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SOME CHARACTERIZATIONS OF RIGHT co-H-RINGS

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1. Introduction. In [9] and [10] Harada introduced and investigated the following two conditions about a given ring R:

(I) Every non-small right R-module contains a non-zero injective submodule.

(II) Every non-cosmall right R-module contains a non-zero projectitive direct summand.

Following [15], a ring R is called a *right H-ring* (in honour of Harada's works [9], [10]) if R is a right artinian ring satisfying (I). Dually, R is called a *right co-H-ring* if R satisfies (II) and the ACC on right annihilators. It is shown in [15] that every quasi-Frobenius ring is a right co-H-ring and every right co-H-ring is a semiprimary QF-3 ring. Moreover, the following characterizations of right co-H-rings are known.

Theorem 1 ([15, Theorem 3.18]). *For a ring R the following conditions are equivalent*:

(1) *R is a right co-H-ring.*

(2) *Every projective right R-module is an extending module.*

(3) *Every right R-module is a direct sum of a projective module and a sigular module.*

(4) *The family of all projective right R-modules is closed under taking essential extensions.*

Theorem 2 ([16, Theorem 2]). *A ring R is a right co-H-ring if and only if R satisfies the following three conditions*:

(a) *R is right perfect,*

(b) *R satisfies ACC on right annihilators,*

(c) *$R_R \oplus R_R$ is an extending module.*

In this paper we shall prove the following theorem.

Theorem 3. *Let R be a ring and ω denote the cardinality of the set $\mathbb{N}$ of all natural numbers. The following conditions are equivalent*:

(i) *R is a right co-H-ring.*

(ii) *R is right perfect and $R_R^{(\mathbb{N})}$ is an extending module.*

(iii) *R is right perfect and every ω-generated right R-module is a direct sum of* *a projective module and a singular module.*

(iv) *R is right perfect and every essential extension of $R_R^{(\mathbb{N})}$ is a projective module.*

(v) *R is a right perfect ring satisfying* ACC *on right annihilators and every essential extension of R_R is a projective module.*

Recall that a module M is said to be *semiperfect* if every factor module of M has a projective cover. In view of [19, 43.9], condition (ii) can be replaced by condition

(ii′) *$R_R^{(\mathbb{N})}$ is a semiperfect and extending module.*

Concerning condition (iii) we would like to note that if R is a ring such that every ω-generated right module is a direct sum of a projective module and an injective module, then R is right artinian and each singular right (or left) R-module is injective ([5, Theorem 2]).

2. Definitions and notation. We assume throughout that all rings are associative rings with identity and all modules are unitary. For a module M we denote by $\mathrm{E}(M)$, $\mathrm{J}(M)$ and $\mathrm{Z}(M)$ the injective hull, Jacobson radical and the singular submodule of M, respectively. We write M_R (resp. ${}_RM$) to indicate that M is a right (resp. left) R-module.

Let I be an index set and $\alpha = \mathrm{card}(I)$. Then the direct sum of α copies of a module M is denoted by $\oplus_I M$, $M^{(I)}$ or also $M^{(\alpha)}$. If a module N is generated by α (or fewer) elements, N is called *α-generated*. A module M is called a *local module* if M contains a greatest proper submodule, and M is called an *extending module* if each submodule A of M is an essential submodule of a direct summand of M.

A module M is said to be a *small module* if M is small in $\mathrm{E}(M)$, i.e. for any proper submodule of H of $\mathrm{E}(M)$, $M+H \neq \mathrm{E}(M)$. If M is not small, M is called a *non-small* module. Dually, M is called a *cosmall* module if for any projective module P and any epimophism $f: P \to M$, $\ker(f)$ is essential in P, i.e. for each non-zero submodule H of P, $\ker(f) \cap H \neq 0$. If M is not cosmall, M is called a *non-cosmall module* (see [9], [17]). A module M is defined to be *completely indecomposable* in case $\mathrm{End}({}_RR)$ is a local ring. Let $\{N_i\,;\ i \in I\}$ be an independent set of submodules N_i of M. Then $\oplus_I N_i$ is defined to be a *locally direct summand* of M if for each finite subset F of I, $\oplus_F N_i$ is a direct summand of M.

