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THE INTEGRAL COHOMOLOGY RINGS OF $F_4/\mathrm{Spin}(n)$ AND $E_6/\mathrm{Spin}(m)$

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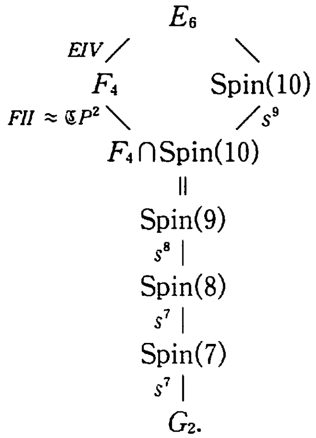
1. Introduction. Let G_2, F_4 and E_6 be the exceptional compact simple Lie groups of rank 2, 4 and 6 respectively, particularly E_6 be simply connected. As is well-known, F_4 has the subgroups $\mathrm{Spin}(n)$ for $n = 7, 8, 9$ and E_6 has the subgroups $\mathrm{Spin}(m)$ for $m = 7, 8, 9, 10$. The following Hasse diagram holds:



For example, F_4 is a subgroup of E_6 and the homogeneous space E_6/F_4 is the compact irreducible symmetric Riemannian space EIV of exceptional type, Spin(9) is a subgroup of Spin(10) and the homogeneous space Spin(10)/Spin(9) is homeomorphic to the 9-dimensional sphere S^9, and so on. These conditions are described in [4] and [5].

The integral cohomology ring structure of the homogeneous space $F_4/\mathrm{Spin}(9)$ is well-known, and L. Conlon determined that of the homogeneous space $E_6/\mathrm{Spin}(10)$; see [3, Corollary 4]. Our aim is to do that of the homogeneous spaces $F_4/\mathrm{Spin}(n)$ for $n = 7, 8$ and $E_6/\mathrm{Spin}(m)$ for $m = 7, 8, 9$.

In this paper, we donote by $\boldsymbol{Z}$ the ring of integers, by $\boldsymbol{R}$ the field of real numbers, by $\boldsymbol{C}$ the field of complex numbers, by $\boldsymbol{Z}_k$ the cyclic group $\boldsymbol{Z}/k\boldsymbol{Z}$ of order k for a positive integer k, by $\boldsymbol{Z}[x_1, \cdots, x_n]$ the polynomial ring over $\boldsymbol{Z}$ generated by variables $x_1, \cdots, x_n$, by $(-)$ the ideal generated by $-$, by $\langle x\rangle_M$ a module M generated by a base x, by $\langle x_1, \cdots, x_n\rangle_M$ the module $\langle x_1\rangle_M \oplus \cdots \oplus \langle x_n\rangle_M$, and by $\wedge_M(x_1, \cdots, x_n)$ the exterior algebra over $\langle x_1, \cdots, x_n\rangle_M$.

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2. The integral cohomology rings of $F_4/\mathrm{Spin}(n)$. In this section, we determine the integral cohomology ring structure of the homogeneous space $F_4/\mathrm{Spin}(n)$ for $n = 7, 8$.

For the homogeneous space $F_4/\mathrm{Spin}(9)$, it is well-known that $F_4/\mathrm{Spin}(9)$ is homeomorphic to the Cayley projective plain $\mathfrak{C}P^2$. Therefore we have

$$\text{(2.1)} \qquad H^*(F_4/\mathrm{Spin}(9)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}[x_8]/(x_8^3)$$

as a graded ring, where $\deg x_8 = 8$. For the homogeneous space $F_4/\mathrm{Spin}(8)$, A. Borel showed that additively

$$\text{(2.2)} \qquad H^*(F_4/\mathrm{Spin}(8)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}[y_8, y_8']/(y_8^3, y_8'^2)$$

as a graded module, where $\deg y_8 = 8$ and $\deg y_8' = 8$; see [2, Lemma 20.4]. Furthermore, if we denote by p: $F_4/\mathrm{Spin}(8) \to F_4/\mathrm{Spin}(9)$ the obvious projection, then we can choose generators such that

$$\text{(2,3)} \qquad p^*(x_8) = y_8.$$

For the homogeneous space F_4/G_2, he also showed that

$$\text{(2.4)} \quad H^*(F_4/G_2\,;\ \boldsymbol{Z}) \cong (\langle 1\rangle_Z \oplus \langle u_8\rangle_{Z_3} \oplus \langle u_8^2\rangle_{Z_3} \oplus \langle u_{23}\rangle_Z) \otimes_Z \boldsymbol{Z}[u_{15}]/(u_{15}^2)$$

as a graded ring, where 1 is the unit, $\deg u_8 = 8$, $\deg u_{23} = 23$ and $\deg u_{15} = 15$; see [2, Proposition 23.1].

For the homogeneous space $F_4/\mathrm{Spin}(7)$, we obtain the following theorem:

Theorem 2.1. *As a graded ring*

$$\text{(2.5)} \qquad H^*(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \langle 1\rangle_Z \oplus \langle z_8\rangle_Z \oplus \langle z_8^2\rangle_{Z_3} \oplus \langle z_{23}\rangle_Z \oplus \langle z_8 z_{23}\rangle_Z,$$

where 1 *is the unit,* $\deg z_8 = 8$ *and* $\deg z_{23} = 23$. *Futhermore, if we donote by* p: $F_4/G_2 \to F_4/\mathrm{Spin}(7)$ *the obvious projection, then we can choose generators such that*

$$\text{(2.6)} \qquad p^*(z_8) = u_8,$$

$$\text{(2.7)} \qquad p^*(z_{23}) = u_{23}.$$

Proof. We consider the Serre spectral sequence $(E_*^{*,*}, d_*)$ associated to the fiber bundle $(F_4/\mathrm{Spin}(7), p, F_4/\mathrm{Spin}(9), \mathrm{Spin}(9)/\mathrm{Spin}(7), \mathrm{Spin}(9))$, whose E_2-term is as follows:

