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UNCOUNTABLY MANY INFINITE LOOP SPACES OF THE SAME N-TYPE FOR ALL N

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Two CW-complexes are said to have the same n-type if their Postnikov n-stages are homotopy equivalent. J.F. Adams [1] gave two spaces which have the same n-type for all n and are not finite type. B. Gray [3] gave two CW-complexes which have the same n-type for all n and are loop spaces of finite type. Let $SNT(X)$ be the collection of spaces of same n-type for all n as X. C. A. McGibbon and J.C. Møller [4, 5] obtained theorems which decide the potency of a set $SNT(X)$. They also investigated $SNT(-)$ for many examples e.g. $K(Z, n)\times S^n$, $\Sigma^k(K(Z, n)\vee S^n)$ and classifying spaces BG of Lie groups G. Especially they proved that $SNT(BG)$ is an uncontable set except for $G = T^k$ ($k \geqq 0$), $SU(n)$ or $PSU(n)$ ($n = 2, 3$) (cf. Theorem 4 of [4]), and $SNT(BU)$ and $SNT(BSp)$ are trivial (cf. Example E of [4]). We note that $BU = \varinjlim BU(n)$ and $BSp = \varinjlim BSp(n)$ are infinite loop spaces. Hence, we set the following conjecture,
"If two infinite loop spaces have the same n-type for all n, they have the same homotopy type."

On the other hand, the author proved the next theorem in [6].

Theorem. *Let $f, g : \Sigma^k CP^\infty \to S^{k+3}$ be continuous maps $k \geqq 0$ and $C(f)$, $C(g)$ mapping cones of f and g respectively. $\Omega^m C(f)$ and $\Omega^m C(g)$ are homotopy equivalent if and only if f and $\pm g$ are homotopic for $0 \leqq m \leqq k$.*

In the above theorem, the case $k = \infty$ is an open problem. Since the homotopy set $[\Sigma^k CP^\infty, S^{k+3}] = \hat{Z}/Z$ is a set of phantom maps, the spaces $\{\Omega^m C(f)\}$ have the same n-type for all n and are finite type ($0 \leqq m \leqq k$). Moreover we generalized the above theorem for connected Lie groups and gave examples of unstable complexes of the same n-type for all n [7].

In this paper, we obtain the solution of the open problem for $k = \infty$ and the negative answer for the above conjecture. Our main theorems are stated as follows. We remark that the next results are easily generalized for Lie groups by our paper [7].

Theorem. *Let $f, g : \Sigma^k CP^\infty \to S^{k+3}$ be continuous maps and $C(f)$, $C(g)$ mapping cones of f and g respectively. $Q(C(f))$ and $Q(C(g))$ are homotopy* *equivalent if and only if f and $\pm g$ are homotopic, where* $Q(X) = \varinjlim \Omega^n \Sigma^n X$.

Theorem. *Let $f, g: \Sigma^k CP^\infty \to S^{k+3}$ be continuous maps $Q^q(C(f))$ and $Q^q(C(g))$ are homotopy equivalent if and only if f and $\pm g$ are homotopic, where $Q^q(X)$ is the q-time iteration of the functor Q on X.*

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1. Preliminary. In this paper, we work in the category ***Cw*** of based spaces with homotopy type of CW-complexes and based continuous maps, and the category ***Spec*** of CW-spectra. We shall use the terminologies and the notations of [6], [8]. Now we review the results of [6, 7]. Let CP^∞ and S^n be the infinite dimensional complex projective space and n-sphere respectively. Let $j: S^{k+2} \to \Sigma^k CP^\infty$ be the canonical inclusion. This induces a map of inverse systems $(j_n)_*: [\Sigma^k CP^n, S^{k+2}] \to [\Sigma^k CP^n, \Sigma^k CP^\infty]$ and $j_*: [\Sigma^{k-1} CP^\infty, S^{k+2}] \to [\Sigma^{k-1} CP^\infty, \Sigma^k CP^\infty]$. The group $[\Sigma^{k-1} CP^\infty, S^{k+2}]$ is $\hat{Z}/Z$ by Theorem D of [8] and $[\Sigma^k CP^n, \Sigma^k CP^\infty]$ is a finitely generated nilpotent group for $k > 0$. We obtained the next lemma in [6].

