Math. J. Okayama Univ. 34 (1992), 241—248

CURVATURE FORMS AND EINSTEIN-LIKE METRICS ON SASAKIAN MANIFOLDS

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1. Introduction. Let $\mathscr{E}$ be the class of Einstein manifolds, $\mathscr{P}$ the class of Riemannian manifolds with parallel Ricci tensor and $\mathscr{C}$ the class of Riemannian manifolds with constant scalar curvature. Then

$$\mathscr{E} \subset \mathscr{P} \subset \mathscr{C}.$$

Following A. Gray, [4], it is possible to define two interesting classes lying between $\mathscr{P}$ and $\mathscr{C}$. If (M, g) is a Riemannian manifold, $\mathrm{Ric}(g)$ its Ricci tensor and ∇ the Levi Civita connection of g, we donote with $\mathscr{A}$ and $\mathscr{B}$ the classes of all Riemannian manifolds satisfying the following conditions, respectively,

$$\mathfrak{S}_{XYZ}[\nabla_X \mathrm{Ric}(g)](Y, Z) = 0,$$
$$[\nabla_X \mathrm{Ric}(g)](Y, Z) = [\nabla_Y \mathrm{Ric}(g)](X, Z), \quad X, Y, Z \in \mathfrak{X}(M)$$

where $\mathfrak{S}$ is cyclic sum and $\mathfrak{X}(M)$ is the Lie algebra of the $\mathscr{C}^\infty$ vector fields on M. The manifolds which belong to the above classes are called Einstein-like because

$$\mathscr{E} \subset \mathscr{P} \subset \mathscr{A} \cup \mathscr{B} \subset \mathscr{C}.$$

The Einstein-like conditions have been studied for some particular types of Riemannian manifolds; for a survey about recent results and examples see [4], [2] and their references.

The purpose of this paper is to study Einstein-like metrics on a Sasakian manifold (M, ϕ, η, ξ, g). The definitions are given in the next section.

In section 2, the connection and the curvature forms, the Ricci tensor and its covariant derivative of a Sasakian manifold are computed using the technique of Cartan's moving frame. The connection and the curvature forms are written as matrix-valued forms, in this way some problems can be directly attacked and the calculations can be performed in a compact form. For example, in [1] it is shown how to construct Einstein metrics by means of a deformation of the contact metric in the direction of the contact form. In this way we obtain in a nice way a result of Tanno, [5].

Finally, we find the conditions which ensure that a Sasakian space is of class $\mathscr{A}$ and we show that all Sasakian manifolds of class $\mathscr{B}$ are Einstein.

*Work supported by GNSAGA of CNR and by MURST of Italy.

This last result should be compared with Theorem 6.1 of [4] where it is proved that a Kähler manifold of class $\mathscr{B}$ has parallel Ricci tensor, but it is not necessarily Einstein.

2. Sasakian manifolds of class $\mathscr{A}$ and $\mathscr{B}$. Let (M, ϕ, η, ξ, g) be a Riemannian manifold endowed with an almost contact metric structure. More precisely, M is a $\mathscr{C}^\infty$ differentiable manifold of dimension $2n+1$, g is a Riemannian metric, ϕ, η, ξ are tensor fields of type $(1, 1)$, $(0, 1)$, $(1, 0)$, respectively, such that

$$\phi^2 = -I+\eta\otimes\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi(X), \phi(Y)) = g(X, Y)-\eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M), \tag{2.2}$$

where I is the Kronecker tensor of type $(1, 1)$. From (2.1) and (2.2) it follows

$$\phi(\xi) = 0, \quad \eta\circ\phi = 0, \tag{2.3}$$

$$\eta(\xi) = g(X, \xi), \quad X \in \mathfrak{X}(M). \tag{2.4}$$

The structure (ϕ, η, ξ, g) is called a contact metric structure if η is a contact form compatible with g, i.e.

$$g(X, \phi(Y)) = d\eta(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.5}$$

A contact metric structure is K-contact if ξ is a Killing vector field, which is equivalent to

$$\nabla_X\xi = -\phi(X), \quad X \in \mathfrak{X}(M).$$

On the other hand, a contact metric structure is called Sasakian if

$$(\nabla_X\phi)(Y) = g(X, Y)\xi-\eta(Y)X, \quad X, Y \in \mathfrak{X}(M), \tag{2.6}$$

where ∇ is Levi-Civita connection of g. It follows immediately that Sasakian manifolds are K-contact manifolds. Sasakian spaces are, in some sense, the odd dimensional analogue of Kähler manifolds (see [3]).