Let $\{M_i\,;\ i \in I\}$ be a system of completely indecomposable modules. Then $\{M_i\,;\ i \in I\}$ is defined to be *locally semi-T-nilpotent* (cf. [11]) if for any countable system of non-isomorphisms $\{f_{i_n}: M_{i_n} \to M_{i_{n-1}}\,;\ n \geq 1\}$ with $i_n \neq i_{n'}$ for $n \neq n'$

and for any $x \in M_{i_1}$ there exists an integer m depending on x such that $f_{i_m}f_{i_{m-1}}\cdots f_{i_1}(x) = 0$. It is shown in [13] that $\{M_i ; i \in I\}$ is locally semi-T-nilpontent if and only if for any independent set $\{N_j ; j \in J\}$ of submodules N_j of $M = \oplus_I M_i$, $\oplus_J N_j$ is a direct summand of M whenever it is a locally direct summand of M.

A module M is called *Σ-injective* if for any index set I, $M^{(I)}$ is injective. The following results are useful for our investigation.

Lemma 1 ([7, Proposition 20.3A]). *For an injective right R-module M the following conditions are equivalent :*

(1) *M is Σ-injective.*
(2) *$M^{(\mathbb{N})}$ is injective.*
(3) *R satisfies* ACC *on annihilators of subsets of M.*

Lemma 2 ([9, Theorem 3.6]). *Let R be a semiperfect ring. Then R satisfies* (II) *if and only if*

$$R_R = e_1R\oplus \cdots \oplus e_nR\oplus f_1R\oplus \cdots \oplus f_mR$$

where $\{e_1, \cdots, e_n\}\cup(f_1, \cdots, f_m\}$ is a set of mutually orthogonal primitive idempotents such that the following conditions are satisfied :

(a) *$n \geq 1$ and, for each $1 \leq i \leq n$, e_iR is injective,*
(b) *for any j, $1 \leq j \leq m$, there exists an $e_i \in \{e_1, \cdots, e_n\}$ such that f_iR is isomorprhic to a submodule of e_iR,*
(c) *for each i, $1 \leq i \leq n$, there exists an integer t_i such that e_iJ^t is projective for each $t \leq t_i$ and $e_iJ^{t_i+1}R$ is singular where $J = \mathrm{J}(R_R)$.*

Let M be a module. By [9] and [17] we see that M is non-cosmall if $M \neq \mathrm{Z}(M)$, and, if M contains a non-zero projective submodule, then M is non-cosmall. From this and the definition of non-cosmall modules we have :

Lemma 3. *Let R be a ring and α be a cardinal. Then the following conditions are equivalent :*

(i) *$R_R^{(\alpha)}$ is an extending module.*
(ii) *Every α-generated right R-module is a direct sum of a projective module and a singular module.*

We note that M is an extending module if and only if every closed submodule of M is a direct summand of M. Hence we have :

Lemma 4. *If M is an extending module then every direct summand of M is also an extending module.*

3. Semiprimary QF-3 rings. The class of rings each of which is perfect and contains a faithful injective right ideal and a faithful injective left ideal (i.e. perfect QF-3 rings) has been investigated in [4] and in [18]. It is shown in [4] that such a ring R is semiprimary and by [18], $E(R_R)$ and $E({}_RR)$ are both projective.

Proposition 5 (cf. [4, Theorem 1.3]). *For a ring R the following conditions are equivalent :*

(i) *R is a perfect* QF-*3 ring (i.e. a semiprimary* QF-*3 ring).*
(ii) *R is right perfect and $E(P)$ is projective for each projective module P.*
(iii) *R is right perfect and R contains a faithful Σ-injective right ideal.*

Here we prove the following theorem.

Theorem 6. *For a ring R the following conditions are equivalent* :
(a) *R is a semiprimary* QF-*3 ring.*
(b) *R is right perfect and $E(R_R^{(N)})$ is projective.*
(c) *R is a right perfect ring satisfying* ACC *on right annihilators and $E(R_R)$ is projective.*

Proof. (a) $\Longrightarrow$ (b) by Proposition 5.