Lemma 1.1. *If a map $h: \Sigma^k CP^n \to \Sigma^k CP^\infty$ satisfies $H^m(h) = 0$ for $m > k+2$, the degree of $H^{k+2}(h): H^{k+2}(\Sigma^k CP^\infty) \to H^{k+2}(\Sigma^k CP^n)$ is divided by $\prod_p \mathrm{Max}\{\nu_p(j) \mid j = 1, 2, \cdots\cdots, n\}$ where $\nu_p(j)$ is the factor of a prime p of j.*

In the above statement, we remark that the number $\prod_p \mathrm{Max}\{\nu_p(j) \mid j = 1, 2, \cdots\cdots, n\}$ does not depend on k. If we take S^{k+2} instead of $\Sigma^k CP^\infty$, we can get the similar result. Hence we obtained the next proposition.

Proposition 1.2. *The canonical inclusion $j: S^{k+2} \to \Sigma^k CP^\infty$ induces a monomorphism $j_*: [\Sigma^{k-1} CP^\infty, S^{k+2}] \to [\Sigma^{k-1} CP^\infty, \Sigma^k CP^\infty]$ for $k \geqq 1$.*

Now we prove the analogous theorem for $J = Q(j): Q(S^{k+2}) \to Q(\Sigma^k CP^\infty)$ induced by the canonical inclusion $j: S^{k+2} \to \Sigma^k CP^\infty$. It is difficult to determine the homotopy sets $[\Sigma^{k-1} CP^\infty, Q(S^{k+2})]$ and $[\Sigma^{k-1} CP^\infty, Q(\Sigma^k CP^\infty)]$. But it is comparatively easy to determine sets of phantom maps, because they are determined by the systems of homotopy groups. We set as follows:

$$S(k, j, n) = [\Sigma^{k+j} CP^n, S^{k+j+2}] \qquad S(k, n) = [\Sigma^k CP^n, Q(S^{k+2})]$$
$$T(k, j, n) = [\Sigma^{k+j} CP^n, \Sigma^{k+j} CP^\infty)] \qquad T(k, n) = [\Sigma^k CP^n, Q(\Sigma^k CP^\infty)]$$

Since the inverse systems $\{S(k,\ n)\}$, $\{T(k,\ n)\}$ are the stable homotopy groups of $\{S(k,\ j,\ n)\}$, $\{T(k,\ j,\ n)\}$, they are evaluated by Lemma 1.1. Hence by using the method of Proposition 1.2 in [6], we have sets of phantom maps $\theta[\Sigma^{k-1}CP^{\infty}, Q(S^{k+2})] = \varprojlim^1 S(k,\ n) = \hat{Z}/Z$ and $\theta[\Sigma^{k-1}CP^{\infty}, Q(\Sigma^{k}CP^{\infty})] = \varprojlim^1 T(k,\ n)$ which contains $\hat{Z}/Z$. Hence we get the next theorem.

Theorem 1.3. *The cononical inclusion* $J: Q(S^{k+2}) \to Q(\Sigma^{k}CP^{\infty})$ *induces a monomorphism* $J_*: \theta[\Sigma^{k-1}CP^{\infty}, Q(S^{k+2})] \to \theta[\Sigma^{k-1}CP^{\infty}, Q(\Sigma^{k}CP^{\infty})]$ *for* $k \geqq 1$.

There exists a natural equivalence between ***Cw*** and ***Spec***.

$$[X,\ Q(Y)] = \{\boldsymbol{\Sigma}^{\infty}X,\ \boldsymbol{\Sigma}^{\infty}Y\} \tag{1.4}$$

where $\boldsymbol{\Sigma}^{\infty}X$, $\boldsymbol{\Sigma}^{\infty}Y$ are suspension spectra of CW-complexes X, Y of finite type respectively and $\{\boldsymbol{\Sigma}^{\infty}X,\ \boldsymbol{\Sigma}^{\infty}Y\}$ is a homotopy set in the category ***Spec*** of CW-spectra. Hence we get the next theorem by Theorem 1.3.

