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ON SOME PRODUCTS OF β-ELEMENTS IN THE HOMOTOPY OF THE MOORE SPECTRUM

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1. Introduction. Throughout this paper, the prime p is greater than 3. In the homotopy groups $\pi_*(S^0)$ of spheres, H. Toda first constructed β-elements β_t for small value of t and then L. Smith expanded them into positive integers t. S. Oka [10] [11] [12] also gave an infinite families in it. In their paper [9], H. Miller, D. Ravenel, and S. Wilson gave not only a method to construct a generalized β-elements but all generalized β-elements in the E_2-term of the Adams-Novikov spectral sequence for spheres. In their definition, Oka's elements in the stable homotopy groups of spheres are detected by generalized β-elements of the form $\beta_{sp^r/j}$ in the E_2-term. At this stage, we had little information on the permanent cycles but Toda's, Smith's and Oka's elements. Recently, Jinkun Lin found many of them are permanent cycles ([3], [4]), which motivated us to work on the subjects which we treat here. Unfortunately, the proof of [4] seems to have an error on signs, which is pointed out by H. Sadofsky. (Lin has again got the same results as that of [4] in [7].) In spite of this, results of Sadofsky's and Lin's ([17], [3]) add more β-elements in the homotopy groups. Lin also studied the products of those elements ([6]). Restrict our interest to the product of the form $\beta_s\beta_{tp^r/j}$. Then together with the results of Oka's and the last named author's ([13], [18], [19], [20], [21]), we know that there remain the following products undetermined whether or not they are trivial:

a) $\beta_s\beta_{p^n/p^{n-1}} \in \pi_*(S^0)$ $(n \geq 2, s \geq 1)$.

b) $\beta_s\beta_{tp^n/p^n} \in \pi_*(S^0)$ $(n \geq 2, s \geq 1, p \nmid t \geq 2)$.

c) $\beta_{sp+1}\beta_{tp/p} \in \pi_*(S^0)$ $(s \geq 1, t \geq 2$ and $s+t = (up-1)p^n$ for some u and $n \geq 1)$.

Let M be the mod p Moore spectrum which is defined to be the cofiber of the map of degree p on the sphere S^0, and denote $i: S^0 \to M$ and $\pi: M \to S^1$ the inclusion to the bottom cell and the projection pinching the bottom cell, respectively. Then there exists a pre-image $\beta'_E \in \pi_*(M)$ of $\beta_E \in \pi_*(S^0)$ such that $\pi\beta'_E = \beta_E$ for $E = s$ or sp^n/r, since the order of the element β_E is p (*cf.* [13], see also §2).

Here, in this paper, we study about a product (or a composition) of these β-elements in $\pi_*(M)$ relating to a product of the form b) above.

Theorem A. *Let s, t and r be positive integers with s and $t \geq 2$ and $p \nmid st$. Then, in the homotopy group $\pi_*(M)$ of the Moore spectrum M, we have*

1) $\beta'_{tp-1}\beta_{sp^r/p^r} \neq 0$ *for even r, and*
2) $\beta'_t\beta_{sp^r/p^r} \neq 0$ *for odd r,*

if these elements are homotopy elements.

Since the Moore spectrum M is a ring spectrum, we have the product of the elements induced by the structure map $\mu\colon M \wedge M \to M$. With this product, we have

Theorem B. *Let s, t and r be positive integers with s and $r \geq 2$ and suppose that the β-elements below are homotopy elements. Then in $\pi_*(M)$,*

a) *for even r,*
  1) $\beta'_{tp-1}\beta'_{sp^r/p^r} \neq 0$ *if* $p \nmid s$; *and*

b) *for odd r,*
  2) $\beta_t{}'\beta'_{sp^r/p^r} \neq 0$ *if* $p \nmid st(t-1)$,
  3) $\beta'_{tp^2-sp/k}\beta'_{sp/j} \neq 0$ *if* $p \nmid s, 1 \leq k \leq p$ *and* $p-k < j \leq p$,
  4) $\beta'_{tp^2-p/k}\beta'_{sp^r/j} \neq 0$ *if* $p \nmid s, 1 \leq k \leq p, r \geq 3$ *and* $p^r-k < j \leq p^r$,

and

  5) *Put* $a = up^m - sp^r + p^{n-1} - p^{n-2} + \cdots - p + 1$ *for some u and $m \geq 1$, and we have*

$$\beta'_a\beta'_{sp^r/p^r} \neq 0$$

if $p \nmid s$ and $p \nmid u(u+1)$, or if $p \nmid s$ and $p^2 \mid u+1$.

