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# DEGREES OF SELF-MAPS

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**0. Introduction.** As well-known, one of the basic problems in algebraic topology is to determine the homology representation

$$H : [X, Y] \to \mathrm{Hom}(H_*(X), H_*(Y))$$

where $[X, Y]$ denotes the set of homotopy classes of maps from $X$ to $Y$.

In this paper we shall consider it for the case $X = Y$. Previously, Sullivan and Quillen proved the famous theorem for $X = \mathrm{HP}^\infty$ in [7] and C. A. Mcgibbon determined the image of $H$ for $X = \mathrm{RP}^n$, $\mathrm{CP}^n$ and $\mathrm{HP}^n$ in the stable case in [3]. Furthermore D. M. Davis investigated it for $X = \mathrm{CP}^{n+2}/\mathrm{CP}^{n-1}$, the stunted projective space in [1] and also S. Sasao and M. Nagaishi determined it for $X = \mathrm{HP}^3$ in [6]. In this paper we shall consider the case $X = S^n \cup e^{n+2} \cup e^{n+4}$, which is a generalization of Davis's case and contains the following:

(1) The total spaces of $S^2$-bundles over $S^4$.

(2) the Thom complexes of real $n$-vector bundles over a 2-cell complex $S^2 \cup e^4$, which contain the stunted complex projective spaces $\mathrm{CP}^{n+2}/\mathrm{CP}^{n-1}$.

(3) the iterated suspension of (1) or (2).

Let $X$ be a 3-cell complex of the form $S^n \cup e^{n+2} \cup e^{n+4}$, and $e_j$ be the corresponding generator of $H_j(X) \simeq \boldsymbol{Z}$ for $j = n, n+2, n+4$. Then for each self-map $f \in [X, X]$, the endomorphism

$$H(f) = f_* : H_*(X) \to H_*(X)$$

is uniquely determined by a triple of degrees $(d_1, d_2, d_3)$ which is defined by

$$f_*(e^{(n-2)+2j}) = d_j e^j$$

for $j = 1, 2, 3$. We call $f$ a self-map of $X$ of degrees $(d_1, d_2, d_3)$.

Hence our problem is reduced to characterize a triple of integers $(d_1, d_2, d_3)$ which is a triple of degrees of a self-map of $X$. This note is organized as follows: In §1, we shall consider the case $n = 2$, and investigate the case $n = 3, 4$ in §2. In §3, we shall treat the case $n > 4$ which is belonging to the stable range, and some examples shall be given in §4.

**Remark.** *In a subsequent paper, we shall consider the kernel of H.*

Here we state a part of our results.

**Theorem A.** *For $X = S^2 \cup e^4 \cup e^6$, there exists a self-map of $X$ of degrees $(d_1, d_2, d_3)$ if and only if the followings hold:*

(0) *If $e^2 \cdot e^2 \neq 0$, $Sq^2(e^4) = 0$, then $d_2 = d_1^2$ and $d_3 = d_1^3 \bmod h_6(X)$.*

(1) *If $e^2 \cdot e^2 \neq 0$, $Sq^2(e^4) \neq 0$, $e^2 \cdot e^4 \neq 0$, then $d_2 = d_1^2$ and $d_3 = d_1^3$.*

(2) *If $e^2 \cdot e^2 \neq 0$, $Sq^2(e^4) \neq 0$, $e^2 \cdot e^4 = 0$, then $d_2 = d_1^2$ and $d_3 \equiv d_1 \bmod 2$.*

(3) *If $e^2 \cdot e^2 = 0$, $e^2 \cdot e^4 \neq 0$, $Sq^2(e^4) = 0$, then $d_3 = d_1 d_2$ and $d_3 \equiv d_1 \bmod h_7(X)$.*

(4) *If $e^2 \cdot e^2 = 0$, $e^2 \cdot e^4 \neq 0$, $Sq^2(e^4) \neq 0$, then $d_3 = d_1 d_2$ and $d_3 \equiv d_2 \bmod 2$.*

(5) *If $e^2 \cdot e^2 = e^2 \cdot e^4 = 0$, $Sq^2(e^4) = 0$, then $d_3 \equiv d_1 \bmod h_6(X)$.*

(6) *If $e^2 \cdot e^2 = e^2 \cdot e^4 = 0$, $Sq^2(e^4) \neq 0$, then $d_3 \equiv d_2 \bmod 2$.*

*Here $\cdot$ denotes the cup product in the cohomology ring and $Sq$ is the Steenrod squaring operation, and $h_n(X)$ denotes the image of the Hurewicz homomorphism at dimension $n$.*

