

REAL HYPERSURFACES IN $P_n(\mathbb{C})$ WITH CONSTANT PRINCIPAL CURVATURES

Dedicated to Professor Hisao Nakagawa on his 60th birthday

U-HANG KI* and RYOICHI TAKAGI

Introduction. Let M be a real hypersurface in a complex projective space $P_n(\mathbb{C})$ of complex dimension n with the metric of constant holomorphic sectional curvature. We shall say M to be *d-isoparametric* if all principal curvatures of M are constant, and the number of different principal curvatures is equal to d . So far, T.E. Cecil and P.J. Ryan [1] proved that any real hypersurface in $P_n(\mathbb{C})$ with at most two principal curvatures must be a part of a geodesic hypersurface, and one of the present authors [3] and Q.M. Wang [4] classified 3-isoparametric hypersurfaces in $P_n(\mathbb{C})$. In this paper we consider a $d(\geq 4)$ -isoparametric real hypersurface in $P_n(\mathbb{C})$. Our main result can be stated as follows:

Main Theorem. *Let M be a $d(\geq 4)$ -isoparametric real hypersurface in $P_n(\mathbb{C})$. Assume that $\dim M > 3 \cdot 2^{d-2}$. Then at each point of M there exists a principal subspace whose image by the complex structure of $P_n(\mathbb{C})$ is again tangent to M .*

Note that if M is a d -isoparametric real hypersurface in $P_n(\mathbb{C})$ and at each point of M there exist $d-1$ principal subspaces all of whose images by the complex structure of $P_n(\mathbb{C})$ are again tangent to M , then M is congruent to one of examples given in [3], which was proved by M. Kimura [2]. Thus our theorem gives a step to see whether a d -isoparametric real hypersurface in $P_n(\mathbb{C})$ exists except for known examples or not. Moreover, we observe that for $d \geq 4$ ($d \neq 5$) we have no examples of a d -isoparametric real hypersurface in $P_n(\mathbb{C})$.

1. Preliminaries. Let M be a real hypersurface in $P_n(\mathbb{C})$. Choose a local field $\{\tilde{e}_1, \dots, \tilde{e}_{2n}\}$ of orthonormal frame in such a way that, restricted to M , the vectors $\tilde{e}_1, \dots, \tilde{e}_{2n-1}$ are tangent to M . Hereafter the indices $i, j, k, l, \alpha, \beta, \gamma$ run from 1 to $2n-1$, and the indices A, B, C, D run from 1 to $2n$. Let ω_A be the 1-forms dual to \tilde{e}_A . Then the connection forms ω_{AB} of $P_n(\mathbb{C})$ are the 1-forms uniquely determined by

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$$(1.1) \quad d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0.$$

The curvature forms of $P_n(\mathcal{C})$ are given by

$$(1.2) \quad d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} = c\omega_A \wedge \omega_B + c \sum_{C,D} (I_{AC}I_{BD} + I_{AB}I_{CD})\omega_C \wedge \omega_D,$$

where $4c$ denotes the constant holomorphic sectional curvature of $P_n(\mathcal{C})$ and $I = (I_{AB})$ does the complex structure of $P_n(\mathcal{C})$. Moreover I satisfies

$$(1.3) \quad \sum_C I_{AC}I_{CB} = -\delta_{AB}, \quad I_{AB} + I_{BA} = 0,$$

$$(1.4) \quad dI_{AB} = \sum_C I_{AC}\omega_{CB} - \sum_C I_{BC}\omega_{CA}.$$

In the sequel we denote by e_A , θ_A , θ_{AB} and J_{AB} the restriction of \bar{e}_A , ω_A , ω_{AB} and I_{AB} to M respectively. Then we have

$$(1.5) \quad d\theta_i = -\sum_j \theta_{ij} \wedge \theta_j, \quad \theta_{ij} + \theta_{ji} = 0,$$

which implies that θ_{ij} are the connection forms of M . By Cartan's lemma $\theta_{2n,i}$ can be written as

$$(1.6) \quad \phi_i := \theta_{2n,i} = \sum_j h_{ij}\theta_j, \quad h_{ij} = h_{ji}.$$

The quadric form $\sum \phi_i \theta_i$ is called the second fundamental form of M with respect to the normal vector e_{2n} . The symmetric matrix (h_{ij}) of degree $2n-1$ is called the shape operator. An eigenvalue of the shape operator is called a principal curvature.

