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GLOBAL DIMENSION OF 2×2 GENERIC TRACE ZERO MATRICES AND INVARIANTS OF 2×2 GENERIC MATRICES

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0. Introduction. Let K be a field and

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{pmatrix}$$

be 2×2 generic trace zero matrices. Then the algebra $R = K\{X, Y\}$, generated by X and Y, is a graded K-algebra with X-degree (1, 0) and Y-degree (0, 1). Moreover, $R \equiv K\langle x, y\rangle/T$, where $K\langle x, y\rangle$ is the noncommuting free algebra and T is the weak identity of $\mathrm{Mat}_2(K)$ in $K\langle x, y\rangle$, that is, the set of all polynomials $f(x, y) \in K\langle x, y\rangle$ such that $f(X, Y) = 0$.

The notion of a weak identity was introduced by Razmyslov in connection with the study of central polynomials. P. Halpin [3] calculate the Poincare series of T and R. In section 1, we calculate the global dimension of R. In section 2, we give an example of group G acting on the 2×2 generic matrix algebra $S = K\{X_1, \ldots, X_m, Y_1, \ldots, Y_n\}$ such that the fixed subalgebra $S^G = \{s \in S \mid s^g = s$ for all $g \in G\}$ is finitely generated for any integers $m, n \geq 1$.

1. The global dimension of 2×2 generic trace zero matrix algebra. Let $x_1, x_2, x_3, y_1, y_2, y_3$ be algebraically independent indeterminates over K and

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{pmatrix}$$

be 2×2 trace zero matrices. Then $R = K\{X, Y\}$ is a subalgebra of $\mathrm{Mat}_2(K[x_i, y_i \mid i = 1, 2, 3])$, and is a graded K-algebra. Let $Z = XY - YX$; then $\det(Z) = -(x_2y_3 - x_3y_2)^2 - 4(x_1y_2 - x_2y_1)(x_1y_3 - x_3y_1) \neq 0$. Hence Z is not a zero divisor in $\mathrm{Mat}_2(K[x_i, y_i \mid i = 1, 2, 3])$. Furthermore, R has some "Universal Mapping Property" in the sense that, if C is a commutative K-algebra and $A, B \in \mathrm{Mat}_2(C)$ with trace $A = 0$ and trace $B = 0$, then there exists a unique algebra homomorphism $\theta : R \to \mathrm{Mat}_2(C)$ such that $\theta(X) = A$, and $\theta(Y) = B$.

The following Lemma is easy to prove.

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Lemma 1. (1) $X^2 = -det(X)I$, $Y^2 = -det(Y)I$, and $Z^2 = -det(Z)I$, *where*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the 2×2 *identity matrix. In particular,* X^2, Y^2 *and* Z^2 *are contained in the center of* R.

(2) $XZ = -ZX$ *and* $YZ = -ZY$. *In fact,* $X^iZ^j = (-1)^{ij}Z^jX^i$ *and* $Y^iZ^j = (-1)^{ij}Z^jY^i$ *for* $i, j \geq 0$.

(3) $XY = -Z+XY$.

Let $J = [R, R]$ be the commutator ideal of R. Then $R/J \cong K[x, y]$, the commutative polynomial ring in two indetermintates x and y. Also it follows directly from Lemma 1(2) that $J = RZ = ZR$.

Lemma 2. $R = \bigoplus_{i,j,k\geq 0} KZ^kX^iY^j$, *a direct sum of vector spaces over* K.

Proof. Since every element $f(X, Y)$ of R is a sum of monomials in X and Y, by Lemma 1 $f(X, Y) = \sum_{i,j,k\geq 0} a_{ijk}Z^kX^iY^j$ with $a_{ijk} \in K$ and $a_{ijk} = 0$ for all but a finitely many i, j, k. Then it suffices to prove that the set $\{Z^kX^iY^j \mid i, j, k \geq 0\}$ is linearly independent over K. Suppose that $0 = a_1Z^{k_1}X^{i_1}Y^{j_1} + a_2Z^{k_2}X^{i_2}Y^{j_2}+\ldots+a_nZ^{k_n}X^{i_n}Y^{j_n}$ ...... $(*)$, with $n \geq 1$, $a_t \neq 0$ in K and $k_1 \leq k_2 \leq \ldots \leq k_n$. Since R is a graded algebra, we may assume that this sum is homogeneous, that is,

$k_1+i_1 = k_2+i_2 = \ldots = k_n+i_n =$ (the X-degree of the each monomial) and

$k_1+j_1 = k_2+j_2 = \ldots = k_n+j_n =$ (the Y-degree of the each monomial).