**Corollary E.** *For $X = S^n \cup e^{n+2} \cup e^{n+4}$ $(5 \leqq n)$, a triple of integers $(d_1, d_2, d_3)$ is realizable by a self-map of $X$ if and only if the followings hold:*

(1) *$d_3 \equiv d_2 \bmod 2$ and $2d_3 \equiv 2d_1 \bmod h_{n+4}(X)$ if $Sq^2(e^{n+2}) \neq 0$.*

(2) *$d_3 \equiv d_1 \bmod h_{n+4}(X)$ if $Sq^2(e^{n+2}) = 0$ and $Sq^2(e^n) = 0$.*

(3) *$d_2 \equiv d_1 \bmod 2$ and $d_3 \equiv d_1 \bmod h_{n+4}(X)$ if $Sq^2(e^{n+2}) = 0$ and $Sq^2(e^n) \neq 0$.*

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**1. The case $n = 2$.** Let $X$ be a 3-cell complex of the form $S^2 \cup e^4 \cup e^6$. Let $a$ and $b$ be integers and $\varepsilon \in \{0, 1\}$. Then we call $X$ to be of type $(a, b, \varepsilon)$ if and only if

$$e^2 \cdot e^2 = ae^4, \quad e^2 \cdot e^4 = be^6, \quad \text{and} \quad Sq^2(e^4) = \varepsilon e^6$$

where $e^j$ denotes the generator of $H^j(X, \boldsymbol{Z})$ (or $H^j(X, \boldsymbol{Z}/2)$). From now on in this section, we assume that $X$ is of type $(a, b, \varepsilon)$.

First we consider the sub-case

**1.1** $a = 0$ (i.e. $e^2 \cdot e^2 = 0$) By the assumption we may consider that $X$ has a form $X = (S^2 \vee S^4) \cup e^6$. Then, the attaching class $\beta$ for the cell $e^6$ is the following

$$\beta = x\iota_2(\eta_2\eta_3\eta_4) + \iota_4(\eta_4) + b[\iota_2, \iota_4]$$

where $\eta_n$ denotes the Hopf class of $\pi_{n+1}(S^n)$.

Now define two maps $S^2 \vee S^4 \to S^2 \vee S^4$ as follows:

$$\psi_{k,\ell}|S^2 = k\iota_2 \quad \text{and} \quad \psi_{k,\ell}|S^4 = \ell\iota_4,$$
$$\phi_{k,\ell}|S^2 = k\iota_2 \quad \text{and} \quad \phi_{k,\ell}|S^4 = \ell\iota_4 + \iota_2\eta_2\eta_3.$$

**Lemma 1.** *When $b \neq 0$, there exists a self-map of $X$ of degrees $(k, \ell, m)$ if and only if $m = k\ell$ and $m \equiv \ell \bmod 2$.*

*Proof.* By the standard argument we can easily know that there is a self-map of $X$ of degrees $(k, \ell, m)$ if and only if $\psi_{k,\ell *}(\beta) = m\beta$ or $\phi_{k,\ell *}(\beta) = m\beta$. Since two endomorphisms

$$\psi_{k,\ell *} \text{ and } \phi_{k,\ell *} : \pi_5(S^2 \vee S^4) \to \pi_5(S^2 \vee S^4)$$

are clearly given by

$$\phi_{k,\ell *}(\beta) = kx\iota_2\eta_2\eta_3\eta_4 + \iota_2\eta_2\eta_3\eta_4 + \ell\iota_4\eta_4 + k\ell b[\iota_2, \iota_4],$$
$$\psi_{k,\ell *}(\beta) = kx\iota_2\eta_2\eta_3\eta_4 + \ell\iota_4\eta_4 + k\ell b[\iota_2, \iota_4] \qquad (1\text{-}1),$$

the condition is equivalent to $mb = k\ell b$, $\ell \equiv m \bmod 2$, and $kx\eta_2\eta_3\eta_4 \equiv mx\eta_2\eta_3\eta_4 \bmod b\eta_2\eta_3\eta_4$. Then the assumption $b \neq 0$ completes the proof.

Now assume that $b = 0$ and $c \neq 0$. Then, from (1-1) we can obtain the following:

**Lemma 2.** *When $\varepsilon = 0$ and $b \neq 0$, there is a self-map of $X$ of degrees $(k, \ell, m)$ if and only if $m = k\ell$ and $m \equiv k \bmod h_7(\Sigma X)$, where $\Sigma$ denotes the suspension functor.*

Analogously we have

**Lemma 3.** *When $b = 0$, there exists a self-map of $X$ of degrees $(k, \ell, m)$ if and only if*

(1) $m \equiv k \bmod h_6(X)$ *if* $\varepsilon = 0$.