(2.8) $$E_2^{p,q} \cong \mathrm{H}^p(F_4/\mathrm{Spin}(9)\,;\ \mathrm{H}^q(\mathrm{Spin}(9)/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}))$$

as a module. Since the homogeneous space $\mathrm{Spin}(9)/\mathrm{Spin}(7)$ is homeomorphic to the Stiefel manifold $\boldsymbol{R}V_{9,2}$, we have

(2.9) $$\mathrm{H}^*(\mathrm{Spin}(9)/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \langle 1\rangle_{\boldsymbol{Z}} \oplus \langle v_8\rangle_{\boldsymbol{Z}_2} \oplus \langle v_{15}\rangle_{\boldsymbol{Z}}$$

as a graded ring, where 1 is the unit, $\deg v_8 = 8$ and $\deg v_{15} = 15$. Since the homogeneous spaces $\mathrm{Spin}(8)/\mathrm{Spin}(7)$ and $\mathrm{Spin}(7)/G_2$ are homeomorphic to the 7-dimensional sphere S^7, for any module M we obtain the following two Gysin exact sequences associated to $(F_4/\mathrm{Spin}(7), p, F_4/\mathrm{Spin}(8), \mathrm{Spin}(8)/\mathrm{Spin}(7), \mathrm{Spin}(8))$ and $(F_4/G_2, p, F_4/\mathrm{Spin}(7), \mathrm{Spin}(7)/G_2, \mathrm{Spin}(7))$:

(2.10) $$\cdots \to \mathrm{H}^{p-8}(F_4/\mathrm{Spin}(8)\,;\ M) \xrightarrow{d_8} \mathrm{H}^p(F_4/\mathrm{Spin}(8)\,;\ M) \xrightarrow{p^*} \mathrm{H}^p(F_4/\mathrm{Spin}(7)\,;\ M) \to \mathrm{H}^{p-7}(F_4/\mathrm{Spin}(8)\,;\ M) \to \cdots,$$

(2.11) $$\cdots \to \mathrm{H}^{p-8}(F_4/\mathrm{Spin}(7)\,;\ M) \xrightarrow{d_8} \mathrm{H}^p(F_4/\mathrm{Spin}(7)\,;\ M) \xrightarrow{p^*} \mathrm{H}^p(F_4/G_2\,;\ M) \to \mathrm{H}^{p-7}(F_4/\mathrm{Spin}(7)\,;\ M) \to \cdots.$$

By the Serre spectral sequence, (2.1) and (2.9), we have

(2.12) $$\mathrm{H}^p(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \begin{cases} \boldsymbol{Z} & \text{for } p = 0, 23, 31, \\ 0 & \text{for } p \neq 0, 8, 15, 16, 23, 24, 31. \end{cases}$$

For $p = 8$, we have

(2.13) $$\mathrm{H}^8(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z} \quad \text{or} \quad \boldsymbol{Z} \oplus \boldsymbol{Z}_2.$$

By (2.4) and (2.11) for $M = \boldsymbol{Z}$, we see that

(2.14) $$\mathrm{H}^8(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}.$$

Furthermore we can choose a generator z_8 of $\mathrm{H}^8(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z})$ such that (2.6) holds.

For $p = 24$, we have

(2.15) $$\mathrm{H}^{24}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong 0 \quad \text{or} \quad \boldsymbol{Z}_2.$$

By (2.1) and (2.9), we obtain the following Serre exact sequence associated to $(F_4/\mathrm{Spin}(7), p, F_4/\mathrm{Spin}(9), \mathrm{Spin}(9)/\mathrm{Spin}(7), \mathrm{Spin}(9))$:

(2.16) $$\mathrm{H}^7(\mathrm{Spin}(9)/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \to \mathrm{H}^8(F_4/\mathrm{Spin}(9)\,;\ \boldsymbol{Z}) \xrightarrow{p^*} \mathrm{H}^8(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \xrightarrow{i^*} \mathrm{H}^8(\mathrm{Spin}(9)/\mathrm{Spin}(7)\,;\ \boldsymbol{Z})$$

$$\to H^9(F_4/\mathrm{Spin}(9)\,;\ \boldsymbol{Z}),$$

where i: Spin(9)/Spin(7) → F_4/Spin(7) is the obvious map induced by the inclusion map i: Spin(9) → F_4. By (2.1), (2.9) and (2.14), we can choose generators such that

$$p^*(x_8) = 2z_8, \tag{2.17}$$

and hence, by (2.3), we see that

$$p^*(y_8) = 2z_8. \tag{2.18}$$

Therefore, by (2.18), we have

$$p^*(y_8{}^2y_8') = 4z_8{}^2p^*(y_8'). \tag{2.19}$$

Hence, by (2.2) and (2.10) for $M = \boldsymbol{Z}$, we see that $H^{24}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z})$ has no 2-torsion submodules, and so, by (2.15), we see that

$$H^{24}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) = 0. \tag{2.20}$$

For $p = 15, 16$, there are six possibilities as follows:

(2.21) $H^{15}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) = 0$, $H^{16}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) = 0$,
(2.22) $H^{15}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) = 0$, $H^{16}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}_2$,
(2.23) $H^{15}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}$, $H^{16}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}$,
(2.24) $H^{15}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}$, $H^{16}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}\oplus\boldsymbol{Z}_2$,
(2.25) $H^{15}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) = 0$, $H^{16}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}_k$ for some k,
(2.26) $H^{15}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) = 0$, $H^{16}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}_k\oplus\boldsymbol{Z}_2$ for some even number k.