But Z is not a zero divisor in R, we may also assume that $k_1 = 0$. If $n \geq 2$, then $0 < k_2$ because $k_2 = 0$ $(= k_1)$ implies that $i_1 = i_2$ and $j_1 = j_2$. By specializing

$$X \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } Y \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ we have } 0 = a_1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{i_1+j_1}$$

and this is a contradiction to the assumption that $a_1 \neq 0$. So n must be 1 and $(*)$ is reduced to $0 = a_1X^{i_1}Y^{j_1}$. However, this is an impossibility since $a_1 \neq 0$. Therefore the set $\{Z^kX^iY^j \mid i, j, k \geq 0\}$ is linearly independent and this completes the proof.

Lemma 3. *Let* $R_1 = K[Z]$ *be the polynomial ring in* Z *over* K. *Then*

(1) $R_2 = K\{Z, X\}$, *the subalgebra of* R *generated by* Z *and* X *is an Ore extension of* R_1, *and*

(2) $R = K\{X, Y\}$ *is an Ore extension of* R_2.

Proof. (1) Let $\alpha : R_1 \to R_1$ be the K-automorphism induced by the correspondence $Z \to -Z$. Then it is straightfoward to show that $R_2 = K\{Z, X\} = R_1[X ; \alpha]$ is an Ore extension of automorphism type. (2) Since every element of R_2 can be expressed uniquely in the form $\sum_{i,k\geq 0} a_{ik} Z^k X^i$ ($a_{ik} \in K$), the mapping $\beta : R_2 \to R_2$ defined by $\beta(\sum_{i,k\geq 0} a_{ik} Z^k X^i) = \sum_{i,k\geq 0} (-1)^k a_{ik} Z^k X^i$ is a K-algebra automorphism of R_2. Now define a linear mapping $\delta : R_2 \to R_2$ by

$$\delta(Z^k X^i) = \begin{cases} 0 & \text{for even } i \\ (-1)^{k+1} Z^{k+1} X^{i-1} & \text{for odd } i. \end{cases}$$

Then for any element $f = \sum_{i,k\geq 0} a_{ik} Z^k X^i$ of R_2, we have $Yf = \beta(f) Y + \delta(f)$. Therefore, by [1, Theorem 12.2.1] and Lemma 2 $R = K\{X, Y\} = R_2[Y ; \beta, \delta]$ is an Ore extension of R_2.

Corollary. $R = K\{X, Y\}$ *is a Noetherian Ore domain.*

Theorem 4. *gl. dim* $R = 3$.

Proof. Let $J = [R, R]$ be the commutator ideal. Then $J = ZR = RZ$ and $R/J \equiv K[x, y]$, the commutative polynomial ring in two indeterminates. Since Z is regular, normal, and non-unit in R, by [4, Theorem 3.5], gl. dim $R \geq$ (gl. dim R/J)+1 = gl. dim $K[x, y]+1 = 3$. On the other hand, R is an iterated Ore extension of K, so by [4, Theorem 5.3], gl. dim $R \leq$ (gl. dim R_2)+1 = (gl. dim R_1)+1+1 = 3.

2. Invariants of groups acting on the 2×2 generic matrix algebras. Let m and n be (arbitrary) positive integers and $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be 2×2 generic matrices, that is,

$$X_i = \begin{pmatrix} x_{11}(i) & x_{12}(i) \\ x_{21}(i) & x_{22}(i) \end{pmatrix} \quad \text{and} \quad Y_j = \begin{pmatrix} y_{11}(j) & y_{12}(j) \\ y_{21}(j) & y_{22}(j) \end{pmatrix},$$

where the entries $\{x_{pq}(i), y_{pq}(j) \mid 1 \leq p, q \leq 2, 1 \leq i \leq m, 1 \leq j \leq n\}$ are algebraically independent variables over K. Then the generic matrix algebra $S = K\{X_1, \ldots, X_m, Y_1 \ldots, Y_n\}$ generated by X_i and Y_j is a graded algebra with X_i-degree (1, 0) and Y_j-degree (0, 1) for all i, j ($1 \leq i \leq m, 1 \leq j \leq n$). For the case when $m+n = 2$, there are some examples of nonscalar group G acting linearly on S such that the invariant subalgebras S^G are finitely generated. But for the case of $m+n \geq 3$, no example of nonscalar group G acting on S where

S^G is finitely generated have been given as yet.