(2) $m \equiv \ell \bmod 2$ *if* $\varepsilon \neq 0$.

**1.2** $a \neq 0$ (i.e. $e^2 \cdot e^2 \neq 0$). Let $A_a$ be the 2 cell-complex $S^2 \cup e^4$ which has a $a\eta_2$ as the attaching class for the cell $e^4$. First we quote the following ((2.13) of [8]):

**Lemma 4.**

$$\pi_5(A_a)=\begin{cases}\boldsymbol{Z} & a\equiv 1 \pmod 2\\ \boldsymbol{Z}+\boldsymbol{Z}/4 & a\equiv 2 \pmod 4\\ \boldsymbol{Z}+\boldsymbol{Z}/2+\boldsymbol{Z}/2 & a\equiv 0 \pmod 4\end{cases}$$

Let $\psi_k: A_a\to A_a$ be a map of degrees $(k, k^2)$, then other self-maps of $A_a$ of degrees $(k, k^2)$ is only one, which is given by the composite

$$\psi'_k:\ A_a \underset{c}{\to} A_a\vee S^4\to A_a=\psi_k\vee\iota_2\eta_2\eta_3\cdot c$$

Let $\beta$ be the attaching class for the cell $e^6$ of $X$ and $\beta'$ be the image of $\beta$ by the pinching map $A_a\to S^4=A_a/S^2$. Clearly $\beta'$ is 0 or $\eta_4$, which is determined by $Sq^2(e^4)$. Using the fact $[\iota_2, \eta_2\eta_3]=0$, we can easily obtain the following:

$$\psi_{k*}(\beta)=k^3\beta+\iota_2\gamma \quad\text{and}\quad \psi'_{k*}(\beta)=\psi_{k*}(\beta)+\iota_2\eta_2\eta_4\beta' \tag{1-2}$$

where $\gamma$ is an element of $\pi_5(S^2)$, i.e. 0 or $\eta_2\eta_3\eta_4$.

**Lemma 5.** *If $a\equiv 1$ mod 2, then we have $\psi_{k*}(\beta)=\psi'_{k*}(\beta)=k^3\beta$.*

*Proof.* Lemma 4 implies that $\gamma$ and $\beta'$ in the formula (1-2) is always 0. Hence the proof is complete.

Next we investigate the case $a\equiv 0$ mod 2 $(a\neq 0)$. First we prove

**Lemma 6.** *For $a=2$, we can choose $\psi_k$ satisfying $\psi_{k*}(\beta)=k^3\beta$.*

*Proof.* We may regard $A_2$ as the 4-skelton of the reduced product $S^2_\infty$ of $S^2$ in [2]. The map $k\iota_2: S^2\to S^2$ induces the map $S^2_\infty\to S^2_\infty$ whose restriction on its 4-skelton is the desired map $\psi_k$, and $\psi_{k*}(\alpha)=k^3\alpha$ holds for $\alpha$, the attaching class for the 6-cell of $S^2_\infty$. On the other hand, from lemma 4 and the homotopy exact sequence of the pair $(A_a, S^2)$ we can know that there is a class $\alpha_1$ of $\pi_5(A_2)$ satisfying $j_*(\alpha_1)=[\chi_4, \iota_2]_r$, where $[\ ,\ ]_r$ denotes the relative Whitehead product and $\chi_4$ is the characteristic map for the cell $e^4$ of $S^2_\infty$. Since we have $e^2\cdot e^4=3e^6$ in $H(S^2_\infty)$ (see [2]), the following relations hold

$$3\alpha_1=\alpha \quad\text{or}\quad 3\alpha_1=\alpha+\eta_2\eta_3\eta_4.$$

Then we have

$$3\psi_{k*}(\alpha_1)=\psi_{k*}(\alpha)=k^3\alpha=3k^3\alpha_1 \quad\text{or}$$
$$3\psi_{k*}(\alpha_1)=\psi_{k*}(\alpha)+k\eta_2\eta_3\eta_4=k^3\alpha+k\eta_2\eta_3\eta_4=k^3(\alpha+\eta_2\eta_3\eta_4)=3k^3\alpha_1 \quad\text{i.e.}$$

$\psi_{k*}(\alpha_1) = k^3\alpha_1.$

Hence, for $\beta(\in \pi_5(A_2))$ with $\beta' = 0$, it holds $\psi_{k*}(\beta) = k^3\beta$. Moreover, if $\beta' = 0$, we obtain from the formula (1-2) that $\psi_{k*}(\beta) = k^3\beta$ or $\psi'_{k*}(\beta) = k^3\beta$. Thus the proof is complete.

Secondly, we prove the general case.

