k)}(p)$ for any $k(\geq 3)$ cannot span a nondegenerate subspace in $E(p, t)$, which is a contradiction.

We now define a function L defined by

$$L(p, t) = L_p(t) = \langle h(t, t), h(t, t)\rangle$$

on $U_pM_r^n$, where $U_pM_r^n = \{t \in T_pM_r^n \mid |\langle t, t\rangle|^{1/2} = 1\}$.

Note. If $L = 0$, then M_r^n does not have nondegenerate pointwise normal sections.

By a vertex of curve γ we mean a point p on γ such that its curvature κ satisfies $d\kappa^2(0)/ds = 0$.

Theorem 2.4. *Let M_r^n be an n-dimensional spacelike or timelike submanifold with nonvanishing L in a pseudo-Euclidean space E_s^m. Then the following are equivalent.*

(a) $(\bar{\nabla}_t h)(t, t) = 0$ *for all t tangent to M,*

(b) $\bar{\nabla} h = 0$,

(c) *M_r^n has nondegenerate pointwise planar normal sections and each normal section at $p \in M_r^n$ has one of its vertices at p.*

Proof. By linearization we easily see that (a) $\Longleftrightarrow$ (b). We now prove (b) implies (c). Since $\bar{\nabla} h = 0$, M_r^n has pointwise planar normal sections by Theorem 2.1. Since L is nonvanishing, every normal section is nondegenerate. Let γ be the normal section of M_r^n at $\gamma(0) = p$ in a given direction $t \in T_pM$. We may assume γ is parametrized by arc length, that is, $\langle\gamma'(s), \gamma'(s)\rangle = \varepsilon$, $\varepsilon = \pm 1$. Since $\varepsilon\kappa^2(s) = \langle\nabla_T T, \nabla_T T\rangle+\langle h(T, T), h(T, T)\rangle$, we get

$$\frac{1}{2}\varepsilon\frac{d\kappa^2}{ds} = \langle \nabla_T\nabla_T T, \nabla_T T\rangle + \langle D_T h(T, T), h(T, T)\rangle,$$

where $T = \gamma'(s)$, which together with Lemma 2.2 yields

$$\frac{1}{2}\frac{d\kappa^2}{ds}(0) = \langle(\bar{\nabla} h)(t, t, t), h(t, t)\rangle = 0, \tag{2.6}$$

that is, p is one of its vertices of γ.

We shall show that (c) implies (a). Let γ be a nondegenerate pointwise planar normal section at p in a given direction t. Then, we have by Theorem 2.1

$$(\bar{\nabla} h)(t, t, t)\wedge h(t, t) = 0.$$

Since p is a vertex of γ, (2.6) gives rise to

$$(\bar{\nabla} h)(t, t, t) = 0.$$

This completes the proof.

Theorem 2.5. *Let M^n_r be an n-dimensional pseudo-Riemannian submanifold of E^m_s. If every normal section of M^n_r in nonnull direction is planar and has the same constant curvature, then the function L defined on a unit tangent bundle is constant and normal sections are one of the following:*

(a) $L > 0$: *a part of circle $S^1 \subset E^2$ of radius $1/\sqrt{L}$,*

(b) $L > 0$: *a part of $S^1_1 \subset E^2_1$ of radius $1/\sqrt{L}$,*

(c) $L < 0$: *a part of $H^1 \subset E^2_1$ of radius $1/\sqrt{-L}$,*

(d) $L < 0$: *a part of $H^1_1 \subset E^2_2$ of radius $1/\sqrt{-L}$,*

(e) $L = 0$: *a straight line segment or a curve in a degenerate plane $E^2_{0,1}$ or $E^2_{1,1}$.*

Proof. Let p be a point of M^n_r. Let $O_p = \{u \in U_pM^n_r \mid L(u) = \langle h(u, u), h(u, u)\rangle \neq 0\}$. Suppose $O_p \neq \phi$. Let γ be the normal section of M^n_r at p in the direction $t \in O_p$. Since $\nabla_t T = 0$, $L(t) = \langle h(t, t), h(t, t)\rangle = \varepsilon\kappa^2$, where $T = \gamma'(s)$, $\gamma'(0) = t$, $\gamma(0) = p$ and κ is the curvature of γ. By continuity, O_p is closed and thus $U_pM^n_r = O_p$. Choose $t \in U_pM^n_r$ and let γ be denoted by the normal section of M^n_r at p in the direction t. We may assume that γ is parametrized by arc length s. Since γ is a plane curve, we may write

$$\gamma(s) = \gamma(0) + f(s)t + g(s)h(t, t) \tag{2.7}$$

for some smooth functions f and g, from which we have

$$\begin{aligned}
&(f'(s))^2\varepsilon+(g'(s))^2L \;= \varepsilon,\\
&(f''(s))^2\varepsilon+(g''(s))^2L = L,\\
&f(0) \;= g(0) = 0,\\
&f'(0) = 1, \quad g'(0) \;= 0,\\
&f''(0) = 0, \quad g''(0) = 1.
\end{aligned}$$