By (2.4), (2.11) for $M = \boldsymbol{Z}$ and (2.12), there are no possibilities for (2.21) and (2.22). By (2.18), we have

$$p^*(y_8{}^2) = 4z_8{}^2, \tag{2.27}$$
$$p^*(y_8y_8') = 2z_8p^*(y_8'). \tag{2.28}$$

Therefore, by (2.2), (2.10) for $M = \boldsymbol{Z}$ and (2.12), $H^{16}(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z})$ is not a free module and has no 2-torsion submodules, and hence, there are no possibilities for (2.23), (2.24) and (2.26). Hence (2.25) holds, and, by (2.4) and (2.11) for $M = \boldsymbol{Z}$, we can choose a generator u_{23} in $H^{23}(F_4/G_2\,;\ \boldsymbol{Z})$ such that (2.7) holds. By (2.4), (2.11) for $M = \boldsymbol{Z}$ and (2.12), we see that k is divisible by 3.

Assume that $k > 3$. By (2.4) and (2.11) for $M = \boldsymbol{Z}_k$, it holds that

$$H^*(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}_k) \cong \boldsymbol{Z}_k[\bar{z}_8]/(\bar{z}_8{}^3)\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_k[\bar{z}_{15}]/(\bar{z}_{15}{}^2) \tag{2.29}$$

as a ring, where 1 is the unit, deg $\bar{z}_8 = 8$ and deg $\bar{z}_{15} = 15$, since it holds that

(2.30) $\mathrm{H}^*(F_4/G_2;\ \boldsymbol{Z}_k)$
$$\cong (\langle 1\rangle_{Z_k}\oplus\langle \bar{u}_7\rangle_{Z_3}\oplus\langle \bar{u}_8\rangle_{Z_3}\oplus\langle \bar{u}_7\bar{u}_8\rangle_{Z_3}\oplus\langle \bar{u}_8{}^2\rangle_{Z_3}\oplus\langle \bar{u}_7\bar{u}_8{}^2\rangle_{Z_k})\otimes_Z\langle \bar{u}_{15}\rangle_{Z_k}$$

as a ring, where 1 is the unit, deg $\bar{u}_7 = 7$, deg $\bar{u}_8 = 8$ and deg $\bar{u}_{15} = 15$. Furthermore we can choose generators such that

(2.31) $$p^*(\bar{z}_8) = \bar{u}_8,$$
(2.32) $$p^*(\bar{z}_{15}) = \bar{u}_{15}.$$

Let $\beta:\ \mathrm{H}^p(F_4/\mathrm{Spin}(7);\ \boldsymbol{Z}_k)\rightarrow \mathrm{H}^{p+1}(F_4/\mathrm{Spin}(7);\ \boldsymbol{Z}_k)$ and $\beta:\ \mathrm{H}^p(F_4/G_2;\ \boldsymbol{Z}_k)\rightarrow \mathrm{H}^{p+1}(F_4/G_2;\ \boldsymbol{Z}_k)$ be the Bockstein operations associated to the exact sequence

(2.33) $$0\rightarrow \boldsymbol{Z}_k \xrightarrow{k\cdot} \boldsymbol{Z}_{k^2}\rightarrow \boldsymbol{Z}_k\rightarrow 0.$$

We see that

(2.34) $$\beta(\bar{z}_{15}) = \bar{z}_8{}^2,$$
(2.35) $$\beta(\bar{u}_7) = \bar{u}_8.$$

Consider the following commutative diagram:

(2.36) $$\begin{array}{ccc} \mathrm{H}^{15}(F_4/\mathrm{Spin}(7);\ \boldsymbol{Z}_k) & \xrightarrow{p^*} & \mathrm{H}^{15}(F_4/G_2;\ \boldsymbol{Z}_k) \\ \beta\downarrow & & \beta\downarrow \\ \mathrm{H}^{16}(F_4/\mathrm{Spin}(7);\ \boldsymbol{Z}_k) & \xrightarrow{p^*} & \mathrm{H}^{16}(F_4/G_2;\ \boldsymbol{Z}_k). \end{array}$$

By (2.31), (2.32) and (3.34), we see that

(3.37) $$\beta(\bar{u}_{15}) = \bar{u}_8{}^2.$$

Then, by (2.35) and (2.37), we have

(2.38) $$\beta(\bar{u}_7\bar{u}_{15}) = \bar{u}_8\bar{u}_{15} - \bar{u}_7\bar{u}_8{}^2.$$

This contradicts (2.30) and the fact that the Bockstein operation $\beta:\ \mathrm{H}^{22}(F_4/G_2;\ \boldsymbol{Z}_k)\rightarrow \mathrm{H}^{23}(F_4/G_2;\ \boldsymbol{Z}_k)$ is a homomorphism. Hence $k = 3$, and so we have (2.5). Thus the proof is complete.

Here we establish some notation. Let X be any topological space. We denote by KO(X) and K(X) the KO- and K-ring of X respectively; they are the Grothendieck rings of classes of real and complex vector bundles over X respectively. For any real vector bundle ξ over X, we denote by $\boldsymbol{c}(\xi)$ the complex vector bundle $\xi\otimes_R 1_C$ over X, where 1_C is the trivial complex vector bundle over X of degree 1. Then, as is well-known, $\boldsymbol{c}$ defines a ring homomorphism

$$c: \mathrm{KO}(X) \to \mathrm{K}(X)$$

which is called complexification.