**Lemma 7.** *If* $a \equiv 0 \bmod 2$, *then there is a map* $\phi_k : A_a \to A_a$ *satisfying* $\phi_{k*}(\beta) = k^3\beta$ *for any* $\beta$ *of* $\pi_5(A_a)$.

*Proof.* For $\beta$ with $\beta' \neq 0$, the proof follows from the formula (1-2). If $\beta' = 0$ we put $a = 2a'$. Since, for any map $\phi(1, a') : A_a \to A_2$ of degrees $(1, a')$, there exists a map $\phi_k : A_a \to A_2$ which makes the following diagram commutative:

$$\begin{array}{ccc} A_a = A_{2a} & \xrightarrow[\phi(1,a')]{} & A_2 \\ \downarrow \phi_k & & \downarrow \psi_k \\ A_a & \xrightarrow[\phi(1,a')]{} & A_2 \end{array}$$

Then, the proof follows from the formula (1-1), lemma 6, and the restriction $\phi(1, a)|S^2 =$ the identity.

**Proposition 1.** *For* $a \neq 0$, *a triple* $(d_1, d_2, d_3)$ *is realizable by a self-map of* $X$ *if and only if the followings hold*:

(0) $d_2 = k^2$.

(1) *If* $\varepsilon = 0$, *then* $d_3 \equiv d_1^3 \bmod h_6(X)$.

(2) *If* $\varepsilon \neq 0$ *and* $b \neq 0$, *then* $d_3 = d_1^3$.

(3) *If* $\varepsilon \neq 0$ *and* $b = 0$, *then* $d_3 \equiv d_1^3 \bmod 2$.

*Proof.* (0) is easy by the assumption $a \neq 0$. Using lemma 5 and 7, we may take $\gamma = 0$ in the formula (1-1). Then our desired condition is

$$m\beta = k^3\beta \quad \text{or} \quad m\beta = k^3\beta + \iota_2\eta_2\eta_3\beta'.$$

Thus the proof of (1), (2), and (3) follows from using that $\beta'$ is determined by $Sq^2(e^4)$.

Now we prove Theorem A. Namely, (3) follows from lemma 2. (4) follows from lemma 1. (5) and (6) follow from lemma 3. The others follow from lemma 5 and proposition 1.

**2. The case $n = 3, 4$.** The case can be divided into two ones by the formula

$Sq^2Sq^2 = Sq^3Sq^1$:

(1) $Sq^2(e^{n+2}) \neq 0$, then $X = (S^n \vee S^{n+2}) \cup e^{n+4}$ because of $Sq^2(e^n) = 0$.

(2) $Sq^2(e^{n+2}) = 0$, then $X = e^{n+4} \cup S^n \cup e^{n+2}$.

Let $\beta$ be the attaching class for the $(n+4)$-cell of $X$. First we consider the case (1). Define two maps $\psi_{k,\ell}$ and $\psi'_{k,\ell}$ as follows:

$$\begin{aligned}&\psi_{k,\ell},\ \psi'_{k,\ell}:\ S^n \vee S^{n+2} \to S^n \vee S^{n+2},\\ &\psi_{k,\ell}|S^n = k\iota_n \quad \text{and} \quad \psi_{k,\ell}|S^{n+2} = \ell\iota_{n+2},\\ &\psi'_{k,\ell}|S^n = k\iota_n \quad \text{and} \quad \psi'_{k,\ell}|S^{n+2} = \ell\iota_{n+2} + \iota_n\eta_n\eta_{n+1}.\end{aligned}$$

Then we have

$$\psi_{k,\ell *}(\beta) = (k\iota_n)_*(\beta_1) + \iota_{n+2}(\ell\eta_{n+2}),$$

and

$$\psi'_{k,\ell *}(\beta) = (k\iota_n)_*(\beta_1) + \iota_n\eta_n\eta_{n+1}\eta_{n+2} + \iota_{n+2}(\ell\eta_{n+2}),$$

where $\beta = \beta_1 + \iota_{n+2}\eta_{n+2}$ ($\in \pi_{n+3}(S^n \vee S^{n+2}) = \pi_{n+3}(S^n) + \pi_{n+3}(S^{n+2})$). Hence, a triple $(k, \ell, m)$ is realizable by a self-map of $X$ if and only if the following equality holds:

$$m\beta_1 + m\iota_{n+2}\eta_{n+2} = m\beta = (k\iota_n)_*\beta_1 + \ell\iota_{n+2}\eta_{n+2} \mod \iota_n\eta_n\eta_{n+1}\eta_{n+2}. \quad (2\text{-}1)$$

**Lemma 8.** *For* $n = 3$ ($Sq^2(e^5) \neq 0$), *there exists a self-map of* $X$ *of degrees* $(k, \ell, m)$ *if and only if* $\ell \equiv m \bmod 2$ *and* $2m \equiv 2k \bmod h_7(X)$.