Then we have only the following cases:

(a) $\varepsilon = 1, \quad L > 0,$ (b) $\varepsilon = -1, \quad L > 0,$
(c) $\varepsilon = 1, \quad L < 0,$ (d) $\varepsilon = -1, \quad L < 0,$
(e) $L = 0.$

By the straightforward computation, we obtain:

For (a), $\gamma(s) = \gamma(0)+(\sin\sqrt{L}\,s)t/\sqrt{L}-(\cos\sqrt{L}\,s-1)L^{-1}h(t,\,t)$, which is a part of S^1 with radius $1/\sqrt{L}$.

For (b), $\gamma(s) = \gamma(0)+(\sinh\sqrt{L}\,s)t/\sqrt{L}+(\cosh\sqrt{L}s-1)L^{-1}h(t,\,t)$, which is a part of $S_1^1 \subset E_1^2$ with radius $1/\sqrt{L}$.

For (c), $\gamma(s) = \gamma(0)+(\sinh\sqrt{-L}\,s)t/\sqrt{-L}-(\cosh\sqrt{-L}\,s-1)L^{-1}h(t,\,t)$, which is a part of $H^1 \subset E_1^2$ with radius $1/\sqrt{-L}$.

For (d), $\gamma(s) = \gamma(0)+(\sin\sqrt{-L}\,s)t/\sqrt{-L}+(\cos\sqrt{-L}\,s-1)L^{-1}h(t,\,t)$, which is a part of $H_1^1 \subset E_2^2$ with radius $1/\sqrt{-L}$.

For (e), it is obvious that γ is a straight line segment or a curve in a degenerate plane $E_{0,1}^2$ or $E_{1,1}^1$.

Corollary 2.6. *Let M_r^n be an n-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidean space E_s^m with nonvanishing L. If every normal section of M_r^n in nonnull direction is planar and has the same constant curvature κ, then M_r^n is a parallel submanifold, that is, $\bar{\nabla}h = 0$.*

Proof. In the proof of Theorem 2.5 we see that the function L defined on the unit tangent bundle $UM_r^n = \bigcup_{p\in M}U_pM_r^n$ is constant. Let t be a nonnull unit vector tangent to M_r^n at p. Let γ be the normal section of M_r^n at p in the direction t. Without loss of generality we may assume that γ is parametrized by arc length s. Since the normal section γ has the forms (a)-(d) in Theorem 2.5, $\gamma'(s)$ and $\gamma'''(s)$ are proportional. Making use of this fact and (2.3), we see that

$$(\bar{\nabla}h)(T,\,T,\,T)+3h(\nabla_T T,\,T) = 0. \tag{2.8}$$

On the hand, L = constant and Lemma 3.1 which will be discussed in §3 imply that $\langle h(\nabla_T T,\,T),\,h(T,\,T)\rangle = 0$. Thus, Theorem 2.1 and (2.8) give $(\bar{\nabla}h)(T,\,T,$

$T) = 0$. This holds for all nonnull unit vector fields and thus h is parallel, i.e., $\bar{\nabla} h = 0$ by linearization.

Theorem 2.7. *Let M_r^n be an n-dimensional spacelike or timelike submanifold of a pseudo-Euclidean space E_s^m. Then M_r^n has planar geodesics if and only if every normal section of M_r^n is planar and has the same constant curvature κ.*

Proof. (Sufficiency). Let γ be the normal section of M_r^n at p in the direction t. We may assume that γ is parametrized by arc length s. Let $\gamma(0) = p$, $\gamma'(s) = T$ and $T(0) = t$. Then we have

$$\kappa^2 \varepsilon = \langle \nabla_T T, \nabla_T T \rangle + \langle h(T, T), h(T, T) \rangle.$$

According to (2.5), we get

$$\kappa^2 \varepsilon = \langle h(t, t), h(t, t) \rangle.$$

Since κ is constant and L is constant by Theorem 2.5, we have $L = \kappa^2 \varepsilon = \langle h(u, u), h(u, u) \rangle$ for any unit vector u. Thus, $\langle \nabla_T T, \nabla_T T \rangle = 0$. Since M_r^n is spacelike or timelike, $\nabla_T T = 0$, that is, γ is a part of geodesic. Thus, M_r^n has planar geodesics.