The curvature forms Θ_{ij} of M are defined by

$$(1.7) \quad \Theta_{ij} = d\theta_{ij} + \sum_k \theta_{ik} \wedge \theta_{kj}.$$

Then from (1.2) we have the equations of Gauss

$$(1.8) \quad \Theta_{ij} = \phi_i \wedge \phi_j + c\theta_i \wedge \theta_j + \sum_{k,l} c(J_{ik}J_{jl} + J_{ij}J_{kl})\theta_k \wedge \theta_l.$$

Put $f_i = J_{2n,i}$. Then from (1.2) and (1.6) we obtain the equations of Codazzi

$$(1.9) \quad d\phi_i + \sum_j \phi_j \wedge \theta_{ji} = c \sum_{j,k} (f_j J_{ik} + f_i J_{jk})\theta_j \wedge \theta_k.$$

Moreover it follows from (1.3) and (1.4) that

$$(1.10) \quad J_{ij} + J_{ji} = 0, \quad \sum_k J_{ik}J_{kj} - f_i f_j = -\delta_{ij},$$

$$\sum_j J_{ij}f_j = 0, \quad \sum_i f_i^2 = 1,$$

$$(1.11) \quad dJ_{ij} = \sum_k J_{ik}\theta_{kj} - \sum_k J_{jk}\theta_{ki} - f_i\phi_j + f_j\phi_i,$$

$$(1.12) \quad df_i = \sum_j f_j \theta_{ji} - \sum_j J_{ji} \phi_j.$$

2. Formulas. In this section we assume that all principal curvatures x_1, \dots, x_{2n-1} of M are constant. Denote by $m(x_i)$ the multiplicity of x_i . We may set $\phi_i = x_i \theta_i$. Then by (1.9) we can write the connection forms θ_{ji} in the form

$$(2.1) \quad (x_i - x_j) \theta_{ij} = c \sum_k (A_{ijk} + f_i J_{jk} + f_j J_{ik}) \theta_k,$$

where $A_{ijk} = A_{jik} = A_{ikj}$ (cf. [3]). In particular we have

$$(2.2) \quad A_{ijk} = -f_i J_{jk} - f_j J_{ik} \text{ if } x_i = x_j,$$

$$(2.3) \quad f_i J_{jk} = 0 \text{ if } x_i = x_j = x_k.$$

For an index i , we denote by $[i]$ the set of indices j with $x_j = x_i$. Then it is obvious that the vector $F_i = \sum_{j \in [i]} f_j e_j$ is independent of the choice of orthonormal frame $\{e_j \mid j \in [i]\}$ for the eigenspace belonging to x_i . Therefore for any index i we can indicate a special index i' so that the vector F_i linearly depends on $e_{i'}$. In other words, $f_j = 0$ for $j \in [i] \setminus \{i'\}$. Note that $f_{i'} = 0$ is possible.

Lemma 2.1. *Assume $J_{ij} = 0$ for $i, j \in [i']$. Then*

$$(2.4) \quad \sum_{\alpha \in [i]} \frac{2c}{x_\alpha - x_i} (f_i J_{ja} - f_j J_{ia}) J_{ka} - f_i x_j \delta_{jk} + f_j x_i \delta_{ik} = 0$$

for $i, j, k \in [i']$,

$$(2.5) \quad \sum_{\alpha \in [i]} \left(\frac{J_{ia}}{x_\alpha - x_i} (A_{a\beta} + f_\alpha J_{j\beta} + f_j J_{a\beta}) - \frac{J_{ja}}{x_\alpha - x_i} (A_{ai\beta} + f_\alpha J_{i\beta} + f_i J_{a\beta}) \right) = 0$$

for $\beta \in [i']$ and $i, j \in [i']$.