Let G be any Lie group and H any closed subgroup of G. Furthermore let $\theta: H \to \mathrm{GL}(n, \boldsymbol{R})$ be any real representation of H of degree n, where $\mathrm{GL}(n, \boldsymbol{R})$ is the general linear group over the field $\boldsymbol{R}$. Then we denote by $\alpha_{(G,H)}(\theta)$ the real vector bundle $(G \times_H \boldsymbol{R}^n, p', G/H, \boldsymbol{R}^n, \mathrm{GL}(n, \boldsymbol{R}))$ of degree n over the homogeneous space G/H associated to the principal H-bundle $(G, p, G/H, H)$ via $\theta: H \to \mathrm{GL}(n, \boldsymbol{R})$. We denote by $\mathrm{RO}(H)$ the real representation ring of H; it is the Grothendieck ring of classes of real representations of H. Then, as is well-known, $\alpha_{(G,H)}$ defines a ring homomorphism

$$\alpha_{(G,H)}: \mathrm{RO}(H) \to \mathrm{KO}(G/H)$$

which is called α-construction.

We can show the following lemma by an elementary way:

Lemma 2.2. *If H_1 and H_2 are closed subgroups of a Lie group G with $H_1 \subset H_2$, then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{RO}(H_2) & \xrightarrow{i^*} & \mathrm{RO}(H_1) \\ \alpha_{(G,H_2)} \downarrow & & \alpha_{(G,H_1)} \downarrow \\ \mathrm{KO}(G/H_2) & \xrightarrow{p^*} & \mathrm{KO}(G/H_1), \end{array} \tag{2.39}$$

where $i^*: \mathrm{RO}(H_2) \to \mathrm{RO}(H_1)$ is the induced homomorphism of the real representation rings by the inclusion homomorphism $i: H_1 \to H_2$ and $p^*: \mathrm{KO}(G/H_2) \to \mathrm{KO}(G/H_1)$ is the induced homomorphism of the KO*-rings by the obvious projection $p: G/H_1 \to G/H_2$.*

For the homogeneous space $F_4/\mathrm{Spin}(8)$, we obtain the following theorem:

Theorem 2.3. *As a graded ring*

$$\mathrm{H}^*(F_4/\mathrm{Spin}(8); \boldsymbol{Z}) \cong \boldsymbol{Z}[y_8, y_8']/(y_8^3, y_8^2 + y_8 y_8' + y_8'^2), \tag{2.40}$$

where $\deg y_8 = 8$ *and* $\deg y_8' = 8$. *Furthermore, if we denote by $p: F_4/\mathrm{Spin}(7) \to F_4/\mathrm{Spin}(8)$ the obvious projection, then we can choose generators such that*

$$p^*(y_8) = 2z_8, \tag{2.41}$$
$$p^*(y_8') = -z_8. \tag{2.42}$$

Proof. We have already obtained (2.41) in the proof of Theorem 2.1, that is, (2.18). By (2.2), (2.5), (2.10) and (2.18), we can choose a generator y_8' in $\mathrm{H}^8(F_4/\mathrm{Spin}(8)\,;\,\boldsymbol{Z})$ such that (2.42) holds, furthermore,

$$d_8(1) = y_8 + 2y_8'. \tag{2.43}$$

By (2.1) and (2.3), we have

$$y_8{}^3 = p^*(x_8{}^3) = 0. \tag{2.44}$$

We denote by $p_{\mathrm{Spin}(8)}:\mathrm{Spin}(8)\to\mathrm{SO}(8)$ and $p_{\mathrm{Spin}(9)}:\mathrm{Spin}(9)\to\mathrm{SO}(9)$ the covering group homomorphisms respectively. Since $F_4/\mathrm{Spin}(7)$ is homeomorphic to $F_4\times_{\mathrm{Spin}(8)}(\mathrm{Spin}(8)/\mathrm{Spin}(7))$, and hence, to $F_4\times_{\mathrm{Spin}(8)}S^7$, the fiber bundle $(F_4/\mathrm{Spin}(7), p, F_4/\mathrm{Spin}(8), \mathrm{Spin}(8)/\mathrm{Spin}(7), \mathrm{Spin}(8))$ is equivalence to the sphere bundle $(F_4\times_{\mathrm{Spin}(8)}S^7, p, F_4/\mathrm{Spin}(8), S^7, \mathrm{SO}(8))$ associated to the real vector bundle $\alpha_{(F_4,\mathrm{Spin}(8))}(p_{\mathrm{Spin}(8)})$ as a fiber space. Therefore, if we denote by e and e' the Euler classes of $(F_4/\mathrm{Spin}(7), p, F_4/\mathrm{Spin}(8), \mathrm{Spin}(8)/\mathrm{Spin}(7), \mathrm{Spin}(8))$ and $\alpha_{(F_4,\mathrm{Spin}(8))}(p_{\mathrm{Spin}(8)})$ respectively, then we have

$$e = e' \tag{2.45}$$

in $\mathrm{H}^8(F_4/\mathrm{Spin}(8)\,;\,\boldsymbol{Z})$. Let $i^*:\mathrm{RO}(\mathrm{Spin}(9))\to\mathrm{RO}(\mathrm{Spin}(8))$ be the induced ring homomorphism of the real representation rings by the inclusion homomorphism $i:\mathrm{Spin}(8)\to\mathrm{Spin}(9)$. We see that

$$i^*(p_{\mathrm{Spin}(9)}) = 1 + p_{\mathrm{Spin}(8)} \tag{2.46}$$

in $\mathrm{RO}(\mathrm{Spin}(8))$, where 1 is the trivial real representation of $\mathrm{Spin}(8)$ of degree 1. Therefore, by (2.46), the naturality of the complexification $\boldsymbol{c}$ and Lemma 2.2, we have