Theorem 1.5. *The canonical inclusion* $J: \boldsymbol{\Sigma}^{\infty}S^{k+2} \to \boldsymbol{\Sigma}^{\infty}\Sigma^{k}CP^{\infty}$ *induces a monomorphism* $J_*: \theta\{\boldsymbol{\Sigma}^{\infty}\Sigma^{k-1}CP^{\infty}, \boldsymbol{\Sigma}^{\infty}S^{k+2}\} \to \theta\{\boldsymbol{\Sigma}^{\infty}\Sigma^{k-1}CP^{\infty}, \boldsymbol{\Sigma}^{\infty}\Sigma^{k}CP^{\infty}\}$ *in the category of spectra.*

If suspension spectra $\boldsymbol{\Sigma}^{\infty}X$, $\boldsymbol{\Sigma}^{\infty}Y$ of CW-complexes X, Y of finite type are homotopy equivalent, infinite loop spaces $Q(X)$, $Q(Y)$ are homotopy equivalent. This is geometricaly clear for specialists [2]. For the completeness, we shall prove it in Appendix by pure categorical method.

2. Main theorems. A map $f: X \to Y$ is called a phantom map if the restriction $f \mid X^n$ on the skeleton X^n is homotopic to the constant map for all n. We remark that the suspension functor Σ induces the isomorphism Σ: $\theta\{\boldsymbol{\Sigma}^{\infty}CP^{\infty}, \boldsymbol{\Sigma}^{\infty}S^3\} = \theta\{\boldsymbol{\Sigma}^{\infty}\Sigma CP^{\infty}, \boldsymbol{\Sigma}^{\infty}S^4\}$. By Theorem D of A. Zabrodsky [8], the homotopy set $[\Sigma^{k}CP^{\infty}, S^{k+3}]$ is equal to $\mathrm{Ext}(H_{k+2}(\Sigma^{k}CP^{\infty}: Q), \pi_{k+3}(S^{k+3}))$ $= \mathrm{Ext}(Q,\ Z) = \hat{Z}/Z$ and all maps $f: \Sigma^{k}CP^{\infty} \to S^{k+3}$ are phantom maps. By the results of section 1, the functor $\boldsymbol{\Sigma}^{\infty}$ induces the isomorphism $[\Sigma^{k}CP^{\infty}, \Sigma^{k}S^3]$ $= \theta\{\boldsymbol{\Sigma}^{\infty}CP^{\infty}, \boldsymbol{\Sigma}^{\infty}S^3\}$ for all $k \geqq 0$, and the isomorphism between the components $\hat{Z}/Z$ of $[\Sigma^{k}CP^{\infty}, \Sigma^{k}\Sigma CP^{\infty}]$ and $\theta\{\boldsymbol{\Sigma}^{\infty}CP^{\infty}, \boldsymbol{\Sigma}^{\infty}\Sigma CP^{\infty}\}$ for all $k \geqq 0$. Hence we can identify phantom maps $f: \Sigma^{k}CP^{\infty} \to S^{k+3}$ and $\boldsymbol{\Sigma}^{\infty} f: \boldsymbol{\Sigma}^{\infty}CP^{\infty} \to \boldsymbol{\Sigma}^{\infty}S^3$ (resp. $g: \Sigma^{k}CP^{\infty} \to \Sigma^{k+1}CP^{\infty}$ and $\boldsymbol{\Sigma}^{\infty}g: \boldsymbol{\Sigma}^{\infty}CP^{\infty} \to \boldsymbol{\Sigma}^{\infty}\Sigma CP^{\infty}$). A symbol "$\sim$" means a homotopy relation.