In this section, we give a nonscalar group G acting linearly on S such that the invariant subalgebra S^G is finitely generated for arbitary positive integers m and n. In the finite generation problem of invariant subalgebras of groups acting on generic matrix algebras, it is well-known that the base field K can be replaced by its algebraic closure so that we may assume that K is algebraically closed. Now suppose that char $K \neq 3$. Let ω $(\in K)$ be a primitive 3rd root of unity and let

$$g = \begin{pmatrix} \omega I_m & 0 \\ 0 & \omega^2 I_m \end{pmatrix} = \begin{pmatrix} \omega & & & & & \\ & \ddots & & & 0 & \\ & & \omega & & & \\ & & & \omega^2 & & \\ & 0 & & & \ddots & \\ & & & & & \omega^2 \end{pmatrix}$$

be the diagonal matrix of size $m+n$, where I_m and I_n are $m\times m$ and $n\times n$ identity matrices, respectively. Then g acts linearly on S as an K-automorphism with the action defined by $X_i^g = \omega X_i$ and $Y_j^g = \omega^2 Y_j$ $(i = 1, 2, \dots, m, j = 1, 2, \dots, n)$.

From now on, we will show that the invariant subalgebra $S^G = \{s \in S \mid s^g = s\}$ is finitely generated over K, where $G = \langle g\rangle$. To do this we need some preliminaries.

Lemma 5. *For any 2×2 matrices A and B with entries in a commutative K-algebra, the following identities hold:*

(1) $A^2 - tr(A)A + det(A)I = 0$

(2) $AB+BA = [tr(AB) - tr(A)tr(B)]I + tr(A)B + tr(B)A.$

(3) $tr(A)(AB-BA) = A(AB-BA)+(AB-BA)A$

and

$tr(B)(AB-BA) = B(AB-BA)+(AB-BA)B.$

(Here $tr(\ )$ and $det(\ )$ denote the trace and the determinant respectively and I is the 2×2 identity matrix.)

Proof. These identities are from the Cayley-Hamilton theorem and its multilinearization.

Corollary. *If A and B are trace zero 2×2 matrices, then $A^2 = -det(A)I$ is a diagonal matrix and $tr(AB) = AB+BA$.*

Proof. The first identity is from Lemma 5(1) and the second is from Lemma 5(2).

For each pair i, j $(1 \leq i \leq m, 1 \leq j \leq n)$, let $W_{ij} = X_iY_j - Y_jX_i$. Then by Corollary to Lemma 5, W_{ij}^2 is a scalar matrix and $tr(W_{ij}W_{kl}) = W_{ij}W_{kl} + W_{kl}W_{ij}$. Note that since there are exactly mn W_{ij}'s, we will relabel them with the indexed set $\{1, 2, ..., mn\}$ for notational simplicity so that $W_1, W_2, ..., W_{mn}$ stand for all the W_{ij}'s.

Now we return to the invariant subalgebra S^G where $G = \langle g \rangle$. Since

$$g = \begin{pmatrix} \omega I_m & 0 \\ 0 & \omega^2 I_n \end{pmatrix}$$

is a diagonal matrix, S^G is generated by monomials in X_i and Y_j. In fact, it is easy to prove that S^G is generated by the set Ω of monomials:

$$X_{i_1}(X_{k_1}Y_{l_1}) \dots (X_{k_\rho}Y_{l_\rho})X_{i_2}X_{i_3}, \qquad X_i(X_{k_1}Y_{l_1}) \dots (X_{k_\rho}Y_{l_\rho})Y_j,$$
$$Y_j(Y_{l_1}X_{k_1}) \dots (X_{l_\rho}X_{k_\rho})X_i, \text{ and } \qquad Y_{j_1}(Y_{l_1}X_{k_1}) \dots (Y_{l_\rho}X_{k_\rho})Y_{j_2}Y_{j_3},$$

for $i, k \in \{1, 2, ..., m\}$ and $j, l \in \{1, 2, ..., n\}$ and $\rho \geq 0$.

Remark 6. (1) Since S is a domain, if $0 \neq u, v$ in S such that $uv \in S^G$ and $u \in S^G$ then $v \in S^G$.

(2) By definition of g, each W_i lies in S^G, hence for any $\rho \geq 0$, $W_{l_1}W_{l_2} \dots W_{l_\rho} \in S^G$.