*Proof.* Since we can replace $(k\iota_n)_*(\beta_1)$ with $k\beta_1$ in the formula (2-1) we have that $\ell \equiv m \bmod 2$ and $(m-k)\beta_1 \equiv 0 \bmod \eta_3\eta_4\eta_5$. On the other hand, $\eta_3\eta_4\eta_5$ is the only one element of order 2 in $\pi_6(S^3)$. Therefore the latter condition is equivalent to $2(m-k)\beta_1 = 0$. Thus the proof is completed by $2\beta_1 = 2\beta$.

**Lemma 9.** *For* $n = 4$ ($Sq^2(e^6) \neq 0$), *our conditions are as follows*:

(1) *If* $e^4 \cdot e^4 = 0$, *then* $\ell \equiv m \bmod 2$ *and* $2(m-k) \equiv 0 \bmod h_8(X)$.

(2) *If* $e^4 \cdot e^4 \neq 0$, *then* $m = k^2$, $\ell \equiv m \bmod 2$, *and* $m \equiv k \bmod h_9(\Sigma X)$.

*Proof.* Since $\beta_1$ has a representation

$$\beta_1 = x\nu + y\Sigma\omega \quad \text{for some integers } x \text{ and } y,$$

where $\nu$ denotes the Hopf map $S^7 \to S^4$ and $\omega$ is the Blaker-Massey map, we get

$$\begin{aligned}(k\iota_4)_*(\beta_1) &= k\beta_1 + k(k-1)/2[\iota_4, \iota_4]H(\beta_1)\\ &= k^2x\nu + \{ky + k(k-1)x/2\}\Sigma\omega.\end{aligned}$$

Hence, the formula (2-1) is equivalent to

$$mx\nu + my\Sigma\omega = k^2x\nu + [\{2ky + k(k-1)x\}/2]\Sigma\omega \quad \mathrm{mod}\ \eta_4\eta_5\eta_6.$$

Moreover this gives that

if $x \neq 0$ (i.e. $e^4 \cdot e^4 \neq 0$), then $m = k^2$ and $\{2my - 2ky - k(k-1)x\}/2\Sigma\omega \equiv 0$ mod $\eta_4\eta_5\eta_6$, and that
if $x = 0$ (i.e. $e^4 \cdot e^4 = 0$), then $(m-k)y\Sigma\omega \equiv 0$ mod $\eta_4\eta_5\eta_6$.

Thus (1) is obtained from the same argument as lemma 8. Next, we consider the case (2). From $m = k^2$ we have that

$$2my - 2ky - k(k-1)x = (m-k)2y - k(k-1)x = k(k-1)(2y-x).$$

On the other hand, we know that $\Sigma\beta_1 = (x-2y)\nu$. Thus the proof of (2) follows from $k(k-1)\Sigma\beta_1 = k(k-1)\Sigma\beta$.

Secondly, we prove the case $Sq^2(e^{n+2}) = 0$. Let $A$ be the subcomplex, $S^n \cup e^{n+2}$, of $X$ and let $\phi_k(\gamma)$ be the map defined by

$$\phi_k(\gamma) = (k1_A \vee \gamma)C_A : A \to A \vee S^{n+2} \to A$$

for $\gamma \in \pi_{n+2}(A)$ where $k1_A$ denotes the $k$ time of the identity of $A$ in the sense of the suspension-addition and $C_A$ is the co-action map of $A$. Here we note that
(0) $\beta = i_*(\beta')$, where $i$ is the inclusion $S^n \to A$.
(1) $\phi_k(\gamma)$ is of degrees $(k, k + h_{n+2}(X))$.
(2) $\phi_k(\gamma)_*(\beta) = i_*((k\iota_n)_*(\beta'))$.

Now consider the following diagram:

$$\begin{array}{ccccc} & & \pi_{n+3}(S^n) & \xrightarrow{i_*} & \pi_{n+3}(A) \\ & & \downarrow (k\iota_n)_* & & \downarrow (k1_A)_* \\ \pi_{n+4}(A, S^n) & \xrightarrow{\partial} & \pi_{n+3}(S^n) & \xrightarrow{i_*} & \pi_{n+3}(A) \end{array}$$

Then, it is easy from $\phi_k(\gamma)_*(\beta) = (k1_A)_*(\beta)$ to obtain the following:

**Lemma 10.** *There exists a self-map of $X$ of degrees $(k, \ell, m)$ if and only if $\ell \equiv k$ mod $h_{n+2}(X)$ and $m(\beta') \equiv (k\iota_n)_*(\beta')$ mod $\partial$-image.*

**Remark.** $h_{n+2}(X) = \boldsymbol{Z}$ *if* $Sq^2(e^n) = 0$ *and* $h_{n+2}(X) = 2\boldsymbol{Z}$ *if* $Sq^2(e^n) \neq 0$.