Once we have shown a nontriviality of a product of β-elements in the E_2-term of the Adams-Novikov spectral sequence, then we have the same nontriviality in the stable homotopy groups, since nothing kills the product in the spectral sequence by the degree reasons. So we prove these theorems by showing the same results in the E_2-term of the Adams-Novikov spectral sequence. These results are, in a sense, a corollary of the computation [22] [18] of $H^*(M_1^1)$. In these theorems, we assume that these β-elements are defined in the homotopy groups of spheres. But if we use the Bousfield localization functor L_2 with respect to the Johnson-Wilson spectrum $E(2)$ whose coefficient group is $Z_{(p)}[v_1, v_2, v_2^{-1}]$, then the theorems are valid in the homotopy ring $\pi_*(L_2M^0)$ of the localized Moore spectrum without the condition that β-elements are homotopy elements.

We define β-elements in §2 and prove theorems above in the last section by using the results of [22] and [18].

2. Definition of β-elements. According to [9], we define the $BP_*(BP)$-comodules N_n^s and M_n^s to give β-elements in the E_2-term of the Adams-Novikov spectral sequence for the sphere and the Moore spectrum. Here BP denotes the Brown-Peterson ring spectrum at the prime p, and the pair $(BP_*, BP_*(BP))$ becomes the Hopf algebroid in a canonical way using the structure maps of the ring spectrum. Furthermore, the coefficient ring $BP_* = \boldsymbol{Z}_{(p)}[v_1, v_2, \cdots]$ over the Hazewinkel generators v_i with degree $2p^i-2$. The BP_*-algebra $BP_*(BP)$ is flat over BP_*, because $BP_*(BP)$ is the polynomial algebra $BP_*[t_1, t_2, \cdots]$ over the generators t_i with degree $2p^i-2$.

Since the right unit $\eta_R: BP_* \to BP_*(BP)$ satisfies $(\eta_R \otimes 1)\eta_R = \Delta\eta_R$, BP_* itself is also a $BP_*(BP)$-comodule. Here 1 and Δ denote the identity map $BP_*(BP) \to BP_*(BP)$ and the coproduct $BP_*(BP) \to BP_*(BP) \otimes_{BP_*} BP_*(BP)$. Furthermore, take an ideal

$$I_n = (v_0, v_1, \cdots, v_{n-1}) \quad (v_0 = p),$$

and we have another $BP_*(BP)$-comodule BP_*/I_n with structure map induced from η_R above, which we also denote by η_R. We now recall [8]: A $BP_*(BP)$-comodule M is said to be I_n-nil if each element x of M has an integer k such that $I_n^k x = 0$. If $BP_*(BP)$-comodule M is I_n-nil, then $v_n^{-1}M = \lim_{\overrightarrow{v_n}} M$ is also a $BP_*(BP)$-comodule and the localization map $M \to v_n^{-1}M$ is a comodule map ([8, Lemma 3.2]). Since a comodule map introduces a comodule structure to its cokernel, we have a comodule

$$M/(v_n^\infty),$$

the cokernal of the map $M \to v_n^{-1}M$. This definition permits us to write an element x of $M/(v_n^\infty)$ to be

$$x = m/v_n^k$$

for some $k > 0$ and $m \in M$, and $x = 0$ if $v_n^k \mid m$.

We now define the $BP_*(BP)$-comodules N_n^s and M_n^s: Put $N_n^0 = BP_*/I_n$, and suppose that N_n^s is an I_{n+s}-nil $BP_*(BP)$-comodule with structure map η_R. Then we define M_n^s to be a localization $v_{n+s}^{-1} N_n^s$ and N_n^{s+1} to be the cokernel of the localization map $N_n^s \to M_n^s$. By this definition, we see that these comodules M_n^s and N_n^{s+1} are I_{n+s}- and I_{n+s+1}-nil, respectively. Thus we complete the inductive definition.