(b) $\Longrightarrow$ (c). Assume (b). Since $E(R_R)$ is isomorphic to a direct summand of $E(R_R^{(N)})$, $E(R_R)$ is projective. Further, by (b) we have

$$E(R_R^{(N)}) \simeq (\oplus_{I_1} e_1R) \oplus \cdots \oplus (\oplus_{I_k} e_kR) \tag{1}$$

where $e_1, \cdots, e_k$ are primitive idempotents of R. By Lemma 1 it is enough to show that $E(R_R^{(N)})$ is injective. By (1) and since $E(R_R)$ is projective, there are subsets F_j of I_j ($j = 1, \cdots, k$) such that

$$E(R_R) \simeq (\oplus_{F_1} e_1R) \oplus \cdots \oplus (\oplus_{F_k} e_kR). \tag{2}$$

Hence

$$E(R_R)^{(N)} \simeq ((\oplus_{I_1} e_1R) \oplus \cdots \oplus (\oplus_{I_k} e_kR))^{(N)}. \tag{3}$$

Since $E(R_R)^{(N)}$ is isomorphic to a submodule of $E(R_R{}^{(N)})$ we can apply (3) and [7, Theorem 21.15] to see that $E(R_R)^{(N)}$ is injective.

(c) $\Longrightarrow$ (a). Assume (c). In order to show (a), by Proposition 5 it is enough to show that R contains a faithful Σ-injective right ideal eR. Note that $R = e_1R\oplus \cdots \oplus e_nR$ where $\{e_i ; 1 \leq i \leq n\}$ is a set of mutually orthogonal primitive idempotents. Since $\mathrm{E}(R_R)$ is projective by (c), there is at least one e_i such that e_iR is injective. We may assume that $e_1R, \cdots, e_kR$ are injective and $e_{k+1}R, \cdots, e_nR$ are not. Put $e = e_1 + \cdots + e_k$. Then $eR = e_1R\oplus \cdots \oplus e_kR$ is a non-zero injective right ideal of R. We may use (2) to have

$$\mathrm{E}(R_R) = (\oplus_{F_1} e_1R)\oplus \cdots \oplus(\oplus_{F_k} e_kR)$$

and this is isomorphic to a submodule of $\oplus_F eR$ where $F = F_1 \cup \cdots \cup F_k$. Hence, if $P = \mathrm{ann}_R(eR)$, then $\mathrm{E}(R_R) \cap P = 0$. It follows that $P = 0$, showing that eR is faithful. By (c) R satisfies ACC on annihilators of subsets of eR. Hence eR is Σ-injective by Lemma 1. This completes the proof.

Corollary 7. *Let R be a right perfect ring with* ACC *on right annihilators. If* $\mathrm{E}(R_R)$ *is projective then* $\mathrm{E}(R_R)$ *is also projective.*

Proof. Let R be as above and assume that $\mathrm{E}(R_R)$ is projective. Then, by Theorem 6, R is a semiprimary QF-3 ring. Hence, by [18], $\mathrm{E}(R_R)$ is projective.

Corollary 8. *Let R be a perfect ring such that* $\mathrm{E}(R_R)$ *is projective. Then* $\mathrm{E}(R_R)$ *is projective if and only if R satisfies* ACC *on right annihilators.*

Proof. One direction is clear, by Corollary 7. Now let R be a (right and left) perfect ring such that $\mathrm{E}(R_R)$ is projective and $\mathrm{E}(R_R)$ are projective. It follows from [4] that R is a semiprimary QF-3 ring. Hence, by Theorem 6, R has ACC on right annihilators.

Remark. By [14, Example 3], there is a semiprimary ring R such that $\mathrm{E}(R_R)$ is projective but $\mathrm{E}({}_RR)$ is not projective.