Theorem 2.1. *Let $f, g : \Sigma^k CP^\infty \to S^{k+3}$ be continuous maps and $C(f)$, $C(g)$ mapping cones of f and g respectively. The suspension spectra $\Sigma^\infty C(f)$ and $\Sigma^\infty C(g)$ are homotopy equivalent if and only if f and $\pm g$ are homotopic. These spaces are finite type and have the same n-type for all n.*

Proof. If f and $\pm g$ are homotopic, $\Sigma^\infty C(f)$ and $\Sigma^\infty C(g)$ are clearly homotopy equivalent. If there exists a homotopy equivalence β: $\Sigma^\infty C(f) \to \Sigma^\infty C(g)$, we set $\beta^*(V) = aV + b\Sigma U$ and $\beta^*(\Sigma U) = cV + d\Sigma^k U$, $ad - bc = \pm 1$ where V is the generator of $H^{k+3}(\Sigma^\infty S^{k+3} : Z)$ and $\Sigma^k U$ is the generator of $H^{k+3}(\Sigma^\infty \Sigma^k CP^\infty ; Z)$. By using the reduced power operation, we have $b = 0$ and $a = \pm 1$, $d = \pm 1$. Moreover, β induces a map α: $\Sigma^\infty S^{k+3} \to \Sigma^\infty S^{k+3}$ of degree ± 1 and $c = 0$ by the following consideration. Now consider the following sequences.

$$\begin{array}{ccccccccc}
\Sigma^\infty\Sigma^k CP^\infty & \xrightarrow{\Sigma^\infty f} & \Sigma^\infty S^{k+3} & \xrightarrow{\Sigma^\infty i} & \Sigma^\infty C(f) & \xrightarrow{\Sigma^\infty p} & \Sigma^\infty\Sigma^{k+1} CP^\infty & \xrightarrow{-\Sigma^\infty f} & \Sigma^\infty S^{k+4} \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} \\
\Sigma^\infty\Sigma^k CP^\infty & \xrightarrow[\Sigma^\infty g]{} & \Sigma^\infty S^{k+3} & \xrightarrow[\Sigma^\infty j]{} & \Sigma^\infty C(g) & \xrightarrow[\Sigma^\infty q]{} & \Sigma^\infty\Sigma^{k+1} CP^\infty & \xrightarrow[-\Sigma^\infty g]{} & \Sigma^\infty S^{k+4}
\end{array}$$

The existence of α is proved as follows. When $f \sim g \sim 0$, the statement is clear. We may assume $f \not\sim 0$. $\Sigma^\infty q\beta\Sigma^\infty i\Sigma^\infty f$ is homotopic to the constant map by $\Sigma^\infty(if) \sim 0$. Since $\Sigma^\infty q\beta\Sigma^\infty i\Sigma^\infty f$ is homotopic to $s\Sigma^\infty f$: $\Sigma^\infty\Sigma^k CP^\infty \to \Sigma^\infty S^{k+3} \to \Sigma^\infty\Sigma^{k+1} CP^\infty$ where s: $\Sigma^\infty S^{k+3} \to \Sigma^\infty\Sigma^{k+1} CP^\infty$ is a map of degree s. A map s is homotopic to the constant map by Theorem 1.5. Hence β induces the above commutative diagram where α, β, γ and δ are maps of homotopy equivalences. We have that $\Sigma^\infty\Sigma f$ and $\Sigma^\infty\Sigma g$ are equivalent under the action of homotopy equivalences. Hence we have $\Sigma^\infty\Sigma f \sim \pm\Sigma^\infty\Sigma g$ and also $f \sim \pm g$.

By Theorem 2.1 and Theorem A.1 of Appendix, we get the next result.

Theorem 2.2. Let $f, g : \Sigma^k CP^\infty \to S^{k+3}$ *be continuous maps and $C(f)$, $C(g)$ mapping cones of f and g respectively. The infinite loop spaces $Q(C(f))$ and $Q(C(g))$ are homotopy equivalent if and only if f and $\pm g$ are homotopic. These spaces are finite type and have the same n-type for all n.*

These results are easily generalized for non-trivial connected Lie groups by considering the results of [7].