(Necessity.) Suppose M_r^n has planar geodesics. Then, L is constant (see [2]). Let γ be a geodesic with initial velocity t. Then, γ is a part of $S^1 \subset E^2$, $S_1^1 \subset E_1^2$ with radius $1/\sqrt{L}$ or part of $H^1 \subset E_1^2$, $H_1^1 \subset E_2^2$ with radius $1/\sqrt{-L}$ or a line segment or a curve in $E_{0,1}^2$ (See [2] for detail). These curves are generated by $\gamma'(0) = t$ and $h(t, t)$ and thus $\gamma(s)$ lies in $\gamma(0) + \mathit{Span}\{t, h(t, t)\} \subset E(p, t)$. Thus, γ is a planar normal section of the same constant curvature.

3. Pseudo-isotropic submanifolds with pointwise planar normal sections. Let M_r^n be a pseudo-Riemannian submanifold of a pseudo-Euclidean space E_s^m. M_r^n is said to be pseudo-isotropic at $p \in M_r^n$, if L_p is independent of the choice of any unit vector t tangent to M_r^n at p. M_r^n is said to be pseudo-isotropic, if M_r^n is pseudo-isotropic at each point p in M_r^n. In particular, if L is independent of points, then M_r^n is said to be constant pseudo-isotropic.

Lemma 3.1 ([5]). M_r^n is pseudo-isotropic if and only if

$$\langle h(t, t), h(t, t^\perp) \rangle = 0 \tag{3.1}$$

for any orthonormal vectors t and $t^\perp$.

Remark. If M_r^n has planar geodesics, then M_r^n is constant pseudo-isotropic.

Proposition 3.2. *Let M_r^n be an n-dimensional pseudo-isotropic pseudo-Riemannian submanifold in E_s^m. If M_r^n has pointwise planar normal sections, then M_r^n is constant pseudo-isotropic.*

Proof. Let $t(\neq 0)$ be a nonnull vector tangent to M_r^n at p. We may assume that $\langle t, t\rangle = \varepsilon$. Let γ be the pointwise planar normal section of M_r^n at p in the direction t. By theorem 2.1, we have

$$(\bar{\nabla} h)(t, t, t)\wedge h(t, t) = 0. \tag{3.3}$$

Let $\gamma'(s) = T(s)$. We want to prove that the function L is constant. Let z be a unit vector orthogonal to t at $p \in M_r^n$ and extend z to Z on a neighborhood of p which is parallel along γ and $\langle T, Z\rangle = 0$. Then

$$\begin{aligned}
\frac{1}{2}z(L) &= \frac{1}{2}\langle h(T, T), h(T, T)\rangle \\
&= \frac{1}{2}Z\langle h(T, T), h(T, T)\rangle\,|_{s=0} \\
&= \langle(\bar{\nabla} h)(Z, T, T), h(T, T)\rangle|_{s=0} + 2\langle h(\nabla_Z T, T), h(T, T)\rangle|_{s=0} \\
&= \langle(\bar{\nabla} h)(Z, T, T), h(T, T)\rangle|_{s=0} \quad \text{(because of } \langle \nabla_z T, t\rangle = 0) \\
&= \langle(\bar{\nabla} h)(T, Z, T), h(T, T)\rangle|_{s=0} \quad \text{(Codazzi equation)} \\
&= \langle(D_T h)(Z, T), h(T, T)\rangle|_{s=0} \quad \text{(because of } \nabla_T Z = 0) \\
&= T\langle h(Z, T), h(T, T)\rangle|_{s=0} - \langle h(Z, T), D_T h(T, T)\rangle|_{s=0} \\
&= 0 \quad \text{because of (3.1) and (3.3).}
\end{aligned}$$

Since $\dim M_r^n \geq 2$ and p is arbitrary, L is constant on M_r^n, that is, M_r^n is constant pseudo-isotropic.

Considering Theorem 2.7 and Proposition 3.2, we have

Theorem 3.3. *If M_r^n be a pseudo-isotropic spacelike or timelike pseudo-Riemannian submanifold of E_s^m with pointwise planar normal section, then M_r^n has planar geodesics.*

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