4. The proof of Theorem 3. Statement (i) implies (ii) by Theorem 2(a) and Theorem 1(2).

(ii) $\Longrightarrow$ (i). Assume (ii). Then in particular $R_R^{(n)}$ is an extending module for each $n \in \mathrm{N}$, by Lemma 4. By Theorem 2 it is enough to show that R has ACC on right annihilators. In order to do so, by Lemma 1 it is suffices to show that $\mathrm{E}(R_R)$ is Σ-injective, or, equivalently, $\mathrm{E}(R_R)^{(\mathrm{N})}$ is injective (see Lemma 1).

Since R is right perfect and $R_R\oplus R_R$ is an entending module, R satisfies (II) by [16, Theorem 1]. Hence, by Lemma 2, R has a decomposition

$$R = e_1R \oplus \cdots \oplus e_nR \oplus f_1R \oplus \cdots \oplus f_mR$$

where $\{e_1, \cdots, e_n\} \cup \{f_1, \cdots, f_m\}$ is a set of mutually orthogonal primitive idempotents with $n \geq 1$ such that $e_1R, \cdots, e_nR$ are injective and each f_jR is not injective. Moreover, for each f_j there is an e_i $(1 \leq i \leq n)$ such that f_jR is isomorphic to a submodule of e_iR. Put $e = e_1 + \cdots + e_n$. Since $\mathrm{E}(R_R) = e_1R \oplus \cdots \oplus e_nR \oplus \mathrm{E}(f_1R) \oplus \cdots \oplus \mathrm{E}(f_mR)$ and since each $\mathrm{E}(f_mR)$ is isomorphic to some e_iR, we have

$$\mathrm{E}(R_R) \simeq (\oplus_{I_1} e_1R) \oplus \cdots \oplus (\oplus_{I_n} e_nR) \tag{4}$$

with finite sets $I_1, \cdots, I_n$. Let $I = I_1 \cup \cdots \cup I_n$. It follows that $\mathrm{E}(R_R)$ is isomorphic to a submodule of $\oplus_I eR$. Put $E = \mathrm{E}(R_R)$. Let U be a submodule of R_R and $\phi: U_R \rightarrow E^{(\mathrm{N})}$ be an R-homomorphism. We shall show that ϕ can be extended to a homomorphism in $\mathrm{Hom}_R(R_R, E^{(\mathrm{N})})$, i.e. $E^{(\mathrm{N})}$ is injective. We may assume that U_R is essential in R_R.

Consider $Q = R_R \oplus E^{(\mathrm{N})}$. Since $E^{(\mathrm{N})}$ is isomorphic to a direct summand of $(\oplus_I eR)^{(\mathrm{N})}$ which in turn is isomorphic to a direct summand of $(\oplus_I R)^{(\mathrm{N})}$ it follows that Q_R is isomorphic to a direct summand of $R_R \oplus (\oplus_I R_R)^{(\mathrm{N})}$. Since I is finite, we have

$$R_R \oplus (\oplus_I R_R)^{(\mathrm{N})} \simeq R_R^{(\mathrm{N})}.$$

By (ii) and Lemma 4, Q_R is then an extending module. Hence there exists a submodule U^* of Q_R such that $\{u - \phi(u)\,;\ u \in U\}$ is an essential submodule of U^* and

$$Q_R = U^* \oplus Q^*. \tag{5}$$

We have $U^* \cap \mathrm{E}^{(\mathrm{N})} = 0$ and moreover U is a submodule of $U^* \oplus \mathrm{E}^{(\mathrm{N})}$. In particular, $U^* \oplus E^{(\mathrm{N})}$ is essential in Q_R. Let p be the projection of Q onto Q^* given by (5).