$$\begin{aligned} p^*\boldsymbol{c}\alpha_{(F_4,\mathrm{Spin}(9))}(p_{\mathrm{Spin}(9)}) &= \boldsymbol{c}p^*\alpha_{(F_4,\mathrm{Spin}(9))}(p_{\mathrm{Spin}(9)}) \\ &= \boldsymbol{c}\alpha_{(F_4,\mathrm{Spin}(8))}i^*(p_{\mathrm{Spin}(9)}) \\ &= \boldsymbol{c}(1+\alpha_{(F_4,\mathrm{Spin}(8))}(p_{\mathrm{Spin}(8)})) \\ &= 1_C + \boldsymbol{c}\alpha_{(F_4,\mathrm{Spin}(8))}(p_{\mathrm{Spin}(8)}) \end{aligned} \tag{2.47}$$

in $\mathrm{K}(F_4/\mathrm{Spin}(8))$, where 1 and 1_C are the trivial real and complex vector bundles of degree 1 respectively. Let c_i and p_i be the Chern and Pontryagin classes of complex and real vector bundles respectively. Here there is an integer k such that

$$c_8\boldsymbol{c}\alpha_{(F_4,\mathrm{Spin}(9))}(p_{\mathrm{Spin}(9)}) = kx_8{}^2 \tag{2.48}$$

in $\mathrm{H}^{16}(F_4/\mathrm{Spin}(9)\,;\,\boldsymbol{Z})$. Since $\alpha_{(F_4,\mathrm{Spin}(8))}(p_{\mathrm{Spin}(8)})$ is orientable and of even number degree, by (2.3), (2.43), (2.45), (2.47), (2.48) and the naturality of the Chern classes, we have

$$
\begin{aligned}
ky_8^2 &= p^*(kx_8^2) \\
&= p^* c_8 ca_{(F_4,\mathrm{Spin}(9))}(p_{\mathrm{Spin}(9)}) \\
&= c_8 p^* ca_{(F_4,\mathrm{Spin}(9))}(p_{\mathrm{Spin}(9)}) \\
&= c_8(1_C + ca_{(F_4,\mathrm{Spin}(8))}(p_{\mathrm{Spin}(8)})) \\
&= c_8 ca_{(F_4,\mathrm{Spin}(8))}(p_{\mathrm{Spin}(8)}) \\
&= p_4 a_{(F_4,\mathrm{Spin}(8))}(p_{\mathrm{Spin}(8)}) \\
&= e'^2 = e^2 = d_8(1)^2 \\
&= y_8^2 + 4y_8y_8' + 4y_8'^2.
\end{aligned}
\tag{2.49}
$$

Therefore, we have

$$
y_8'^2 = \frac{k-1}{4} y_8^2 - y_8 y_8', \tag{2.50}
$$

furthermore $k-1$ is divisible by 4.

By (2.2), (2.5), (2.10) for $M = \boldsymbol{Z}$, (2.18), (2.42), (2.43) and (2.50), we have

$$
p^*(y_8^2) = 4z_8^2 = z_8^2, \tag{2.51}
$$

$$
p^*(y_8y_8') = -2z_8^2 = z_8^2, \tag{2.52}
$$

$$
d_8(y_8) = y_8^2 + 2y_8y_8', \tag{2.53}
$$

$$
d_8(y_8') = \frac{k-1}{2} y_8^2 - y_8y_8'. \tag{2.54}
$$

Since we see that Coker $d_8 \cong \boldsymbol{Z}_3$ as a module, we have $k = \pm 3$. Since we have already seen that $k-1$ is divisible by 4, we have $k = -3$. Therfore, by (2.50), we see that $y_8'^2 = -y_8^2 - y_8y_8'$. Thus the proof is complete.

3. The integral cohomology rings of $E_6/\mathrm{Spin}(m)$. In this section, we determine the cohomology ring structure of the homogeneous space $E_6/\mathrm{Spin}(m)$ for $m = 7, 8, 9$.

For the homogeneous space E_6/F_4, S. Araki showed that

$$
\mathrm{H}^*(E_6/F_4\,;\ \boldsymbol{Z}) \cong \wedge_Z(s_9, s_{17}) \tag{3.1}
$$

as a graded ring, where deg $s_9 = 9$ and deg $s_{17} = 17$; see [1, Proposition 2.5]. For the homogeneous space $E_6/\mathrm{Spin}(10)$, L. Conlon showed that

$$
\mathrm{H}^*(E_6/\mathrm{Spin}(10)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}[t_8, t_{17}]/(t_8^3, t_{17}^2) \tag{3.2}
$$

as a graded ring, where deg $t_8 = 8$ and deg $t_{17} = 17$; see [3, Corollary 4].

For the homogeneous space $E_6/\mathrm{Spin}(9)$, we obtain the following theorem:

Theorem 3.1. *As a graded ring*

(3.3) $$\mathrm{H}^*(E_6/\mathrm{Spin}(9)\,;\,\boldsymbol{Z}) \cong \mathrm{H}^*(F_4/\mathrm{Spin}(9)\,;\,\boldsymbol{Z})\otimes_{\boldsymbol{Z}}\wedge_{\boldsymbol{Z}}(x_9,\,x_{17}),$$

where $\deg x_9 = 9$ *and* $\deg x_{17} = 17$. *Furthermore, if we denote by* p: $E_6/\mathrm{Spin}(9) \to E_6/F_4$ *and* p: $E_6/\mathrm{Spin}(9) \to E_6/\mathrm{Spin}(10)$ *the obvious projections respectively, then we can choose generators such that*

(3.4) $$p^*(s_9) = x_9,$$
(3.5) $$p^*(t_8) = x_8,$$
(3.6) $$p^*(t_{17}) = x_{17}.$$