Since the category of $BP_*(BP)$-comodules has enough injectives (*cf.* [16, Lemma A1.2.2]), we define $\mathrm{Ext}^s_{BP_*(BP)}(BP_*, -)$ to be the s-th right derived functor of the functor $\mathrm{Hom}_{BP_*(BP)}(BP_*, -)$. After [9], we use the abbreviation

$$H^sM = \mathrm{Ext}^s_{BP_*(BP)}(BP_*, M)$$

for a $BP_*(BP)$-comodule M. We compute this group by using a cobar complex Ω^*M, which is explained below. The comodules which appear from here on are only N_n^i and M_n^i with $n+i \le 2$, and so the structure map ψ is η_R. Besides we consider Ext-groups H^sM only for $s \le 2$. Let M denote N_n^i and M_n^i for some n and i. Then M has the structure map η_R. Furthermore we have a cobar complex

$$0 \to M \xrightarrow{d_0} \Omega^1 M = M \otimes_{BP_*} BP_*(BP)$$
$$\xrightarrow{d_1} \Omega^2 M = M \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} BP_*(BP) \xrightarrow{d_2}$$

in which the differential d_i are given by

(2.1) $$d_0(m) = \eta_R(m) - m, \; d_1(x) = 1 \otimes x - \Delta(x) + x \otimes 1 \quad \text{and}$$
$$d_1(mx) = d_0(m) \otimes x + m d_1(x)$$

for $m \in M$ and $x \in BP_*(BP)$. Thus we have

(2.2) $$H^0M = \mathrm{Ker}\, d_0 \quad \text{and} \quad H^1M = \mathrm{Ker}\, d_1/\mathrm{Im}\, d_0.$$

The definition of the comodules N_n^i and M_n^i indicates the short exact sequences

(2.3) $$0 \to N_0^0 \to M_0^0 \to N_0^1 \to 0,$$
$$0 \to N_0^1 \to M_0^1 \to N_0^2 \to 0, \quad \text{and}$$
$$0 \to N_1^0 \to M_1^0 \to N_1^1 \to 0,$$

and we have the associated boundary homomorphisms

(2.4) $$\delta_1 : H^1N_0^1 \to H^2N_0^0,$$
$$\delta_0 : H^0N_0^2 \to H^1N_0^1 \quad \text{and}$$
$$\delta_0' : H^0N_1^1 \to H^1N_1^0.$$

We note that while we compute on the complex $\Omega^*N_1^*$, we use the formula $p=0$, and so we sometimes compute some cochains of the complex $\Omega^*N_0^0$ mod(p). By Quillen's formula (*cf.* [1], [16]), we easily get the formula $\eta_R(v_1) = v_1 + pt_1$. Besides Quillen's gives Landweber's formula $\eta_R(v_2) \equiv v_2 + v_1 t_1^p - v_1^p t_1$ mod(p), by which we compute to obtain

(2.5) $$v_2^{sp^r}/pv_1^j \in H^0N_0^2 \quad \text{and}$$
$$v_2^{sp^r}/v_1^j \in H^0N_1^1 \quad \text{if } j \le p^r$$

as follows: $d_0(v_2^{sp^r}/v_1^j) = d_0(v_2^{sp^r})/v_1^j = v_1^{p^r}x/v_1^j = 0$, since $v_1^j \mid v_1^{p^r}$, where we write $d_0(v_2^{sp^r}) \equiv v_1^{p^r}x$ mod(p) for some $x \in BP_*(BP)$. Thus we have the second one of (2.5). Similarly we obtain the first one. By virtue of the result (2.5), we define the β-elements using the maps in (2.4) as follows:

(2.6)
$$\begin{aligned} \beta_{sp^r/j} &= \delta_1\delta_0(v_2^{sp^r}/pv_1^j) \in H^2N_0^0 \quad \text{for } j \le p^r, \text{ and} \\ \beta'_{sp^r/j} &= \delta'_0(v_2^{sp^r}/v_1^j) \quad \in H^1N_1^0 \quad \text{for } j \le p^r. \end{aligned}$$

As usual, we abbreviate $\beta_{sp^r/1}$ to β_{sp^r} and $\beta'_{sp^r/1}$ to β'_{sp^r}.

Lemma 2.7. *In the cobar complex $\Omega^2N_0^0 = \Omega^2BP_*$, the element β_{sp^r/p^r} in* (2.6) *is represented by a cocycle*

$$sv_2^{(s-1)p^r}T^{p^r}$$

mod$(p, v_1^{p^{r-1}})$. *Similarly, $\beta'_{sp^r/j}$ in* (2.6) *is*

$$sv_1^{p^r-j}v_2^{(s-1)p^r}t_1^{p^{r+1}}$$

mod$(v_1^{2p^r-j})$ *in the cobar complex $\Omega^1N_1^0 = \Omega^1BP_*/(p)$.*

Here we use the notation used in [22]: $T = -\sum_{i=1}^{p-1}\binom{p}{i}t_1^{p-i}\otimes t_1^i$.