The following index conventions will be used

$$i, j, h, \ldots = 1, \ldots, n; \quad \alpha, \beta, \gamma, \ldots = 1, \ldots, 2n; \quad A, B, C, \ldots = 1, \ldots, 2n+1.$$

Let (M, ϕ, η, ξ, g) be a Sasakian manifold; it is always possible to consider, around each point of M, a coordinate neighbourhood U with a local orthonormal frame $(E_1, E_2, \ldots, E_{2n+1})$ which is adapted to the almost contact structure, i.e.

$$\phi(E_i) = E_{n+i}, \quad E_{2n+1} = \xi.$$

If $(\omega^1, \omega^2, \ldots, \omega^{2n-1})$ denotes the dual frame, from (2.1) and (2.3) it follows

$$\omega^{n+i} = -\omega^i \circ \phi, \quad \omega^{2n+1} = \eta. \tag{2.7}$$

Let us consider the matrices

$$\Theta = \begin{pmatrix} \omega^1 \\ \omega^2 \\ \vdots \\ \omega^{2n} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n is the identity matrix of order n, Θ is a $\boldsymbol{R}^n$-valued 1-form and J is an element of $Gl(2n, \boldsymbol{R})$.

The connection 1-forms of ∇ are defined by

$$\omega_B^A(X) = \omega^A(\nabla_X E_B). \tag{2.8}$$

Since ∇ is a metric connection, the matrix $\omega = (\omega_B^A)$ is an $\mathfrak{so}(2n+1)$-valued 1-form, ($\mathfrak{so}(2n+1)$: Lie algebra of the skew-symmetric real matrices of order $2n+1$). Hence ω can be written as

$$\omega = \begin{pmatrix} \Gamma & \Psi \\ -{}^t\Psi & 0 \end{pmatrix}, \tag{2.9}$$

where Γ is an $\mathfrak{so}(2n)$-valued 1-form, Ψ is a $\boldsymbol{R}^{2n}$-valued 1-form, and ${}^t\Psi$ is the transposed matrix of Ψ. The condition (2.6) is equivalent to

$$\Gamma J = J\Gamma, \quad \Psi = J\Theta, \tag{2.10}$$

that is

$$\omega_j^i = \omega_{n+j}^{n+i}, \quad \omega_{n+j}^i = -\omega_j^{n+i}, \quad \omega_{2n+1}^i = \omega^{n+i}, \quad \omega_{2n+1}^{n+i} = -\omega^i. \tag{2.11}$$

The connection 1-form ω is the unique solution of the first Cartan's structure equation

$$d\begin{pmatrix} \Theta \\ \eta \end{pmatrix} = -\omega \wedge \begin{pmatrix} \Theta \\ \eta \end{pmatrix} = -\begin{pmatrix} \Gamma & \Psi \\ -{}^t\Psi & 0 \end{pmatrix} \wedge \begin{pmatrix} \Theta \\ \eta \end{pmatrix}. \tag{2.12}$$

Here the wedge product of two matrix-valued 1-forms is defined as the natural extension of the usual product of two ordinary matrices. The equation (2.12) implies

$$\begin{cases} d\Theta = -\Gamma \wedge \Theta - \Psi \wedge \eta \\ d\eta = {}^t\Psi \wedge \Theta. \end{cases} \tag{2.13}$$

