Math. J. Okayama Univ. 34 (1992), 175—179

A NOTE ON THE LIE GROUP G_2 AS A FRAMED BOUNDARY

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1. Introduction. Let G_2 be the exceptional compact simple Lie group of rank 2. We consider the problem of finding a parallelizable compact smooth manifold whose boundary is diffeomorphic to G_2. Our candidate is the total space of the disk bundle of the canonical complex or quaternionic line bundle over a homogeneous space G_2/S, where S is a closed subgroup of G_2 that is isomorphic to the group S^1 of unit complex numbers or S^3 of unit quaternions as a Lie group.

Some motivation is provided by the problem of identifying the elements represented in the stable homotopy group of spheres by a Lie group with various framings. K. Knapp shows in the last remark of [2] that any Lie group is framed cobordant to the boundary of a parallelizable compact smooth manifold. Our aim is to further clarify this situation by giving specific framed null cobordism of G_2.

It is a pleasure to express my gratitude to Professor M. Mimura who suggested the problem and Professor N. Iwase who gave me helpful comments.

2. The Lie group G_2 as a framed boundary. We first establish some notation.

We denote by $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ the fields of real numbers, complex numbers and quaternions respectively. Furthermore $\mathfrak{C}$ is the Cayley algebra that is an 8-dimensional $\boldsymbol{R}$-module with additive basis $e_0, e_1, \cdots, e_7$, and the ring structure is given as follows: e_0 is the unit 1, $e_i^2 = -1$ for $i \neq 0$, $e_ie_j = -e_je_i$ for $i, j \neq 0$, $i \neq j$, $e_1e_2 = e_3$, $e_2e_5 = e_7$, $e_2e_4 = -e_6$ and so on. We consider $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ as

$$\boldsymbol{R} = \boldsymbol{R}1,$$
$$\boldsymbol{C} = \boldsymbol{R}1 \oplus \boldsymbol{R}e_1$$

and

$$\boldsymbol{H} = \boldsymbol{R}1 \oplus \boldsymbol{R}e_1 \oplus \boldsymbol{R}e_2 \oplus \boldsymbol{R}e_3$$

respectively. Then the following inclusion holds:

$$\boldsymbol{R} \subset \boldsymbol{C} \subset \boldsymbol{H} \subset \mathfrak{C}.$$

As is well-known, G_2 is given as the group of automorphisms of $\mathfrak{C}$:

$$G_2 = \{x \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}, \mathfrak{C}) \mid x(uv) = x(u)x(v) \text{ for any } u, v \in \mathfrak{C}\},$$

where $\mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{C}, \mathfrak{C})$ is the group of all $\boldsymbol{R}$-isomorphisms from $\mathfrak{C}$ to $\mathfrak{C}$ itself, and the group structure is given by the composition of maps.

We denote by S^1 and S^3 the Lie groups of unit complex numbers and of unit quaternions respectively:

$$S^1 = \{a_0 + a_1 e_1 \in \boldsymbol{C} \mid a_0^2 + a_1^2 = 1\},$$
$$S^3 = \left\{\sum_{i=0}^{3} a_i e_i \in \boldsymbol{H} \mid a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1\right\}.$$

Furthermore we denote by $\sigma_1 : S^1 \to \mathrm{SO}(2)$ the canonical real representation of S^1 defined by

$$a_0 + a_1 e_1 \mapsto \begin{pmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{pmatrix}$$

and by $\sigma_3 : S^3 \to \mathrm{SO}(4)$ the canonical real representation of S^3 defined by

$$\sum_{i=0}^{3} a_i e_i \mapsto \begin{pmatrix} a_0 & -a_1 & -a_2 & a_3 \\ a_1 & a_0 & -a_3 & -a_2 \\ a_2 & a_3 & a_0 & a_1 \\ -a_3 & a_2 & -a_1 & a_0 \end{pmatrix},$$

where SO(2) and SO(4) are the special orthogonal groups.