Proof. By the definition of the boundary homomorphism and the β-elements (2.6), we have

$$\beta_{sp^r/p^r} = \delta_1(d_0(v_1^{-p^r}v_2^{sp^r}/p)).$$

The Landweber's formula and the binomial theorem give $d_0(v_1^{-p^r}v_2^{sp^r}/p) = (sv_2^{(s-1)p^r}t_1^{p^{r+1}} + v_1^{p^r}X)/p \in \Omega^1N_0^1$ for some $X \in BP_*(BP)$. The definition (2.1) of the differential d_1 also gives

$$\begin{aligned} &d_1(p^{-1}(v_2^{(s-1)p^r}t_1^{p^{r+1}})) \\ &\quad \equiv (s-1)v_1^{p^{r-1}}v_2^{(s-1)p^r-p^{r-1}}t_1^{p^r}\otimes t_1^{p^{r+1}} + p^{-1}v_1^{2p^{r-1}}Y + v_2^{(s-1)p^r}T^{p^r} \end{aligned}$$

mod(p), and

$$d_1(p^{-1}v_1^{p^r}X) = p^rW + p^{-1}v_1^{p^r}d_1(X).$$

Here we see that $v_1^{2p^{r-1}}Y + v_1^{p^r}d_1(X_1) \equiv 0 \bmod(p)$ since $d_1(d_0(v_2^{sp^r})) \equiv 0 \bmod(p)$, which indicates the existence of an element U satisfying $v_1^{2p^{r-1}}Y + v_1^{p^r}d_1(X_1) = pv_1^{2p^{r-1}}U$. Besides, we have

$$d_1(v_1^{p^{r-1}}v_2^{(s-1)p^r-p^{r-1}}t_2^{p^r}) \equiv -v_1^{p^{r-1}}v_2^{(s-1)p^r-p^{r-1}}t_1^{p^r}\otimes t_1^{p^{r+1}}$$

mod$(p, v_1^{2p^{r-1}})$, since we have a formula of the diagonal $\Delta(t_2) \equiv t_2\otimes 1 + t_1\otimes t_1^p + 1\otimes t_2 \bmod(p, v_1)$. Therefore we have the first statement.

A similar computation shows the second half. In fact, $\beta'_{sp^r/j} = d_0(v_1^{-j}v_2^{sp^r})$

by definition. Besides we see that $\eta_R(v_1)=v_1$ and $\eta_R(v_2^{sp^r})=(v_2^{p^r}+v_1^{p^r}t_1^{p^{r+1}}-v_1^{p^{r+1}}t_1^{p^r})^s$ in this complex. Thus the binomial theorem is applied to show the results.

As is shown in [9], the β-elements given in (2.6) are the same as those defined by

$$\begin{aligned}\beta_{sp^r/j}&=\partial_1\partial_0(v_2^{sp^r})\in H^2N_0^0 \quad \text{for } j\le p^r, \text{ and}\\ \beta'_{sp^r/j}&=\partial_0(v_2^{sp^r})\in H^1N_1^0 \quad \text{for } j\le p^r,\end{aligned}$$

where ∂_1, and ∂_0 are the boundary homomorphism associated to the short exact sequences

$$0\to N_0^0\xrightarrow{p}N_0^0\to N_1^0\to 0,\quad \text{and}$$

$$0\to N_1^0\xrightarrow{v_1^j}N_1^0\to BP_*/(p,v_1^j)\to 0,$$

respectively. Thus we have

$$\beta_{sp^r/j}=\partial_1(\beta'_{sp^r/j}) \tag{2.8}$$

in the E_2-term of the Adams-Novikov spectral sequence for computing $\pi_*(S^0)$. The boundary homomorphism ∂_1 corresponds to the geometric boundary π in the cofiber sequence $S^0\xrightarrow{p}S^0\xrightarrow{i}M\xrightarrow{\pi}S^1$ defining the Moore spectrum M. Then the geometric boundary theorem [2] certifies that the relation (2.8) implies the one

$$\beta_{sp^r/j}=\pi\beta'_{sp^r/j} \tag{2.9}$$

in the homotopy group, where $\beta_{sp^r/j}\in\pi_*(S^0)$ and $\beta'_{sp^r/j}\in\pi_*(M)$.