The curvature tensor of the Levi-Civita connection ∇, defined by

$$R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z, \quad X, Y, Z \in \mathfrak{X}(M),$$

gives rise to the curvature two forms Ω_B^A such that

$$2\Omega_B^A(X, Y) = -\omega^A(R_{XY}E_B), \quad X, Y \in \mathfrak{X}(M).$$

Since they are skew-symmetric with respect to the indices A and B, $\Omega = (\Omega_B^A)$ is a $\mathfrak{so}(2n+1)$-valued form which satisfies the second Cartan's structure equation

$$\Omega = d\omega + \omega \wedge \omega. \tag{2.14}$$

Theorem 2.1. *The curvature 2-form Ω is given by*

$$\Omega = \begin{pmatrix} d\Gamma + \Gamma \wedge \Gamma - \Psi \wedge {}^t\Psi & \Theta \wedge \eta \\ -{}^t(\Theta \wedge \eta) & 0 \end{pmatrix}. \tag{2.15}$$

Proof. From (2.14) and (2.9), the curvature 2-form Ω can be written as

$$\Omega = \begin{pmatrix} d\Gamma + \Gamma \wedge \Gamma - \Psi \wedge {}^t\Psi & d\Psi + \Gamma \wedge \Psi \\ -d({}^t\Psi) - {}^t\Psi \wedge \Gamma & -{}^t\Psi \wedge \Psi \end{pmatrix}.$$

First of all, note that if ${}^t\Psi = (\Psi^1, \Psi^2, \dots, \Psi^{2n})$, then

$${}^t\Psi \wedge \Psi = \textstyle\sum_a \Psi^a \wedge \Psi^a = 0.$$

Because of (2.10) and (2.13), one has

$$d\Psi = d(J\Theta) = Jd(\Theta) = -J(\Gamma \wedge \Theta) - J(\Psi \wedge \eta).$$

It is easy to check directly that

$$J(\Gamma \wedge \Theta) = (J\Gamma) \wedge \Theta = (\Gamma J) \wedge \Theta = \Gamma \wedge (J\Theta)$$

and

$$J(\Psi \wedge \eta) = (J\Psi) \wedge \eta = -\Theta \wedge \eta,$$

where, again (2.10) is used. Hence

$$d\Psi = -\Gamma \wedge (J\Theta) + \Theta \wedge \eta = -\Gamma \wedge \Psi + \Theta \wedge \eta \tag{2.16}$$

Finally, recall that if A and B are matrix-valued 1-forms, then

$${}^t(A \wedge B) = -{}^tB \wedge {}^tA.$$

Since Γ is $\mathfrak{so}(2n)$-valued form, the above identity gives

$$ {}^t\Psi \wedge \Gamma = -{}^t\Psi \wedge {}^t\Gamma = {}^t(\Gamma \wedge \Psi), $$

and the proof of the Theorem is complete.

Remark 1. As described in the Introduction, formula (2.9) and (2.15) are particulary useful for computations about Sasakian manifolds. For example, it is possible to construct Einstein metrics on Sasakian manifolds by deforming the given metric along the contact form (see [1] and [5]). Using (2.9) and (2.15) the proof of this fact becomes particularly nice.

Remark 2. Some curvature properties of Sasakian manifolds can be easily deduced from (2.15). In particular, we get the relation

$$ \Omega^{\alpha}_{2n+1} = \omega^{\alpha} \wedge \eta \tag{2.17} $$

which implies

$$ 2\Omega^{\alpha}_{2n+1}(E_\beta, E_{2n+1}) = \delta_{\alpha\beta}, \quad \Omega^{\alpha}_{2n+1}(E_\beta, E_\gamma) = \Omega^{\beta}_{\gamma}(E_\alpha, E_{2n+1}) = 0, \tag{2.18} $$

($\delta_{\alpha\beta}$ is the usual Kronecker symbol). Then it follows immediately that

$$ R_{X\xi}X = \xi, \tag{2.19} $$

for any unit vector field X orthogonal to ξ and, in general,

$$ R_{XY}\xi = \eta(X)Y - \eta(Y)X, \quad X, Y \in \mathfrak{X}(M). \tag{2.20} $$

From (2.11), (2.15) and (2.20) the following identity can be obtained

$$ g(R_{XY}Z, W) = g(R_{\phi(X)\phi(Y)}\phi(Z), \phi(W)) \tag{2.21} $$

where X, Y, Z, W are vector fields orthogonal to ξ (see also [3]).