Since $(k\iota_3)_*(\beta') = k\beta'$ (for $n = 3$), we have

**Lemma 11.** *For $n = 3$, a triple $(k, \ell, m)$ is realizable by a self-map of $X$*

*if and only if* $\ell \equiv k \bmod h_5(X)$ *and* $m \equiv k \bmod h_7(X)$.

If $n = 4$ we can take $x\nu + y\Sigma\omega$ as $\beta'$ and use the formula,

$$(k\iota_4)_*(\beta') = k(\beta') + \{k(k-1)/2\}x[\iota_4, \iota_4]. \tag{2-2}$$

**Lemma 12.** *If* $e^4 \cdot e^4 = 0$, *then there is a self-map of* $X$ *of degrees* $(k, \ell, m)$ *if and only if* $\ell \equiv k \bmod h_6(X)$ *and* $m \equiv k \bmod h_8(X)$.

*Proof.* Since $e^4 \cdot e^4 = 0$ is equivalent to $x = 0$, we have

$$(k\iota_4)_*(\beta') = k(\beta'), \quad \text{i.e.} \quad (k1_A)_*(\beta) = k\beta$$

from (2-2). Hence $(m-k)(\beta) = 0$, which completes the proof.

**Lemma 13.** *If* $x \neq 0$, *then there is a self-map of* $X$ *of degrees* $(k, \ell, m)$ *if and only if* $\ell \equiv k \bmod h_6(X)$, $m = k^2$, *and* $m \equiv k \bmod h_9(\Sigma X)$.

*Proof.* First, we have $m = k^2$ from (2-2) and $[\iota_4, \iota_4] = 2\nu + \Sigma\omega$. Then, it holds $i_*\{(m-k)(x-2y)/2\}\Sigma\omega = 0$, which gives

$$i_*\{(m-k)(x-2y)\nu\} = i_*\{(m-k)\Sigma\beta'\} = (m-k)\Sigma\beta = 0$$

from applying the suspension functor. Thus the proof is complete.

Now from lemmas 8, 9, 11, 12, and 13 we have

**Theorem B.** *Let* $X$ *be a complex of the form* $S^3 \cup e^5 \cup e^7$. *Then a triple* $(d_1, d_2, d_3)$ *is realizable by a self-map of* $X$ *if and only if*

(1) *If* $Sq^2(e^5) \neq 0$, *then* $d_3 \equiv d_2 \bmod 2$ *and* $2d_3 \equiv 2d_2 \bmod h_7(X)$.

(2) *If* $Sq^2(e^5) = 0$ *and* $Sq^2(e^3) = 0$, *then* $d_3 \equiv d_1 \bmod h_7(X)$.

(3) *If* $Sq^2(e^5) = 0$ *and* $Sq^2(e^3) \neq 0$, *then* $d_3 \equiv d_1 \bmod h_7(X)$ *and* $d_2 \equiv d_1 \bmod 2$.

**Theorem C.** *Let* $X$ *be a complex of the form* $S^4 \cup e^6 \cup e^8$. *Then a triple* $(d_1, d_2, d_3)$ *is realizable by a self-map of* $X$ *if and only if*

(1) *If* $Sq^2(e^6) \neq 0$ *and* $e^4 \cdot e^4 = 0$, *then* $d_3 \equiv d_2 \bmod 2$ *and* $2d_3 \equiv 2d_1 \bmod h_8(X)$.

(2) *If* $Sq^2(e^6) \neq 0$ *and* $e^4 \cdot e^4 \neq 0$, *then* $d_3 \equiv d_2 \bmod 2$, $d_3 \equiv d_1^2$, *and* $d_3 \equiv d_2 \bmod h_9(\Sigma X)$.

(3) *If* $Sq^2(e^6) = 0$ *and* $e^4 \cdot e^4 = 0$, *then* $d_2 \equiv d_1 \bmod h_6(X)$ *and* $d_3 \equiv d_1 \bmod h_8(X)$.

(4) *If* $Sq^2(e^6) = 0$ *and* $e^4 \cdot e^4 \neq 0$, *then* $d_2 \equiv d_1 \bmod h_6(X)$, $d_3 = d_1^2$, *and* $d_3 \equiv d_1 \bmod h_9(\Sigma X)$.