Theorem 2.3. *Let* $f, g : \Sigma^k BU(n) \to S^{k+3}$ *(or* $\Sigma^k\ BSp(n) \to S^{k+5}$*) be continuous maps and* $C(f)$, $C(g)$ *mapping cones of* f *and* g *respectively. The infinite loop spaces* $Q(C(f))$ *and* $Q(C(g))$ *are homotopy equivalent if and only if* f *and* $\pm g$ *are homotopic. These spaces are finite type and have the same n-type for all* n.

By studying the proof of Theorem 2.1 carefully, we get the next corollaries.

Corollary 2.4. *Let* $f, g : \Sigma^k CP^\infty \to S^{k+3}$ *be continuous maps which are not homotopic to the constant maps. If there exists a map* $\phi : \Sigma^\infty C(f) \to \Sigma^\infty C(g)$, *with* $H_{k+3}(\phi) \neq 0$, *it holds* $nf \sim mg$ *for some non-zero integers* m, n.

Corollary 2.5. *Let* $f, g : \Sigma^k CP^\infty \to S^{k+3}$ *be continuous maps which are not homotopic to the constant maps. If there exists a map* $\phi : Q(C(f)) \to Q(C(g))$ *with* $H_{k+3}(\phi) \neq 0$, *it holds* $nf \sim mg$ *for some non-zero integers* m, n.

Proof. Consider the following map,

$$\varepsilon\Sigma^\infty\phi\Sigma^\infty\eta : \Sigma^\infty C(f) \to \Sigma^\infty Q(C(f)) \to \Sigma^\infty Q(C(g)) \to \Sigma^\infty C(g)$$

where ε and η induce isomorphisms of homology groups at $k+3$ dimension. See Appendix for ε, η. Hence we get the result by Corollary 2.4.

Theorem 2.6. *Let* $f, g : \Sigma^k CP^\infty \to S^{k+3}$ *be continuous maps.* $Q^q(X)$ *is the q-time iteration of the functor* Q *on* X. *If* $Q^q(C(f))$ *and* $Q^q(C(g))$ *are homotopy equivalent, if and only if* f *and* $\pm g$ *are homotopic. These spaces are finite type and have the same n-type for all* n.

Proof. It is sufficient to prove for $q = 2$. For large q, it is inductively proved. When $f \sim g \sim 0$, it is clear. We may assume $f \not\sim 0$. Let $\phi : Q^2(C(f)) \to Q^2(C(g))$ be a homotopy equivalence. Now consider a map,

$$\Phi = \varepsilon\phi\eta : Q(C(f)) \to Q^2(C(f)) \to Q^2(C(g)) \to Q(C(g))$$

μ and η induce isomorphisms of homology groups at $k+3$ dimension. Hence we have the isomorphism $H_{k+3}(\Phi)$. Hence the result follows by the methods of Theorem 2.1 and Corollary 2.5.

Appendix. We prove Theorem A.1 by the pure categorical theorem A.2 and the natural equivalence (1.4).

Theorem A.1. *If suspension spectra $\Sigma^\infty X$ and $\Sigma^\infty X$ of CW-complexes X and Y are homotopy equivalent in* **Spec**, *$Q(X)$ and $Q(Y)$ are homotopy equivalent.*

Theorem A.2. *Let $F: \boldsymbol{C} \to \boldsymbol{D}$, $G: \boldsymbol{C} \to \boldsymbol{C}$ be functors which satisfy the natural equivalence $\boldsymbol{C}(X, G(Y)) = \boldsymbol{D}(F(X), F(Y))$. Then $F(X) = F(Y)$ implies $G(X) = G(Y)$.*

Proof. By $\boldsymbol{C}(G(X), G(X)) = \boldsymbol{D}(FG(X), F(X))$, we have a natural transformation $\varepsilon: FG \to F$ in $\boldsymbol{D}$ which corresponds $Id: G \to G$. By $\boldsymbol{C}(X, G(X)) = \boldsymbol{D}(F(X), F(X))$, we have a natural trasformation $\eta: Id \to G$ in $\boldsymbol{C}$ which corresponds $Id: F \to F$. By $\boldsymbol{C}(G^2(X), G(X)) = \boldsymbol{D}(FG^2(X), F(X))$, we have a natural trasformation $\mu: G^2 \to G$ in $\boldsymbol{C}$ which corresponds $\varepsilon\varepsilon: FG^2 \to F$. These transformations satisfy the next relations by the definitions:

$$\text{(A.3)} \qquad \begin{aligned} \varepsilon\varepsilon &= \varepsilon F(\mu), \quad \varepsilon F(\eta) = Id \\ \mu\eta &= Id = \mu G(\eta) \\ \mu\mu &= \mu G(\mu) \end{aligned}$$