By definition, the Ricci tensor and the scalar curvature of the metric g are

$$ \mathrm{Ric}(g) = 2\textstyle\sum_{A,B,C} \Omega^{A}_{B}(E_C, E_B)\omega^A \otimes \omega^C, \tag{2.22} $$

$$ \mathrm{Scal}(g) = \textstyle\sum_A \mathrm{Ric}(g)(E_A, E_A). \tag{2.23} $$

Theorem 2.2. *The Ricci tensor and the scalar curvature of a Sasakian manifold M have the following expressions*

$$ \mathrm{Ric}(g) = \textstyle\sum_{\alpha,\gamma}\left[2\sum_\beta \Omega^{\alpha}_{\beta}(E_\gamma, E_\beta) + \delta_{\alpha\gamma}\right]\omega^\alpha \otimes \omega^\gamma + 2n\omega^{2n+1} \otimes \omega^{2n+1}, \tag{2.24} $$

$$ \mathrm{Scal}(g) = \textstyle\sum_\alpha \mathrm{Ric}(g)(E_\alpha, E_\alpha) + 2n. \tag{2.25} $$

Proof. (2.24) is the direct consequence of (2.18) and (2.22). (2.25) follows immediately from (2.24).

Remark 3. The following relations between the components of the Ricci tensor, which will be crucial in the last part of the paper, are a direct consequence of (2.21)

$$\begin{cases} \mathrm{Ric}(g)(E_i, E_j) = \mathrm{Ric}(g)(E_{n+i}, E_{n+j}) \\ \mathrm{Ric}(g)(E_i, E_{n+j}) = -\mathrm{Ric}(g)(E_{n+i}, E_j). \end{cases} \tag{2.26}$$

Theorem 2.3. *The covariant derivative of the Ricci tensor of a Sasakian manifold M is*

$$\begin{aligned} \nabla_X(\mathrm{Ric}(g)) = {} & 2\sum_{\alpha,\gamma} X\Big[\sum_\beta \Omega^\alpha_\beta(E_\gamma, E_\beta)\omega^\alpha\otimes\omega^\gamma\Big] \\ & +\sum_{\alpha,\gamma}\Big[2\sum_\beta \Omega^\alpha_\beta(E_\gamma, E_\beta)+\delta_{\alpha\gamma}\Big][\nabla_X\omega^\alpha\otimes\omega^\gamma+\omega^\alpha\otimes\nabla_X\omega^\gamma] \\ & +2n[\nabla_X\omega^{2n+1}\otimes\omega^{2n+1}+\omega^{2n+1}\otimes\nabla_X\omega^{2n+1}], \end{aligned} \tag{2.27}$$

where

$$\begin{cases} \nabla_X\omega^i = -\sum_\alpha \omega^i_\alpha(X)\omega^\alpha-\omega^{n+i}(X)\omega^{2n+1} \\ \nabla_X\omega^{n+i} = -\sum_\alpha \omega^{n+i}_\alpha(X)\omega^\alpha+\omega^i(X)\omega^{2n+1} \qquad X\in\mathfrak{X}(M). \\ \nabla_X\omega^{2n+1} = \sum_h (\omega^{n+h}(X)\omega^h-\omega^h(X)\omega^{n+h}) \end{cases} \tag{2.28}$$

Proof. (2.27) comes from (2.24); the relations (2.28) follow from (2.11).