Lemma 7. *Let S_1 be the subalgebra of S generated by the set Ω_1 of elements:*

$$X_{i_1}W_{l_1} \dots W_{l_\rho}X_{i_2}X_{i_3}, \qquad X_iW_{l_1} \dots W_{l_\rho}Y_j$$
$$Y_jW_{l_1} \dots W_{l_\rho}X_i, \text{ and } \qquad Y_{j_1}W_{l_1} \dots W_{l_\rho}Y_{j_2}Y_{j_3},$$

for $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq mn$ and $\rho \geq 0$. Then $S_1 = S^G$.

Proof. By Remark 6(2) above, Ω_1 is a subset of S^G, so $S_1 \subset S^G$, For the opposite inclusion, it is enough to show that each element of Ω is contained in S_1. By definition of Ω_1, clearly X_iY_j, Y_jX_i, $X_{i_1}X_{i_2}X_{i_3}$, $Y_{j_1}Y_{j_2}Y_{j_3}$ (when $\rho = 0$) and W_l are contained in S_1. Consider any monomial u in Ω. Suppose that $u = X_i(X_{k_1}Y_{l_1})(X_{k_2}Y_{l_2}) \dots (X_{k_\rho}Y_{l_\rho})Y_j$. We will show that $u \in S_1$ by induction on $\rho \geq 0$. The case when $\rho = 0$ is trivial. Now assume that $\rho > 0$ and every element of Ω involving λ number of factors X_kY_l is contained in S_1 for each λ, where $0 \leq \lambda < \rho$.

For any λ $(0 \leq \lambda < \rho)$, substituting X_iY_j with $W_{ij} + Y_jX_i$, we have

$$X_iW_{l_1} \dots W_{l_\lambda}(X_{k_{\lambda+1}}Y_{l_{\lambda+1}}) \dots (X_{k_\rho}Y_{l_\rho})Y_j$$

$$= X_i W_{l_1} \dots W_{l_\lambda} W_{k_{\lambda+1} l_{\lambda+1}} (X_{k_{\lambda+2}} Y_{l_{\lambda+2}}) \dots (X_{k_\rho} Y_{l_\rho}) Y_j$$
$$+ X_i W_{l_1} \dots W_{l_\lambda} (Y_{l_{\lambda+1}} X_{k_{\lambda+1}})(X_{k_{\lambda+2}} Y_{l_{\lambda+2}}) \dots (X_{k_\rho} Y_{l_\rho}) Y_j.$$

Here the last term in the previous equation is an element of S_1. That is so because the $X_i W_{l_1} \dots W_{l_\lambda} Y_{l_{\lambda+1}} \in \Omega_1 \subset S_1$ by definition and $X_{k_{\lambda+1}}(X_{k_{\lambda+2}} Y_{l_{\lambda+2}}) \dots (X_{k_\rho} Y_{l_\rho}) Y_j \in S_1$ by induction hypothesis. Therefore,

$$X_i W_l \dots W_{l_\lambda} (X_{k_{\lambda+1}} Y_{l_{\lambda+1}}) \dots (X_{k_\rho} Y_{l_\rho}) Y_j$$
$$\equiv X_i W_{l_1} \dots W_{l_\lambda} W_{k_{\lambda+1} l_{\lambda-1}} (X_{k_{\lambda+2}} Y_{l_{\lambda+2}}) \dots (X_{k_\rho} Y_{l_\rho}) Y_j (\text{mod } S_1) \text{ for any } (0 \leq \lambda < \rho).$$

If we begin this process from the case of $\lambda = 0$, then

$$\begin{aligned} u &= X_i (X_{k_1} Y_{l_1})(X_{k_2} Y_{l_2}) \dots (X_{k_\rho} Y_{l_\rho}) Y_j \\ &\equiv X_i W_{k_1 l_1} (X_{k_2} Y_{l_2}) \dots (X_{k_\rho} Y_{l_\rho}) Y_j \\ &\equiv \dots \\ &\equiv X_i W_{k_1 l_1} W_{k_2 l_2} \dots W_{k_\rho l_\rho} Y_j \equiv 0 \ (\text{mod } S_1). \end{aligned}$$

Thus $u \in S_1$. A similar argument shows that the other types of elements of Ω are contained in S_1. This completes the proof.