3. Proof of Theorems in §1. The definition of comodules of the previous section implies the short exact sequence

$$0\to N_1^1\xrightarrow{\lambda}M_1^1\to N_1^2\to 0,$$

and we have an induced map $\lambda: H^iN_1^1\to H^iM_1^1$. Here the structure of $H^iM_1^1$ is determined in [22] and [18]. Consider first the following diagram

$$\begin{array}{ccccc} H^2M_1^0 & \to & H^2N_1^1 & \xrightarrow{\delta_2'} & H^3N_1^0\\ & & \downarrow\lambda & & \\ & & H^2M_1^1, & & \end{array}$$

in which the raw is the exact sequence associated to the third one of (2.3). Ravenel shows in [15] that the module on the left is trivial and so δ_2' is monomorphic. Furthermore $\delta_*': H^*N_1^1\to H^*N_1^0$ is a map of H^*BP_* modules and $\beta'_t = \delta'_0(v_2^t/v_1)$. Therefore,

$$\beta'_t\beta_{sp^r/p^r} = \delta'_2(v_2^t\beta_{sp^r/p^r}/v_1).$$

Thus, in order to prove Theorem A, it is sufficient to show that the element $\lambda(v_2^t\beta_{sp^r/p^r}/v_1)$ is not trivial in $H^2M_1^1$ whose structure we know. By Lemma 2.7,

$$\lambda(v_2^t\beta_{sp^r/p^r}/v_1) = sv_2^{t+(s-1)p^r}T^{p^r}/v_1$$

in $H^2M_1^1$. For a while, we work in $E(2)_*(E(2))\otimes E(2)_*(E(2)))$. Since $v_2^{p^{i+2}}T^{p^i} \equiv v_2^{p^{i+1}}T^{p^{i+2}} \bmod (p, v_1^{p^i})$ in (*cf.* [22]), we have

$$T^{p^r} \equiv v_2^{a(r)}T^{p^{\varepsilon(r)}} \quad \bmod (p, v_1^{p^{\varepsilon(r)}}),$$

where $r > 1$ and

$$\begin{aligned} a(r) &= \textstyle\sum_{i=0}^{r-\varepsilon(r)-1}(-1)^i p^{r-i} \quad \text{and} \\ \varepsilon(r) &= 2 \quad \text{if } r \text{ is even} \\ \varepsilon(r) &= 1 \quad \text{if } r \text{ is odd.} \end{aligned}$$

Furthermore we have the formulae $d_1(t_3^{p^{\varepsilon(r)-1}}) \equiv -g^{p^{\varepsilon(r)-1}} - v_2^{p^{\varepsilon(r)-1}}T^{p^{\varepsilon(r)}} \bmod (p, v_1^{p^{\varepsilon(r)-1}})$, $g_0 = v_2^{-p}g$ and $g_1 = v_2^{-p^2-1}g^p$ (*cf.* [22]), where $g = t_1\otimes t_2^p + t_2\otimes t_1^{p^2}$. Therefore, from these results in $E(2)_*(E(2))\otimes E(2)_*(E(2))$, we deduce the formula in $H^2M_1^1$:

$$\lambda(v_2^t\beta_{sp^r/p^r}/v_1^{p^r}) = sv_2^{t+a(s,r)}g_{\varepsilon(r)-1}/v_1, \tag{3.10}$$

where

$$\begin{aligned} a(s, r) &= sp^r - p^r + a(r) + p^2 - p + 1 \quad \text{for even } r \text{ and} \\ a(s, r) &= sp^r - p^r + a(r) + p - 1 \qquad\quad \text{for odd } r. \end{aligned}$$

Recall [18] that

(3.11) *$H^2M_1^1$ is the direct sum of $F_p\{h_0\otimes\zeta^{p^j}/v_j\}$ and*
$F_p[v_1]\langle y_m\otimes\zeta^{p^{n+3}}/v_1^{A(m)}\rangle$ for $m = sp^n \in \Lambda_0$,
$F_p[v_1]\langle x_n^sG_n/v_1^{a_n}\rangle$ for $s+1 \in \boldsymbol{Z} - p\boldsymbol{Z}$ and $n \geq 0$, and
$F_p[v_1]\langle v_2^{tp}V\otimes\zeta^p/v_1^{p-1}\rangle$ for $t \in \boldsymbol{Z}$.

Here we need only the generator $x_n^sG_n/v_1$ appeared in the third module, which equals to $v_2^{sp^n-p^{n-2}-\cdots-1}g_1/v_1$ if $n > 0$, and to $v_2^sg_0/v_1$ if $n = 0$. Thus comparing the generators above and (3.10) we get Theorem A. In other words, we deduce that $\lambda(v_2^t\beta_{sp^r/p^r}/v_1^{p^r}) \neq 0$ if $p \nmid s$ and $sp^n - p^{n-2} - \cdots -1 = t + a(s, r)$, since it is a generator of $H^2M_1^1$.