Proof. From (1.11) we have

$$0 = dJ_{ij} = \sum_\alpha J_{ia} \theta_{a\beta} - \sum_\alpha J_{ja} \theta_{a\alpha} - f_i x_j \theta_j + f_j x_i \theta_i.$$

It follows from this and (2.1) that

$$\sum_\beta \sum_{\alpha \in [i]} \frac{c J_{ia}}{x_\alpha - x_j} (A_{a\beta} + f_\alpha J_{j\beta} + f_j J_{a\beta}) \theta_\beta$$

$$-\sum_{\beta} \sum_{\alpha \notin [i]} \frac{cJ_{j\alpha}}{x_{\alpha} - x_i} (A_{\alpha i\beta} + f_{\alpha}J_{i\beta} + f_iJ_{\alpha\beta})\theta_{\beta} - f_i x_j \theta_j + f_j x_i \theta_i = 0.$$

Taking account of the coefficients of $\theta_k (k \in [i])$ and $\theta_{\beta} (\beta \notin [i])$, we have (2.4) and (2.5).

Lemma 2.2. *Assume $f_{i'} = 0$. Then*

$$(2.6) \quad J_{ij} \left(\sum_{\alpha \notin [i]} \frac{cf_{\alpha}^2}{x_{\alpha} - x_i} + x_i \right) = 0 \text{ for } i, j \in [i'],$$

$$(2.7) \quad \sum_{\beta \notin [i]} \frac{cf_{\beta}}{x_{\beta} - x_i} A_{\beta ia} + J_{ia} \left(\sum_{\beta \notin [i]} \frac{cf_{\beta}^2}{x_{\beta} - x_i} + x_a \right) = 0$$

for $\alpha \notin [i]$ and $i \in [i']$.

Proof. From $df_i = 0$ and (1.12) we have

$$\sum_{\alpha} f_{\alpha} \theta_{\alpha i} - \sum_{\alpha} J_{\alpha i} x_{\alpha} \theta_{\alpha} = 0,$$

which together with (2.1) implies that

$$\sum_{\beta} \sum_{\alpha \notin [i]} \frac{cf_{\alpha}}{x_{\alpha} - x_i} (A_{\alpha i\beta} + f_{\alpha}J_{i\beta} + f_iJ_{\alpha\beta})\theta_{\beta} - \sum_{\alpha} J_{\alpha i} x_{\alpha} \theta_{\alpha} = 0.$$

Taking account of the coefficients of $\theta_j (j \in [i'])$ and $\theta_{\alpha} (\alpha \notin [i'])$ and making use of (2.1), we have (2.6) and (2.7).

Lemma 2.3. *Assume $f_{i'} \neq 0$ and $m(x_i) \geq 2$. Then*

$$\sum_{\alpha \notin [i]} \frac{2c}{x_{\alpha} - x_i} J_{ja} J_{ka} - x_i \delta_{jk} = 0 \text{ for } i, j, k \in [i'] \text{ and } j \neq i'.$$

Proof. Put $i = i'$ and take $j \in [i'] \setminus \{i'\}$ in (2.4).

Lemma 2.4. *Let x_i be a positive maximal or negative minimum principal curvature. Then $f_{i'} = 0$ or $m(x_i) = 1$.*

Proof. For example, let x_i be a positive maximal principal curvature. If $f_{i'} \neq 0$ and $m(x_i) \geq 2$, we put $j = k \in [i'] \setminus \{i'\}$ in Lemma 2.3. Then we have

$$\sum_{\alpha \notin [i]} \frac{2c}{x_{\alpha} - x_i} J_{ja} J_{ja} = x_i > 0,$$

which contradicts the fact that $x_{\alpha} - x_i < 0$ for $\alpha \notin [i']$.