Theorem 8. *Let S and G be defined as before. Then S^G is finitely generated.*

Proof. Let Ω_2 be the set of elements:

$$X_{i_1} W_{l_1} \dots W_{l_\rho} X_{i_2} X_{i_3}, \qquad X_i W_{l_1}, \dots W_{l_\rho} Y_j,$$
$$Y_j W_{l_1} \dots W_{l_\rho} X_i, \text{ and } \qquad Y_{j_1} W_{l_1} \dots W_{l_\rho} Y_{j_2} Y_{j_3},$$

for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $l = 1, 2, \dots, mn$ and $0 \leq \rho \leq mn$, and let S_2 be the subalgebra of S generated by Ω_2. Then clearly S_2 is a finitely generated subalgebra of S^G. Now we need to show that $S^G \subset S_2$. It also suffices to prove that $\Omega_1 \subset S_2$ because Ω_1 generaters S^G. Note that for each i and each j, $X_i Y_j$, $Y_j X_i$ (when $\rho = 0$) are contained in S_2 hence $W_l \in S_2$ for $l = 1, 2, \dots$, mn. For an element $u = X_i W_{l_1} \dots W_{l_\rho} Y_j \in \Omega_1$, we will show that $u \in S_2$ by induction on $\rho \geq 0$. If $\rho \leq mn$, then $u \in \Omega_2 \subset S_2$ by definition of Ω_2. If $\rho > mn$, then since there are precisely mn W_l's, there exist integers λ, μ $(1 \leq \lambda < \mu \leq \rho)$ such that $W_{l_\lambda} = W_{l_\mu}$. Now we consider two cases.

Case 1: If $\mu = \lambda + 1$, then since $W_{l_\lambda} W_{l_\mu} = W_{l_\lambda}^2$ is a scalar matrix and $W_{l_\lambda}^2 \in S_2$, $u = W_{l_\lambda}^2 X_i W_{l_1} \dots W_{l_{\lambda-1}} W_{l_{\lambda+2}} \dots W_{l_\rho} Y_j$ is contained in S_2 by induction hypothesis and by virtue of the fact that $W_l \in S_2$ as stated previously.

Case 2: If $\lambda + 1 < \mu$, then since $W_{l_\lambda} W_{l_{\lambda+1}} = tr(W_{l_\lambda} W_{l_{\lambda+1}}) I - W_{l_{\lambda+1}} W_{l_\lambda}$,
$u = X_i W_{l_1} \dots W_{l_\lambda} W_{l_{\lambda+1}} \dots W_{l_\mu} \dots W_{l_\rho} Y_j$

$$= X_i W_{l_1} \dots \{tr(W_{l_\lambda} W_{l_{\lambda+1}})I - W_{l_{\lambda+1}} W_{l_\lambda}\} \dots W_{l_\mu} \dots W_{l_p} Y_j$$
$$= tr(W_{l_\lambda} W_{l_{\lambda+1}}) X_i W_{l_1} \dots W_{l_{\lambda-1}} W_{l_{\lambda+2}} \dots W_{l_p} Y_j$$
$$- X_i W_{l_1} \dots W_{l_{\lambda+1}} W_{l_\lambda} W_{l_{\lambda+2}} \dots W_{l_\mu} \dots W_{l_p} Y_j.$$

But note that $tr(W_{l_\lambda} W_{l_{\lambda+1}})I = W_{l_\lambda} W_{l_{\lambda+1}} + W_{l_{\lambda+1}} W_{l_\lambda} \in S_2$ and $X_i W_{l_1} \dots W_{l_{\lambda-1}} W_{l_{\lambda+2}} \dots W_{l_p} Y_j \in S_2$ (by induction hypothesis). Then

$$u = X_i W_{l_1} \dots W_{l_\lambda} W_{l_{\lambda+1}} \dots W_{l_p} Y_j$$
$$\equiv -X_i W_{l_1} \dots W_{l_{\lambda+1}} W_{l_\lambda} \dots W_{l_p} W_j \pmod{S_2}.$$

Continuing this process, we get the following:

$$u = X_i W_{l_1} \dots W_{l_\lambda} W_{l_{\lambda+1}} \dots W_{l_\mu} \dots W_{l_p} Y_j$$
$$\equiv -X_i W_{l_1} \dots W_{l_{\lambda-1}} W_{l_\lambda} \dots W_{l_\mu} \dots W_{l_p} Y_j$$
$$\equiv \dots$$
$$\equiv (-1)^{\mu-\lambda-1} X_i W_{l_1} \dots W_{l_{\lambda-1}} W_{l_{\lambda+1}} \dots W_{l_{\mu-1}} W_{l_\lambda} W_{l_\mu} \dots W_{l_p} Y_j \pmod{S_2}.$$

Since $W_{l_\lambda} = W_{l_\mu}$, it follows from Case 1 that $u \in S_2$. A similar argument proves for the other types of elements of Ω_1. Thus $\Omega_1 \subset S_2$ and $S^G \subset S_2$. Therefore S^G $(= S_2)$ is finitely generated.

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