The above natural equivalence $\Phi: \boldsymbol{C}(X, G(Y)) \to \boldsymbol{D}(F(X), F(Y)$ is described by $\Phi(f) = \varepsilon F(f)$. We define

$$\begin{aligned} \hat{\Omega}&: \boldsymbol{C}(X, G(Y)) \quad\ \ \to \boldsymbol{C}(G(X), G(Y)) \\ \Omega&: \boldsymbol{D}(F(X), F(Y)) \to \boldsymbol{C}(G(X), G(Y)) \end{aligned}$$

by $\hat{\Omega}(f) = \mu G(f)$ and $\Omega = \hat{\Omega}\Phi^{-1}$ respectively. By $\eta^*\hat{\Omega}(f) = \mu G(f)\eta = \mu\eta(f) = f$, $\hat{\Omega}$ and Ω are monomorphisms. The theorem is easily proved by the next two lemmas.

Lemma A.4. *The adjoint of $gf: F(X) \to F(Y) \to F(Z)$ is $\Omega(g)f': X \to G(Y) \to G(Z)$ where f' is the adjoint of f.*

Proof. It is sufficient to prove $\Phi(\Omega(g)f') = \Phi(g')\Phi(f')$ where g' is the adjoint of g. We get the result by the following calculation;

$$\begin{aligned} \Phi(\Omega(g)f') &= \varepsilon F(\Omega(g)f') = \varepsilon F(\mu G(g'))F(f') = \varepsilon F(\mu)F(G(g'))F(f') \\ &= \varepsilon\varepsilon F(G(g'))F(f') = \varepsilon F(g')\varepsilon F(f') = \Phi(g')\Phi(f'). \end{aligned}$$

Lemma A.5. *Ω satisfies $\Omega(gf) = \Omega(g)\Omega(f)$, $\Omega(Id) = Id$ where $f: F(X) \to F(Y)$, $g: F(Y) \to F(Z)$.*

Proof. Since the adjoint of gf is $\Omega(g')f$, we get the result by the following calculation;

$$\Omega(gf) = \mu G(\Omega(g)f') = \mu G(\Omega(g))G(f') = \mu G(\mu G(g'))G(f')$$
$$= \mu G(\mu)G^2(g')G(f') = \mu\mu G^2(g')G(f') = \mu G(g')\mu G(f') = \Omega(g)\Omega(f)$$

$$\Omega(Id) = \mu G(\eta) = Id.$$

REFERENCES

[1] J. F. ADAMS: An example in homtopy theory, Proc. Camb. Phil. Soc. **53** (1957), 922-923.

[2] J. F. ADAMS: Infinite loop spaces, Annals of Math. Studies, No. **90** (1978).

[3] B. GRAY: Spaces of the same n-type for all n, Topology **5** (1966), 241-243.

[4] C. A. MCGIBBON and J. A. MØLLER: On spaces of the same n-types for all n, Topology **31** (1992), 177—201.

[5] C. A. MCGIBBON and J. A. MØLLER: On infinite dimensional spaces that are rationally equivalent to a bouquet of spheres, Proc. 1990 Barcerona Conference on Algebraic Topology, Springer LNM.

[6] Y. SHITANDA: Uncountably many loop spaces of the same n-type for all n, I, Yokohama Math. J. **41** (1993), 17—24.

[7] Y. SHITANDA: Uncountably many loop spaces of the same n-type for all n, **II**, to appear in Yokohama Math. J.

[8] A. ZABRODSKY: On phantom maps and a theorem of H. Miller, Israel J. Math. **58** (1987), 129-143.

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