Let $\lambda_1, \dots, \lambda_d$ be all different principal curvatures with $\lambda_1 < \dots < \lambda_d$, and put $m_r := m(\lambda_r)$.

We may assume $\lambda_1 < 0$ by reversing the unit normal vector e_{2n} if necessary.

Lemma 2.5. (1) *Let $r \in \{1, \dots, d\}$ be an index such that $\lambda_r = x_i$ for some i with $f_{i'} \neq 0$. If $m_r \geq m_1 + \dots + m_{r-1} + 2$, then $\lambda_r > 0$. If $m_r \geq m_{r+1} + \dots + m_d + 2$, then $\lambda_r < 0$.*

(2) *Let $r, s \in \{1, \dots, d\}$ be indices such that $r < s$ and $\lambda_r = x_i, \lambda_s = x_j$ for some i, j with $f_{i'}f_{j'} \neq 0$. Then $m_r \leq m_1 + \dots + m_{r-1} + 1$ or $m_s \leq m_{s+1} + \dots + m_d + 1$.*

Proof. We consider a matrix $A = (J_{ij})$, where the indices i and j take all values such that $x_i = \lambda_r$ and $x_j = \lambda_1$ or, \dots , or λ_{r-1} . Thus A is an $m_r \times (m_1 + \dots + m_{r-1})$ -matrix, which can be considered as a linear mapping of the m_r -dimensional vector space $V(m_r)$ into the $(m_1 + \dots + m_{r-1})$ -dimensional vector space $V(m_1 + \dots + m_{r-1})$. Denote by $V(m_r - 1)$ the orthogonal complement of $\mathbf{Re}_{i'}$ in $V(m_r)$. Putting $\tilde{A} := A | V(m_r - 1)$, we have $\ker \tilde{A} \neq 0$ by the assumption. Then we retake an orthonormal frame $\{e_A\}$ so that $\ker \tilde{A} \ni e_i$ for some $i \in [i']$. Now since $J_{ij} = 0$ for any j with $x_j = \lambda_1$ or, \dots , or λ_{r-1} , by Lemma 2.3 we have $\lambda_r \geq 0$. But $\lambda_r = 0$ cannot occur by (1.10) and (2.3). Similarly we can prove the second statement. Now, (2) is an immediate consequence of (1).

Lemma 2.6. (1) *Let $r \in \{2, \dots, d\}$ be an index such that $\lambda_r = x_i$ for some i with $f_{i'} \neq 0$. Assume $m_r \geq 2$. Then $\lambda_r(\lambda_{r-1} - \lambda_r) < 2c$ and*

$$m_1 + \dots + m_{r-2} + m_{r+1} + \dots + m_d \geq m_r - 1,$$

or $\lambda_r(\lambda_{r-1} - \lambda_r) = 2c$ and

$$m_{r-1} \geq m_r - 1.$$

(2) *Let $r \in \{1, \dots, d-1\}$ be an index such that $\lambda_r = x_i$ for some i with $f_{i'} \neq 0$. Assume $m_r \geq 2$. Then $\lambda_r(\lambda_{r+1} - \lambda_r) < 2c$ and*

$$m_1 + \dots + m_{r-1} + m_{r+2} + \dots + m_d \geq m_r - 1,$$

or $\lambda_r(\lambda_{r+1} - \lambda_r) = 2c$ and

$$m_{r-1} \geq m_r - 1.$$

Proof. (1) According to Lemma 2.3., we have

$$(2-8) \quad \sum_{\alpha \notin [i]} \frac{2c}{x_\alpha - x_i} J_{ja} J_{ka} = x_i \delta_{jk} \quad \text{for } i, j, k \in [i'] \text{ and } j \neq i'.$$

On the other hand, it follows from (1.10) that

$$\frac{2c}{\lambda_{r-1} - \lambda_r} \sum_{\alpha} J_{ja} J_{ka} = \frac{2c}{\lambda_{r-1} - \lambda_r} \delta_{jk}.$$