Proof. Since the homogeneous space $\mathrm{Spin}(10)/\mathrm{Spin}(9)$ is homeomorphic to the 9-dimensional sphere S^9, we obtain the following Gysin exact sequence associated to $(E_6/\mathrm{Spin}(9),\,p,\,E_6/\mathrm{Spin}(10),\,\mathrm{Spin}(10)/\mathrm{Spin}(9),\,\mathrm{Spin}(10))$:

(3.7) $$\cdots \to \mathrm{H}^{p-10}(E_6/\mathrm{Spin}(10)\,;\,\boldsymbol{Z}) \xrightarrow{d_{10}} \mathrm{H}^p(E_6/\mathrm{Spin}(10)\,;\,\boldsymbol{Z}) \xrightarrow{p^*} \mathrm{H}^p(E_6/\mathrm{Spin}(9)\,;\,\boldsymbol{Z}) \to \mathrm{H}^{p-9}(E_6/\mathrm{Spin}(10)\,;\,\boldsymbol{Z}) \to \cdots.$$

By (3.2) and (3.7), we see (3.3), (3.5) and (3.6).

By (2.1) and (3.1), we obtain the following Serre exact sequence associated to $(E_6/\mathrm{Spin}(9),\,p,\,E_6/F_4,\,F_4/\mathrm{Spin}(9),\,F_4)$:

(3.8) $$\mathrm{H}^8(E_6/F_4\,;\,\boldsymbol{Z}) \xrightarrow{p^*} \mathrm{H}^8(E_6/\mathrm{Spin}(9)\,;\,\boldsymbol{Z}) \xrightarrow{i^*} \mathrm{H}^8(F_4/\mathrm{Spin}(9)\,;\,\boldsymbol{Z}) \to \mathrm{H}^9(E_6/F_4\,;\,\boldsymbol{Z}) \xrightarrow{p^*} \mathrm{H}^9(E_6/\mathrm{Spin}(9)\,;\,\boldsymbol{Z}) \xrightarrow{i^*} \mathrm{H}^9(F_4/\mathrm{Spin}(9)\,;\,\boldsymbol{Z}),$$

where i: $F_4/\mathrm{Spin}(9) \to E_6/\mathrm{Spin}(9)$ is the obvious map induced by the inclusion map i: $F_4 \to E_6$. Therefore we see (3.4). Thus the proof is complete.

For the homogeneous space $E_6/\mathrm{Spin}(8)$, we obtain the following theorem:

Theorem 3.2. *As a graded ring*

(3.9) $$\mathrm{H}^*(E_6/\mathrm{Spin}(8)\,;\,\boldsymbol{Z}) \cong \mathrm{H}^*(F_4/\mathrm{Spin}(8)\,;\,\boldsymbol{Z})\otimes_{\boldsymbol{Z}}\wedge_{\boldsymbol{Z}}(y_9,\,y_{17}),$$

where $\deg y_9 = 9$ *and* $\deg y_{17} = 17$. *Furthermore, if we denote by* p: $E_6/\mathrm{Spin}(8) \to E_6/\mathrm{Spin}(9)$ *the obvious projection, then we can choose generators such that*

(3.10) $$p^*(x_8) = y_8,$$
(3.11) $$p^*(x_9) = y_9,$$
(3.12) $$p^*(x_{17}) = y_{17}.$$

Proof. We consider the Serre spectral sequence $(E_*^{*,*}, d_*)$ associated to the fiber bundle $(E_6/\mathrm{Spin}(8), p, E_6/\mathrm{Spin}(10), \mathrm{Spin}(10)/\mathrm{Spin}(8), \mathrm{Spin}(10))$, whose E_2-term is as follows:

$$E_2^{p,q} \cong \mathrm{H}^p(E_6/\mathrm{Spin}(10);\ \mathrm{H}^q(\mathrm{Spin}(10)/\mathrm{Spin}(8);\ \boldsymbol{Z})) \tag{3.13}$$

as a module. Since the homogeneous space $\mathrm{Spin}(10)/\mathrm{Spin}(8)$ is homeomorphic to the Stiefel manifold $\boldsymbol{R}V_{10,2}$, it holds that

$$\mathrm{H}^*(\mathrm{Spin}(10)/\mathrm{Spin}(8);\ \boldsymbol{Z}) \cong \boldsymbol{Z}[w_8, w_9]/(w_8{}^2, w_9{}^2) \tag{3.14}$$

as a graded ring, where $\deg w_8 = 8$ and $\deg w_9 = 9$. Since the homogeneous space $\mathrm{Spin}(9)/\mathrm{Spin}(8)$ is homeomorphic to the 8-dimensional sphere S^8, we obtain the following Gysin exact sequence associated to $(E_6/\mathrm{Spin}(8), p, E_6/\mathrm{Spin}(9), \mathrm{Spin}(9)/\mathrm{Spin}(8), \mathrm{Spin}(9))$:

$$\begin{aligned} \cdots \to \mathrm{H}^{p-9}(E_6/\mathrm{Spin}(9);\ \boldsymbol{Z}) &\xrightarrow{d_9} \mathrm{H}^p(E_6/\mathrm{Spin}(9);\ \boldsymbol{Z}) \\ &\xrightarrow{p^*} \mathrm{H}^p(E_6/\mathrm{Spin}(8);\ \boldsymbol{Z}) \to \mathrm{H}^{p-8}(E_6/\mathrm{Spin}(9);\ \boldsymbol{Z}) \to \cdots \end{aligned} \tag{3.15}$$