Now turn to the proof of Theorem B. It is almost identical to that of

Theorem A. The only difference is that we use the structure of $H^1M_1^1$ instead of $H^2M_1^1$. So we just sketch it. Consider the diagram

$$\begin{array}{ccccc} H^1M_1^0 & \xrightarrow{j_*} & H^1N_1^1 & \xrightarrow{\delta_1'} & H^2N_1^0 \\ & & \downarrow \lambda & & \\ & & H^1M_1^1 & & \end{array}$$

where we know that $H^1M_1^0 = F_p[v_1, v_1^{-1}]\{h_0\}$ by [15]. Here we see that the cokernel of $i_*: H^1N_1^0 \to H^1M_1^0$ is $(F_p[v_1, v_1^{-1}]/F_p[v_1])\{h_0\}$ for the map i_* induced from the inclusion $N_1^0 \to M_1^0$. Therefore every elements in the image of the map j_* in the above diagram has a non-positive degree. Lemma 2.7 shows that

$$v_2^{tp^\ell}\beta'_{sp^r/j}/v_1^k = sv_2^{tp^\ell+(s-1)p^r}t_1^{p^{r+1}}/v_1^{k-p^r+j} + v_2^{tp^\ell}X/v_1^{k-2p^r+j}$$

for some $X \in BP_*(BP)$, which is not in the image of $H^1M_1^0$ by degree reason. In fact, it has a positive degree. Therefore it is also sufficient to study if the element $\lambda(v_2^{tp^\ell}\beta'_{sp^r/j}/v_1^k)$ is non-trivial in $H^1M_1^1$, whose structure is known in [22]. We need here is the following:

(3.12) *$H^1M_1^1$ contains the direct sum of*
$F_p[v_1]\langle y_m/v_1^{A(m)}\rangle$ *for* $m = sp^n \in \Lambda_0$, *and*
$F_p[v_1]\langle v_2^{tp}V/v_1^{p-1}\rangle$ *for* $t \in \boldsymbol{Z}$.

Here Λ_0 is a subset of integers $\{sp^n \mid p \nmid s(s+1) \text{ or } p^2 \mid s+1\}$ and $A(sp^n)$ is an integer $(p+1)(p^n-1)/(p-1)+2$ if $p \nmid s(s+1)$, and $(p+1)(p^{n+1}-p^n+(p^n-1)/(p-1))+2$ if $p^2 \mid s+1$.

By using the relation $v_2^{p^{i-1}}t_1^{p^i} \equiv v_2^{p^i}t_1^{p^{i+2}} \bmod (p, v_1^{p^i})$ in $E(2)_*(E(2))$ (*cf.* [22]), we get

$$\lambda(v_2^{tp^\ell}\beta'_{sp^r/j}/v_1^k) = -sv_2^{tp^\ell+sp^r-p^{r-1}+\cdots+p^2-2p+1}V/v_1^{k-p^r+j} + \cdots$$

for even r, and

$$\lambda(v_2^{tp^\ell}\beta'_{sp^r/j}/v_1^k) = sy_{tp^\ell+sp^r-p^{r-1}+\cdots-p^2+p-1}/v_1^{k-p^r+j} + \cdots$$

for odd r, since $V \equiv -v_2^{p-1}t_1^p \bmod (p, v_1)$ and $y_m \equiv v_2^m t_1 \bmod (p, v_1)$. Here the leading terms on the right of the equations are generators of $H^1M_1^1$ and $\cdots$ denotes an element killed by a smaller power of v_1 than that shown. Now compare the elements above with the generators of $H^1M_1^1$ given in [22] and obtain Theorem B. In other words, we obtain the following from (3.12). For even r, $v_2^{tp^\ell}\beta'_{sp^r/j}/v_1^k \neq 0$ if $p \nmid s$, $0 < k-p^r+j < p$ and $p \mid tp^\ell+sp^r-p^{r-1}+\cdots+p^2-2p+1$. When r is odd, $v_2^{tp^\ell}\beta'_{sp^r/j}/v_1^k \neq 0$ if $p \nmid s$, $tp^\ell+sp^r-p^{r-1}+\cdots-p^2+p-1 \in \Lambda_0$ and $0 < k-p^r+j < A(tp^\ell+sp^r-p^{r-1}+\cdots-p^2+p-1)$. The statement of the Theorem is the one rewritten these conditions.

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