**3. The case $X = \Sigma X'$.** First we note that this case contains the case $5 \leqq n$. Next, let $Y$ be the complex $X/S^n$ and let us consider maps from $Y$ to $X$.

**Lemma 14.** *If* $Sq^2(e^{n+2}) = 0$ *in* $\mathrm{H}^*(X\,;\, \boldsymbol{Z}/2)$, *then there exists a map from* $Y$ *to* $X$ *with degrees* $(\ell, m)$ *if and only if* $\ell \equiv 0 \bmod h_{n+2}(X)$ *and* $m \equiv 0 \bmod h_{n+4}(X)$.

*Proof.* Since the assumption implies $Y = S^{n+2} \vee S^{n+4}$, the proof is clear.

**Lemma 15.** *If* $Sq^2(e^{n+2}) \neq 0$, *then there is a map from* $Y$ *to* $X$ *with degrees* $(\ell, m)$ *if and only if* $m \equiv \ell \bmod 2$ *and* $2m \equiv 0 \bmod h_{n+4}(X)$.

*Proof.* Since $Sq^2(e^{n+2}) \neq 0$ implies $Sq^2(e^n) = 0$, $X$ has a decomposition

$$X = (S^n \vee S^{n+2}) \cup e^{n+4}.$$

Let $f: S^{n+2} \to X$ be the map defined by $f = n\gamma + \ell\iota_{n+2}$ for $\gamma \in \pi_{n+2}(S^n)$ and $\beta = \iota_n\beta' + \iota_{n+2}\eta_{n+2}$ be the attaching class for the $(n+4)$-cell of $X$. Then, we have

$$f_*(\eta_{n+2}) = \iota_n(\gamma\eta_{n+2}) + \ell\iota_{n+2}\eta_{n+2}.$$

Hence the proof follows from $m\beta \equiv m\iota_n\beta' + m\iota_{n+2}\eta_{n+2} = f_*(\eta_{n+2})$,

i.e. $m \equiv \ell \bmod 2$ and $m\beta' \equiv 0 \bmod \iota_n\eta_n\eta_{n+1}\eta_{n+2}$.

Secondly, using the group structure of $[X, X] = [\Sigma X', \Sigma X']$ we have the map $k1_X$, which is of degrees $(k, k, k)$. Let $g: X \to X$ be a self-map of degrees $(k, \ell, m)$. Then the map $g - k1_X$ is of degrees $(0, \ell - k, m - k)$, and lemmas 14 and 15 give the following

**Theorem D.** *Assume that* $X$ *is a suspended space. Then a triple* $(d_1, d_2, d_3)$ *is realizable by a self-map of* $X$ *if and only if*

(1) *If* $Sq^2(e^{n+2}) = 0$, *then* $d_2 \equiv d_1 \bmod h_{n+2}(X)$ *and* $d_3 \equiv d_1 \bmod h_{n+4}(X)$.

(2) *If* $Sq^2(e^{n+2}) \neq 0$, *then* $d_2 \equiv d_1 \bmod 2$ *and* $2d_3 \equiv 2d_1 \bmod h_{n+4}(X)$.

**Remark.** *If* $Sq^2(e^n) = 0$, *then* $h_{n+2}(X) = \boldsymbol{Z}$, *and if* $Sq^2(e^n) \neq 0$, *then* $h_{n+2}(X) = 2\boldsymbol{Z}$.

**4. Examples.**

**Example 1.** Let $X_n$ be the space $CP^{n+2}/CP^{n-1}$ $(1 \leqq n)$. Then, using the results in [5] and $Sq^2(e^i) = ie^{i+1}$ in $H^*(CP^N)$, we have that degrees of self-maps of $X_n$, $(d_1, d_2, d_3)$, is characterlized as follows:

(1) For $3 \leqq n$,

if $n \equiv 1 \bmod 2$, then $d_2 \equiv d_1 \bmod 2$ and $d_3 \equiv d_1 \bmod 24/(n+3, 24)$,

if $n \equiv 0 \bmod 8$, then $d_2 \equiv d_3 \bmod 2$ and $2d_3 \equiv 2d_1 \bmod 96/(n, 48)$,

if $n = 2, 4, 6 \bmod 8$, then $d_2 \equiv d_3 \bmod 2$ and $2d_3 \equiv 2d_1 \bmod 48/(n, 48)$,

where ( , ) denotes the greatest common diviser of integers.

**Remark.** These results also hold for the iterated-suspension $\Sigma^s X_n$.

(2) For $n = 2$, $d_3 \equiv d_2 \bmod 2$ and $d_3 \equiv d_1^2$, i.e. $(k, k^2+2\boldsymbol{Z}, k^2)$, and for the space $\Sigma^s X_2$ $(1 \leqq s)$ we have $(d_1, d_2, d_3) = (k, k+2\boldsymbol{Z}, k+12\boldsymbol{Z})$.