First we show that $p(E^{(\mathrm{N})}) = Q^*$. Clearly, $p_1 = (p \mid E^{(\mathrm{N})})$ is a monomorphism. For convenience, instead of $E^{(\mathrm{N})}$ we write $\oplus_{\alpha \in N} E_\alpha$ with $E_\alpha = E$. Since p_1 is monomorphic, $\{p_1(E_\alpha)\,;\ \alpha \in \mathrm{N}\}$ is an independent set of submodules in Q^*. Since each $p_1(E_\alpha)$ is injective, $Q_1 := \oplus_{\alpha \in N} p_1(E_\alpha)$ is a locally direct summand of Q^*. By (1) and by the definition of Q, Q^* is projective. It follows that $Q^* \simeq \oplus_{t \in T} Q_t$ where each Q_t is isomorphic to some eR with a primitive idempotent of e of R. Hence $\{Q_t\,;\ t \in T\}$ is a set of completely indecomposable projective modules and therefore $\oplus_{t \in T} Q_t$ has the exchange property by [12] or [20]. Thus $\{Q_t\,;\ t \in T\}$ is a locally semi-T-nilpotent set by [10, Corollary 2]. From this and [13] it follows that Q_1 is a direct summand of Q^*, say $Q^* = Q_1 \oplus Q_2$.

Suppose that $Q_2 \neq 0$. Then there is a non-zero element x in $Q_2 \cap (U^* \oplus E^{(N)})$. We have $x = u + v$ where $u \in U^*$ and $v \in E^{(N)}$. Hence $x = p(x) = p(u) + p(v) = p(v) \in Q_1$, a contradiction. Hence $p(E^{(N)}) = Q_1 = Q^*$. Therefore $Q = U^* \oplus E^{(N)}$. Now let p_2 be the projection from $U^* \oplus E^{(N)}$ to $E^{(N)}$. Then $(p_2 \mid R_R)$ is an extension of ϕ. Thus $E^{(N)}$ is injective.

(ii) $\Longleftrightarrow$ (iii) by Lemma 3 and (i) $\Longrightarrow$ (iv) by Theorem 2 (a) and Theorem 1 (4).

(iv) $\Longrightarrow$ (v). Assume (iv). Then $\mathrm{E}(R_R^{(N)})$ is projective. Hence by Theorem 6, R satisfies ACC on right annihilators. Further let C be an essential extension of R_R. Then $C^{(N)}$ is an essential extension of $R_R^{(N)}$. By (iv), $C^{(N)}$ is projective. Hence C_R is projective.

(v) $\Longrightarrow$ (i). Assume (v). Then R is semiprimary by [7, Lemma 24.19]. Note that $R = e_1R \oplus \cdots \oplus e_nR$ where $\{e_1, \cdots, e_n\}$ is a set of mutually orthogonal primitive idempotents of R. We may assume that $e_1R, \cdots, e_kR$ are injective and $e_{k+1}R, \cdots, e_nR$ are not. Since $\mathrm{E}(R_R)$ is projective, we have $k \geq 1$. In order to show (i) it is enough to show that R satisfies (a), (b) and (c) of Lemma 2.

For each j, $k+1 \leq j \leq n$, we show the existence of an $e_i \in \{e_1, \cdots, e_n\}$ such that e_jR is isomorphic to submodule of e_iR. Put $e = e_j$. Since $\mathrm{E}(R_R)$ is projective, $\mathrm{E}(eR)$ is also projective. Hence $\mathrm{E}(eR) \simeq (e_1R)^{(I_1)} \oplus \cdots \oplus (e_kR)^{(I_k)} = B$, say. Let ε be an isomorphism of $\mathrm{E}(eR)$ onto B and put $F = \varepsilon(eR)$. Then $F \simeq eR$ and F is an essential submodule of B. Since eR is a direct summand of R_R, using (v) we see that every essential extension of eR is projective. Hence every essential extension of F_R is also projective. For convenience we write $B = \oplus_{t \in T} B_t$ where $T = I_1 \cup \cdots \cup I_k$ and $B_t = e_iR$ if and only if $t \in I_i$ for $i = 1, \cdots, k$. Let p_t be the projection $\oplus_{t \in T} B_t \to B_t$ for each t and put $F_t = p_t(F)$. Then $F_t \neq 0$ for each t and F is a submodule of $\oplus_{t \in T} F_t$. Since F is essential in B, it follows that F is essential in $\oplus_{t \in T} F_t$. Hence, as we have just seen, $\oplus_{t \in T} B_t$ is projective, implying the projectivity of every F_t.