Let S be a closed subgroup of G_2. Furthermore let $\theta : S \to \mathrm{GL}(n, \boldsymbol{R})$ be any real representation of S of degree n, where $\mathrm{GL}(n, \boldsymbol{R})$ is the general linear group over $\boldsymbol{R}$. Then we denote by $\alpha_{(G_2,S)}(\theta)$ the real vector bundle $(G_2 \times_S \boldsymbol{R}^n, p', G_2/S, \boldsymbol{R}^n, \mathrm{GL}(n, \boldsymbol{R}))$ over the homogeneous space G_2/S of degree n associated to the principal S-bundle $(G_2, p, G_2/S, S)$ via $\theta : S \to \mathrm{GL}(n, \boldsymbol{R})$. We denote by $\mathrm{RO}(S)$ the real representation ring of S and by $\mathrm{KO}(G_2/S)$ the KO-ring of G_2/S; they are the Grothendieck rings of classes of real representations of S and of real vector bundles over G_2/S respectively. Then, as is well-known, $\alpha_{(G_2,S)}$ defines a ring homomorphism $\alpha_{(G_2,S)} : \mathrm{RO}(S) \to \mathrm{KO}(G_2/S)$ which we call α-construction.

Let $f_1 : S \to S^1$ or $f_3 : S \to S^3$ be an isomorphism of the Lie groups. We denote by $\mathrm{D}(\alpha_{(G_2,S)}(\sigma_1 f_1))$ or $\mathrm{D}(\alpha_{(G_2,S)}(\sigma_3 f_3))$ the total space of the disk bundle of the real vector bundle $\alpha_{(G_2,S)}(\sigma_1 f_1)$ or $\alpha_{(G_2,S)}(\sigma_3 f_3)$ respectively. Then $\mathrm{D}(\alpha_{(G_2,S)}(\sigma_1 f_1))$ or $\mathrm{D}(\alpha_{(G_2,S)}(\sigma_3 f_3))$ is a compact smooth manifold whose boundary is diffeomorphic to G_2. These manifolds are our candidates.

For a closed subgroup S of G_2 that is isomorphic to S^1 as a Lie group, we obtain the following proposition:

Proposition 2.1. *There does not exist a closed subgroup S of G_2, isomorphic to S^1 as a Lie group, such that $\mathrm{D}(\alpha_{(G_2,S)}(\sigma_1 f_1))$ is parallelizable, where $f_1: S \to S^1$ is an isomorphism of the Lie groups.*

Proof. Let S be any closed subgroup of G_2. If we denote by $i: S \to G_2$ the inclusion map, then, for the isomorphism $f_1: S \to S^1$ of the Lie groups, the composite map $if_1^{-1}: S^1 \to G_2$ represents an element $[if_1^{-1}]$ in the fundamental group $\pi_1(G_2)$ of G_2. It is well-known that G_2 is simply connected, that is, $\pi_1(G_2) = 0$; see [5, Theorem 5.4]. Therefore we have

$$[if_1^{-1}] = 0 = 2\cdot 0,$$

and so the element $[if_1^{-1}]$ in $\pi_1(G_2)$ is halvable. By [1, Proposition 3.1] the real vector bundle $\alpha_{(G_2,S)}(\sigma_1 f_1)$ is not stably trivial, hence by [1, Proposition 2.2 a)] the manifold $\mathrm{D}(\alpha_{(G_2,S)}(\sigma_1 f_1))$, our candidate, is not parallelizable. Thus the proof is complete.

For a closed subgroup S of G_2 that is isomorphic to S^3 as a Lie group, we obtain the following theorem:

Theorem 2.2. *There exists a closed subgroup S of G_2, isomorphic to S^3 as a Lie group, such that $\mathrm{D}(\alpha_{(G_2,S)}(\sigma_3 f_3))$ is parallelizable, where $f_3: S \to S^3$ is an isomorphism of the Lie groups.*