Subtracting this from (2.8), we have

$$\begin{aligned} 2c \sum_{\alpha \notin [i]} \frac{\lambda_{r-1} - x_\alpha}{(x_\alpha - \lambda_r)(\lambda_{r-1} - \lambda_r)} J_{ja} J_{ka} \\ = \left(\lambda_r - \frac{2c}{\lambda_{r-1} - \lambda_r} \right) \delta_{jk} \text{ for } j, k \in [i'], j \neq i'. \end{aligned}$$

This implies that $\lambda_r(\lambda_{r-1} - \lambda_r) \leq 2c$ and there are $m_r - 1$ linearly independent $(m_1 + \dots + m_{r-2} + m_{r+1} + \dots + m_d)$ -dimensional vectors.

If $\lambda_r(\lambda_{r-1} - \lambda_r) = 2c$, then we have $J_{ja} = 0$ for $j \in [i'] \setminus \{i'\}$ and any α with $x_\alpha \neq \lambda_{r-1}, \lambda_r$. Thus we have

$$\sum_{\alpha} J_{ja} J_{ka} = \delta_{jk} \text{ for } j, k \in [i'] \setminus \{i'\},$$

where the summation is taken over all α 's with $x_\alpha = x_{r-1}$. This implies $m_{r-1} \geq m_r - 1$.

Similarly, by considering λ_{r+1} instead of λ_{r-1} , we have (2).

3. Proof of Theorem. As for special indices i' , confer the beginning of section 2. It is sufficient to prove $f_{i'} = 0$ for some index i . For this, assume the contrary. Then we see $m_1 = 1$ by Lemma 2.4. We need to consider two cases.

Case I: $m_d \geq 2$. Then from Lemma 2.5 (2) we have

$$m_r \leq 1 + m_1 + \dots + m_{r-1} \quad \text{for } r = 1, \dots, d-1.$$

This implies $m_r \leq 2^{r-1}$ for $r = 1, \dots, d-1$. And hence

$$(3.1) \quad m_d = \dim M - m_1 - \dots - m_{d-1} \geq \dim M - 2^{d-1} + 1.$$

On the other hand, it follows from Lemma 2.6 (1) that

$$(3.2) \quad m_d \leq 2^{d-2} + 1,$$

which together with (3.1) implies $\dim M \leq 2^{d-1} + 2^{d-2} = 3 \cdot 2^{d-2}$.

Case II: $m_d = 1$. Define two indices r and s by

$$r := \max\{t \mid m_{t-1} \leq 1 + m_1 + \dots + m_{t-2}\}$$

and

$$s := \min\{t \mid m_t \leq 1 + m_{t+1} + \dots + m_d\}.$$

Then $2 \leq r, s \leq d$. If $r < s$, then owing to Lemma 2.5 (2), we have $s = r + 1$. Then, as in the Case I we have

$$\begin{aligned} m_t &\leq 2^{t-1} \quad \text{for } t = 1, \dots, r-1, \\ m_t &\leq 2^{d-1} \quad \text{for } t = r+1, \dots, d. \end{aligned}$$

If $r \geq s$, then above two inequalities hold obviously. Consequently we see that

$$\begin{aligned} m_r &= \dim M - (m_1 + \dots + \widehat{m}_r + \dots + m_d) \\ &\geq 2n - 1 - 2^{r-1} - 2^{d-r} + 2 > n \end{aligned}$$

since $n \geq 3 \cdot 2^{d-3} + 1$ by the assumption. Thus from (2.3) we have $J_{ij} = 0$ for all i and j with $x_i = x_j = \lambda_r$, which contradicts the fact $\text{rank } J = 2n - 2$.

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U-H. KI
DEPARTMENT OF MATHEMATICS
KYUNGPOOK UNIVERSITY
TAEGU, 702-701 KOREA

R. TAKAGI
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
CHIBA UNIVERSITY
CHIBA-SHI 260, JAPAN

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