Now for a fixed t, $t = 1$ say, we put $q_1 = (p_1 \mid F)$ and consider the exact sequence

$$F \xrightarrow{q_1} F_1 \longrightarrow 0$$

with non-zero projective F_1. Then $F = \ker(q_1) \oplus G$ for some submodule G of F. Since $F \simeq eR$ is indecomposable, $\ker(q_1) = 0$. Hence $F_t = 0$ for all $t \neq 1$. It follows that $\oplus_{t \in T} B_t = B_1$. Thus $\mathrm{E}(eR) \simeq e_1R$. In fact we have seen that (a) and (b) of Lemma 2 are satisfied.

To check condition (c) of Lemma 2 put $e = e_i$ for $1 \leq i \leq k$. First we show that every submodule N_R of eR is either projective or singular. Let $Z = Z(eR)$ and suppose that Z is a proper submodule of N. It is clear that there exists a submodule M of eR such that $Z \subset M \subset N \subset eR$ and M/Z is simple. Assume that there is an element $x \in M$ such that xR is not contained in Z and $xR \neq M$. Then the set

$$\{X\,;\ X \text{ is proper cyclic submodule of } M \text{ and } X+Z = M\}$$

contains a minimal element, X say, since R is semiprimary. Hence X has the following properties:

(d) $X+Z = M$ and

(e) for any $x \in X$ with $xR \neq X$ we have $xR+Z \neq M$.

Let $I = X \cap Z$. Then $X/I \simeq (X+Z)/Z \simeq M/Z$. Hence I is maximal in X_R. Let $y \in X$ with $yR \neq X$. If $y \notin I$ then $y \notin Z$ and hence $yR+Z = M$, contradicting (e). Therefore $y \in I$ and so yR is a submodule of I. It follows that I is the greatest proper submodule of X, i.e. X_R is a cyclic local module. By [7, Proposition 18.23], there exists an $e_j \in \{e_1, \cdots, e_n\}$ such that $X \simeq e_jR/B$ for some submodule B of e_jR. As proved above, e_jR is uniform. Then X is either projective (i.e. $B = 0$) or singular (i.e. $B \neq 0$). The latter case implies that $Z+X$ is singular, a contradiction. Hence $X \simeq e_jR$. From this and (v) we can easily see that every essential extension of X is projective. Hence M_R is projective. It follows that there is a primitive idempotent f of R such that $M_R \simeq fR$. Hence M_R is a local module and so Z is the greatest proper submodule of M_R, a contradiction to our assumption above. Hence for each $x \in M$ with $x \notin Z$ we have $xR = M$. This shows that M_R is local and cyclic. By [7, Proposition 18.23] and the previous argument we see that $M_R \simeq e_jR$ for some $e_j \in \{e_1, \cdots, e_n\}$. By (v) we can see that any essential extension of M_R is projective. Thus N_R is projective, since M is essential in the submodule N of eR.

Now let K be any submodule of eR. Put $U = K \cap Z$. Suppose that $U \neq K$ and $U \neq Z$. Then $(K+Z)/U = K/U \oplus Z/U$. As proved above, $K+Z$ is cyclic, projective and local. Hence K/U and Z/U are cyclic and it follows that K/U and Z/U have maximal submodules. Hence $K+Z$ contains two different maximal submodules, a contradiction. Thus we must have $K \subseteq Z$ or $Z \subseteq K$. Using these facts and the argument used to prove [9, Theorem 3.6] we see that R satisfies (c) of Lemma 2.

This completes the proof of Theorem 3.

Remark. Recently it was proved in Clark and Huynh [3] that a semiperfect ring R is QF if and only if R is right self-injective and every uniform submodule of any projective right R-module P is contained in a finitely generated submodule of P, if and only if R is right quasi-continuous and every projective right R-module is extending. From this and Theorem 3 the following two conjectures are equivalent:

(i) For every right self-injective right perfect ring R, $R_R^{(N)}$ is an extending module.

(ii) Every right self-injective right perfect ring is quasi-Frobenius.

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