By the Serre spectral sequence, (3.2) and (3.14), it holds that

$$\mathrm{H}^p(E_6/\mathrm{Spin}(8);\ \boldsymbol{Z}) \cong \begin{cases} \boldsymbol{Z} & \text{for } p = 0, 9, \\ \boldsymbol{Z}\oplus\boldsymbol{Z} & \text{for } p = 8, \\ 0 & \text{for } p = 1, \cdots, 7 \end{cases} \tag{3.16}$$

as modules. By (2.1), (2.40) and (3.1), we obtain the following two Serre exact sequences associated to $(E_6/\mathrm{Spin}(9), p, E_6/F_4, F_4/\mathrm{Spin}(9), F_4)$ and $(E_6/\mathrm{Spin}(8), p, E_6/F_4, F_4/\mathrm{Spin}(8), F_4)$:

$$\begin{aligned} \mathrm{H}^8(E_6/F_4;\ \boldsymbol{Z}) &\xrightarrow{p^*} \mathrm{H}^8(E_6/\mathrm{Spin}(9);\ \boldsymbol{Z}) \xrightarrow{i^*} \mathrm{H}^8(F_4/\mathrm{Spin}(9);\ \boldsymbol{Z}) \\ \to \mathrm{H}^9(E_6/F_4;\ \boldsymbol{Z}) &\xrightarrow{p^*} \mathrm{H}^9(E_6/\mathrm{Spin}(9);\ \boldsymbol{Z}) \xrightarrow{i^*} \mathrm{H}^9(F_4/\mathrm{Spin}(9);\ \boldsymbol{Z}), \end{aligned} \tag{3.17}$$

$$\begin{aligned} \mathrm{H}^8(E_6/F_4;\ \boldsymbol{Z}) &\xrightarrow{p^*} \mathrm{H}^8(E_6/\mathrm{Spin}(8);\ \boldsymbol{Z}) \xrightarrow{i^*} \mathrm{H}^8(F_4/\mathrm{Spin}(8);\ \boldsymbol{Z}) \\ \to \mathrm{H}^9(E_6/F_4;\ \boldsymbol{Z}) &\xrightarrow{p^*} \mathrm{H}^9(E_6/\mathrm{Spin}(8);\ \boldsymbol{Z}) \xrightarrow{i^*} \mathrm{H}^9(F_4/\mathrm{Spin}(8);\ \boldsymbol{Z}), \end{aligned} \tag{3.18}$$

where $i: F_4/\mathrm{Spin}(9) \to E_6/\mathrm{Spin}(9)$ and $i: F_4/\mathrm{Spin}(8) \to E_6/\mathrm{Spin}(8)$ are the obvious maps induced by the inclusion map $i: F_4 \to E_6$ respectively. By (2.1), (2.40), (3.1), (3.3) and (3.16), we see that the maps

$$\begin{aligned} i^*&: \mathrm{H}^8(E_6/\mathrm{Spin}(9);\ \boldsymbol{Z}) \to \mathrm{H}^8(F_4/\mathrm{Spin}(9);\ \boldsymbol{Z}), \\ i^*&: \mathrm{H}^8(E_6/\mathrm{Spin}(8);\ \boldsymbol{Z}) \to \mathrm{H}^8(F_4/\mathrm{Spin}(8);\ \boldsymbol{Z}) \end{aligned}$$

are isomorphisms. Furthermore the following diagram commutes:

$$
(3.19)\qquad
\begin{array}{ccccc}
\cdots \to \mathrm{H}^{p-9}(E_6/\mathrm{Spin}(9);\ \boldsymbol{Z}) & \xrightarrow{d_9} & \mathrm{H}^{p}(E_6/\mathrm{Spin}(9);\ \boldsymbol{Z}) \\
\downarrow i^* & & \downarrow i^* \\
\cdots \to \mathrm{H}^{p-9}(F_4/\mathrm{Spin}(9);\ \boldsymbol{Z}) & \xrightarrow{d_9} & \mathrm{H}^{p}(F_4/\mathrm{Spin}(9);\ \boldsymbol{Z})
\end{array}
$$

$$
\begin{array}{ccccc}
\xrightarrow{p^*} \mathrm{H}^{p}(E_6/\mathrm{Spin}(8);\ \boldsymbol{Z}) & \to & \mathrm{H}^{p-8}(E_6/\mathrm{Spin}(9);\ \boldsymbol{Z}) \to \cdots \\
\downarrow i^* & & \downarrow i^* \\
\xrightarrow{p^*} \mathrm{H}^{p}(F_4/\mathrm{Spin}(8);\ \boldsymbol{Z}) & \to & \mathrm{H}^{p-8}(F_4/\mathrm{Spin}(9);\ \boldsymbol{Z}) \to \cdots.
\end{array}
$$

Therefore, by (2.1), (2.3) and (2.40), we see (3.9), (3.10), (3.11) and (3.12). Thus the proof is complete.

For the homogeneous space $E_6/\mathrm{Spin}(7)$, we obtain the following theorem:

Theorem 3.3. *As a graded ring*

$$(3.20)\qquad \mathrm{H}^*(E_6/\mathrm{Spin}(7);\ \boldsymbol{Z}) \cong \mathrm{H}^*(F_4/\mathrm{Spin}(7);\ \boldsymbol{Z})\otimes_{\boldsymbol{Z}}\Lambda_{\boldsymbol{Z}}(z_9, z_{17}),$$

where $\deg z_9 = 9$ *and* $\deg z_{17} = 17$. *Furthermore, if we denote by* p: $E_6/\mathrm{Spin}(7) \to E_6/\mathrm{Spin}(8)$ *the obvious projection, then we can choose generators such that*

$$(3.21)\qquad p^*(y_8) = 2z_8,$$
$$(3.22)\qquad p^*(y_8') = -z_8,$$
$$(3.23)\qquad p^*(y_9) = z_9,$$
$$(3.24)\qquad p^*(y_{17}) = z_{17}.$$