Proof. Let $S = \{x \in G_2 \mid x(e_1) = e_1, x(e_2) = e_2\}$, then S is a closed subgroup of G_2; see [5, Example 5.1]. We define a map $g: S^3 \to S$ by

$$\begin{aligned} g\Big(\sum_{i=0}^{3} a_i e_i\Big)\Big(\sum_{i=0}^{7} b_i e_i\Big) = {} & b_0 e_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 \\ & +(\ a_0 b_4 - a_1 b_5 - a_2 b_6 + a_3 b_7) e_4 \\ & +(\ a_1 b_4 + a_0 b_5 - a_3 b_6 - a_2 b_7) e_5 \\ & +(\ a_2 b_4 + a_3 b_5 + a_0 b_6 + a_1 b_7) e_6 \\ & +(-a_3 b_4 + a_2 b_5 - a_1 b_6 + a_0 b_7) e_7 \end{aligned}$$

for any elemests $\sum_{i=0}^{3} a_i e_i$ in S^3 and $\sum_{i=0}^{7} b_i e_i$ in the Cayley algebra $\mathfrak{C}$. Then the map $g(\sum_{i=0}^{3} a_i e_i): \mathfrak{C} \to \mathfrak{C}$ is an element in G_2 and the map $g: S^3 \to S$ is an isomorphism of the Lie groups. Here we put $f_3 = g^{-1}: S \to S^3$.

We define a map $\rho: G_2 \to \mathrm{SO}(7)$ by

$$x \mapsto (a_{i,j}(x))_{i,j=1,\cdots,7},$$

where $x(e_j)=\sum_{i=1}^{7}a_{i,j}(x)e_i$ for $j=1,\cdots,7$ and SO(7) is the special orthogonal group. Then the map $\rho: G_2\to SO(7)$ is a real representation of G_2. Since we have

$$\rho g\Big(\sum_{i=0}^{3}a_ie_i\Big)=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&a_0&-a_1&-a_2&a_3\\0&0&0&a_1&a_0&-a_3&-a_2\\0&0&0&a_2&a_3&a_0&a_1\\0&0&0&-a_3&a_2&-a_1&a_0\end{pmatrix}$$

for any element $\sum_{i=0}^{3}a_ie_i$ in S^3, we see that

$$\rho g=3\oplus\sigma_3,$$

where 3 is the trivial representation of degree 3. Let $i^*: RO(G_2)\to RO(S)$ be the ring homomorphism of the real representation rings induced by the inclusion homomorphism $i: S\to G_2$. Since we have

$$\rho i=\rho g f_3=(3\oplus\sigma_3)f_3=3\oplus\sigma_3 f_3,$$

we see that

$$i^*(\rho-3)=\rho i-3=3+\sigma_3f_3-3=\sigma_3f_3$$

in $RO(S)$. Therefore σ_3f_3 is an element in the image of $i^*: RO(G_2)\to RO(S)$. So by [1, Lemma 2.1] we have

$$\alpha_{(G_2,S)}(\sigma_3f_3)=4$$

in $KO(G_2/S)$, where 4 is the trivial real vector bundle of degree 4. So the real vector bundle $\alpha_{(G_2,S)}(\sigma_3f_3)$ is stably trivial. Hence by [1, Proposition 2.2 b)] the manifold $D(\alpha_{(G_2,S)}(\sigma_3f_3))$, our candidate, is parallelizable. Thus the proof is complete.

When we consider G_2 as a framed manifold with the left invariant framing, G_2 represents an element $[G_2]$ in the 14-stem stable homotopy group π_{14}^S of the spheres via the Pontrjagin-Thom construction, since G_2 is of dimension 14. Let $\lambda: G_2\to SO(n)$ be a real representation of G_2. Then it twists the left invariant framing of G_2 and gives a new element $[G_2,\lambda]$ in π_{14}^S. Let $\rho: G_2\to SO(7)$ be the same real representation of G_2 as the one in the proof of Theorem 2.2. Then we obtain the following corollary:

Corollary 2.3 ([3, Theorem 2]). *We have* $[G_2, 3\rho] = 0$ *in* π_{14}^s.

Proof. In the proof of Theorem 2.2, we have obtained

$$\rho i = 3 \oplus \sigma_3 f_3.$$

By [4, Theorem] we have

$$\mathrm{Ad}(G_2) = \wedge^2(\rho) - \rho.$$

Therefore by [1, Corollary 5.4] the proof is complete.

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(*Received December 3, 1992*)