(3) For $n = 1$, $(d_1, d_2, d_3) = (k, k^2, k^3)$, and for $\Sigma^s X_1$ $(1 \leqq s)$ we have $(d_1, d_2, d_3) = (k, k+2\boldsymbol{Z}, k+6\boldsymbol{Z})$.

**Example 2.** Let $X$ be the 6-skelton of the reduced product of $S^2$. Since it is clear that $\Sigma^s X$ is decomposed into $S^{2+s} \vee S^{4+s} \vee S^{6+s}$ $(1 \leqq s)$ we can know that

(1) For $s = 0$, $(d_1, d_2, d_3) = (k, k^2, k^3)$,

(2) For $1 \leqq s$, $(d_1, d_2, d_3)$ for any $d_1$, $d_2$ and $d_3$.

**Example 3.** Let $X_r$ be the 2-sphere bundle over $S^4$ whose characteristic class is $r$ times of a generator $(\in \pi_3(SO(3)) = \boldsymbol{Z})$. Then it is easy to see

$$e^2 \cdot e^2 = re^4 \quad \text{and} \quad e^2 \cdot e^4 = e^6.$$

Furthermore the suspension $\Sigma^N X_r$ has a decomposition

$$\Sigma^N X_r = S^{N+2} \cup e^{N+4} \cup e^{N+6}$$

in which the attaching class $\beta_N$ for the $(N+6)$-cell is given by

$$\beta_N = i_*(\Sigma^{N-1} J(\xi_r)) = i_*(r\Sigma^{N-1}\omega) = ri_*(\Sigma^{N-1}\omega)$$

where $\xi_r$ is the real vector bundle associated with $X_r$.

On the other hand, it is easy to show that

$Sq^2(e^{N+4}) = 0$ $(0 \leqq N)$ and $Sq^2(e^{N+2}) = 0$ for even $r$, $\neq 0$ for odd $r$ $(1 \leqq N)$.

Since $Sq^2(e^{N+4}) = 0$ implies that $i^*$ is injective, we get

$$h_{N+6}(\Sigma^N X_r) = (r, 12)\boldsymbol{Z}.$$

These facts give the following:

A triple $(d_1, d_2, d_3)$ is realizable by a self-map of $\Sigma^N X_r$ if and only if

(1) The case $1 \leqq N$.

$d_3 \equiv d_1 \bmod (r, 12)$ if $r \equiv 0 \bmod 2$,

$d_2 \equiv d_1 \bmod 2$ and $d_3 \equiv d_1 \bmod (r, 12)$ if $r \equiv 1 \bmod 2$.

(2) The case $N = 0$.

$d_3 \equiv d_1 d_2$ if $r = 0$,

$d_2 = d_1^2$ and $d_3 = d_1^3$ if $r \neq 0$.

**Example 4.** Let $Y_a$ be the $CW$-complex $S^2 \cup e^4$ which has $a\eta_2$ as the attaching class for the cell $e^4$. Assume $5 \leqq N$ and consider an $N$-dim real vector bundle $\xi$ over $Y_a$. For simplicity we suppose that its Stiefel-Whitney class $w_2(\xi)$ is trivial. Since $w_2(\xi) = 0$ implies that the restriction $\xi | S^2$ is trivial, there is a commutative diagram

$$\begin{array}{ccc} \xi & \rightarrow & \xi' \\ \downarrow & & \downarrow \\ Y_a & \rightarrow & S^4 = Y_a/S^2 \end{array}$$

for some $\xi'$.

Then, from $Sq^2(e^{N+2}) = ae^{N+4}$ ([4]), we can know that the Thom complex $T(\xi)$ has a decomposition

$$T(\xi) = (S^N \vee S^{N+2}) \cup e^{N+4}$$

in which $\beta = \iota_n J(\xi') + \iota_{n+2}(a\eta_{n+2})$ $(\in \pi_{N+3}(S^N) + \pi_{N+3}(S^{N+2}))$. Thus, from Theorem D, we have that a triple $(d_1, d_2, d_3)$ is realizable by a self-map of $T(\xi)$ if and only if

(1) *if* $a \equiv 0 \bmod 2$, *then* $d_3 \equiv d_1 \bmod (b, 24)$,

(2) *if* $a \equiv 1 \bmod 2$, *then* $d_3 \equiv d_2 \bmod 2$ *and* $2d_3 \equiv 2d_1 \bmod (b, 12)$,

*where* $2be^4 = p_1(\xi)$, *the first Pontrijagin class of* $\xi$.

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