Proof. We consider the Serre spectral sequence $(E_*^{*,*}, d_*)$ associated to the fiber bundle $(E_6/\mathrm{Spin}(7), p, E_6/\mathrm{Spin}(9), \mathrm{Spin}(9)/\mathrm{Spin}(7), \mathrm{Spin}(9))$, whose E_2-term is as follows:

$$(3.25)\qquad E_2^{p,q} \cong \mathrm{H}^p(E_6/\mathrm{Spin}(9);\ \mathrm{H}^q(\mathrm{Spin}(9)/\mathrm{Spin}(7);\ \boldsymbol{Z}))$$

as a module. Since the homogeneous space $\mathrm{Spin}(8)/\mathrm{Spin}(7)$ is homeomorphic to the 7-dimensional sphere S^7, we obtain the following Gysin exact sequence associated to $(E_6/\mathrm{Spin}(7), p, E_6/\mathrm{Spin}(8), \mathrm{Spin}(8)/\mathrm{Spin}(7), \mathrm{Spin}(8))$:

$$(3.26)\qquad \cdots \to \mathrm{H}^{p-8}(E_6/\mathrm{Spin}(8);\ \boldsymbol{Z}) \xrightarrow{d_8} \mathrm{H}^p(E_6/\mathrm{Spin}(8);\ \boldsymbol{Z})$$
$$\xrightarrow{p^*} \mathrm{H}^p(E_6/\mathrm{Spin}(7);\ \boldsymbol{Z}) \to \mathrm{H}^{p-7}(E_6/\mathrm{Spin}(8);\ \boldsymbol{Z}) \to \cdots.$$

By the Serre spectral sequence, (2.9) and (3.3), it holds that

(3.27) $$\mathrm{H}^p(E_6/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \begin{cases} \boldsymbol{Z} & \text{for } p=0,9, \\ 0 & \text{for } p=1,\cdots,7, \\ \boldsymbol{Z} \text{ or } \boldsymbol{Z}\oplus\boldsymbol{Z}_2 & \text{for } p=8 \end{cases}$$

as modules. By (2.5) and (3.1), we obtain the following Serre exact sequence associated to $(E_6/\mathrm{Spin}(7), p, E_6/F_4, F_4/\mathrm{Spin}(7), F_4)$:

(3.28) $$\mathrm{H}^8(E_6/F_4\,;\ \boldsymbol{Z}) \xrightarrow{p^*} \mathrm{H}^8(E_6/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \xrightarrow{i^*} \mathrm{H}^8(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z})$$
$$\to \mathrm{H}^9(E_6/F_4\,;\ \boldsymbol{Z}) \xrightarrow{p^*} \mathrm{H}^9(E_6/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \xrightarrow{i^*} \mathrm{H}^9(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}),$$

where $i: F_4/\mathrm{Spin}(7) \to E_6/\mathrm{Spin}(7)$ is the obvious map induced by the inclusion map $i: F_4 \to E_6$. By (2.5), (3.1) and (3.27), we see that the map

$$i^*: \mathrm{H}^8(E_6/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \to \mathrm{H}^8(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z})$$

is an isomorphism, and hence

(3.29) $$\mathrm{H}^8(E_6/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) \cong \boldsymbol{Z}.$$

In the proof of Theorem 3.2, we have already obtained that

$$i^*: \mathrm{H}^8(E_6/\mathrm{Spin}(8)\,;\ \boldsymbol{Z}) \to \mathrm{H}^8(F_4/\mathrm{Spin}(8)\,;\ \boldsymbol{Z})$$

is the isomorphism. Furthermore the following diagram commutes:

(3.30) $$\begin{array}{ccccc} \cdots \to \mathrm{H}^{p-8}(E_6/\mathrm{Spin}(8)\,;\ \boldsymbol{Z}) & \xrightarrow{d_8} & \mathrm{H}^p(E_6/\mathrm{Spin}(8)\,;\ \boldsymbol{Z}) \\ i^*\downarrow & & i^*\downarrow \\ \cdots \to \mathrm{H}^{p-8}(F_4/\mathrm{Spin}(8)\,;\ \boldsymbol{Z}) & \xrightarrow{d_8} & \mathrm{H}^p(F_4/\mathrm{Spin}(8)\,;\ \boldsymbol{Z}) \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{p^*} \mathrm{H}^p(E_6/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) & \to & \mathrm{H}^{p-7}(E_6/\mathrm{Spin}(8)\,;\ \boldsymbol{Z}) \to \cdots \\ i^*\downarrow & & i^*\downarrow \\ \xrightarrow{p^*} \mathrm{H}^p(F_4/\mathrm{Spin}(7)\,;\ \boldsymbol{Z}) & \to & \mathrm{H}^{p-7}(F_4/\mathrm{Spin}(8)\,;\ \boldsymbol{Z}) \to \cdots. \end{array}$$

Therefore, by (2.5), (2.40), (2.41), (2.42) and (3.9), we see (3.20), (3.21), (3.22), (3.23) and (3.24). Thus the proof is complete.

REFERENCES

[1] S. ARAKI: Cohomology modulo 2 of the compact exceptional groups E_6 and E_7, J. Math. Osaka City Univ. **12** (1961), 43—65.

[2] A. BOREL: Sur l'homologie et la cohomologie des groupes de Lie compacts connexes, Amer. J. Math. **76** (1954), 273—342.

[3] L. CONLON: On the topology of *EIII* and *EIV*, Proc. Amer. Math. Soc. **16** (1965), 575—581.

[4] I. YOKOTA: Exceptional Lie group F_4 and its representation rings, J. Fac. Sci. Shinshu Univ. **3** (1968), 35—60.

[5] I. YOKOTA: Simply connected compact simple Lie group $E_{6(-78)}$ of type E_6 and its involutive automorphisms, J. Math. Kyoto Univ. **20** (1980), 447—473.

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