In particular, from (2.27), (2.28) and (2.26) we get

$$\begin{aligned} &\nabla_{E_A}\mathrm{Ric}(g)(E_{2n+1}, E_{2n+1}) = \nabla_{E_{2n+1}}\mathrm{Ric}(g)(E_A, E_{2n+1}) = 0, \\ &\nabla_{E_i}\mathrm{Ric}(g)(E_j, E_{2n+1}) = -\nabla_{E_{n+i}}\mathrm{Ric}(g)(E_{n+j}, E_{2n+1}) = \mathrm{Ric}(g)(E_{n+i}, E_j), \\ &\nabla_{E_i}\mathrm{Ric}(g)(E_{n+j}, E_{2n+1}) = -\nabla_{E_{n+i}}\mathrm{Ric}(g)(E_i, E_{2n+1}) = \mathrm{Ric}(g)(E_i, E_j)-2n\delta_{ij}, \\ &\nabla_{E_{2n+1}}\mathrm{Ric}(g)(E_i, E_j) = \nabla_{E_{2n+1}}\mathrm{Ric}(g)(E_{n+i}, E_{n+j}), \\ &\nabla_{E_{2n+1}}\mathrm{Ric}(g)(E_i, E_{n+j}) = -\nabla_{E_{2n+1}}\mathrm{Ric}(g)(E_{n+i}, E_j), \\ &\nabla_{E_\alpha}\mathrm{Ric}(g)(E_\beta, E_\gamma) = \nabla_{\phi(E_\alpha)}\mathrm{Ric}(g)(\phi(E_\beta), \phi(E_\gamma)). \end{aligned}$$

Remark 4. The more general case of K-contact manifolds is slightly different. In fact, if M is a K-contact manifold only the relation

$$\Psi = J\Theta, \tag{2.29}$$

holds but, in general, $\Gamma J \neq J\Gamma$. Then, instead of (2.15), the curvature 2-form is given by

$$\Omega = \begin{pmatrix} d\Gamma + \Gamma \wedge \Gamma - \Psi \wedge {}^t\Psi & \Theta \wedge \eta + (\Gamma J - J\Gamma) \wedge \Theta \\ -{}^t(\Theta \wedge \eta) - {}^t\Theta \wedge (\Gamma J - J\Gamma) & 0 \end{pmatrix}. \tag{2.30}$$

Let (M, ϕ, η, ξ, g) be a Sasakian manifold. The following Theorem holds.

Theorem 2.4. *A Sasakian manifold M is of type $\mathscr{A}$ if and only if*

$$\mathfrak{S}_{\alpha\beta\gamma}[\nabla_{E_\alpha}\mathrm{Ric}(g)](E_\beta, E_\gamma) = 0 \tag{2.31}$$

and

$$[\nabla_{E_{2n+1}}\mathrm{Ric}(g)](E_\alpha, E_\beta) = 0. \tag{2.32}$$

Proof. (2.31) and (2.32) are a direct consequence of the definition of Einstein-like metric of type $\mathscr{A}$ (see the Introduction) and of Theorem 2.3.

For Sasakian manifolds of type $\mathscr{B}$ we have the following

Theorem 2.5. *A Sasakian manifold of type $\mathscr{B}$ is Einstein. In other words, if $\mathscr{S}$ denotes the class of Sasakian manifolds, then*

$$\mathscr{S} \cap \mathscr{B} = \mathscr{S} \cap \mathscr{P} = \mathscr{S} \cap \mathscr{E},$$

(for the notations see the Introduction).

Proof. From Theorem 2.3 and the definition of a metric of type $\mathscr{B}$, it follows

$$\begin{aligned} &\mathrm{Ric}(g)(E_i, E_j) = \mathrm{Ric}(g)(E_{n+i}, E_{n+j}) = 2n\delta_{ij} \\ &\mathrm{Ric}(g)(E_i, E_{n+j}) = -\mathrm{Ric}(g)(E_{n+i}, E_j) = 0, \\ &\mathrm{Ric}(g)(E_{2n+1}, E_{2n+1}) = 2n, \end{aligned}$$

that is g is an Einstein metric.

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(*Received March 24, 1992*)