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π-REGULAR RINGS SATISFYING THE CONVERSE OF SCHUR'S LEMMA

Chol On KIM and Chan HUH

By Schur's Lemma in ring theory, if R is a ring and ${}_RM$ is an irreducible left R-module, then the endomorphism ring $\mathrm{End}({}_RM)$ is a division ring. But there exists a ring R with reducible R-module ${}_RM$ such that the endomorphism ring $\mathrm{End}({}_RM)$ is a division ring, thereby the converse of Schur's Lemma does not hold. For simplicity of notation, we will denote the converse Schur's Lemma by (CS). Thus it might be quite interesting to observe the situation for which the condition (CS) is valid.

By Hirano and Park [4], a certain class of rings, for which the condition (CS) holds, was observed. Indeed they studied rings with the property (CS), i.e., rings for which a given left module is irreducible whenever its endomorphism ring is a division ring. For example, one of their interesting results is that a reduced PI-ring satisfying the condition (CS) is von Neumann regular.

Motivated by this fact, they raised the following question: Is every semiprime PI-ring satisfying the condition (CS) von Neumann regular?

For this question, we give in this paper a negative answer by constructing a semiprime 2-regular and strongly π-regular PI-ring, which satisfies the condition (CS) but is not von Neumann regular.

We begin with the following definition.

Definition. A ring R is π-regular (resp. strongly π-regular) if for each element x in R, there exists a positive integer n depending on x and an element y in R such that $x^n = x^n y x^n$ (resp. $x^n = x^{n+1}y$). For a fixed integer n, a ring R is n-regular if for each element x in R, there exists an element y in R such that $x^n = x^n y x^n$.

Azumaya [1] proved that every strongly π-regular ring is π-regular. Moreover Azumaya [1] and Hirano [3] proved that a ring R is strongly π-regular if and only if for each element x in R, $x^n = yx^{n+1}$ for some positive integer n and an element y in R.

Theorem 1 [4, Theorem 14]. *Let R be an Azumaya algebra. Then the*

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following are equivalent :

(1) *R satisfies the condition (CS).*

(2) *R is π-regular.*

(3) *The center $Z(R)$ of R is π-regular.*

Theorem 1 does not always hold even for PI-ring case. Hirano and Park [4, Example 6] constructed a PI-ring which is π-regular but does not satisfy the condition (CS). They also showed that there is a von Neumann regular ring which does not satisfy the condition (CS). However, they observed that some class of von Neumann regular rings still enjoy the condition (CS).

Theorem 2 [4, Theorem 18]. *Von Neumann regular rings whose primitive factor rings are Artinian satisfy the condition (CS).*

Corollary 3. *For a reduced PI-ring, the following are equivalent :*

(1) *R satisfies the condition (CS).*

(2) *R is von Neumann regular.*

Furthermore, they give a characterization of semiprime Goldie rings which satisfy the condition (CS).

Theorem 4 [4, Proposition 11]. *Semiprime left Goldie rings satisfying the condition (CS) are semisimple Artinian.*

By theorem 4, every prime PI-ring with condition (CS) is simple Artinian. Comparing this fact with the results in Theorem 2, they raised the following question [4, Question 21] :

Is every semiprime PI-ring with the condition (CS) von Neumann regular? We give a negative answer to this question by constructing the following example.

Example 5. Let F be a field and $S = \mathrm{Mat}_2(F)$, the 2×2 matrix ring over F. Let R be the ring consisting of all infinite sequences $x = (r_k)_{k=1}^{\infty}$ in S which are eventually of the form

$$r_k = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = a + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where $a, b \in F$. We claim that R is semiprime PI, 2-regular, strongly π-regular of bounded index 2, and satisfies the condition (CS), but R is not von Neumann regular.

Clearly R is a PI-ring which satisfies the same identities as S does.

For $0 \neq x = (r_k)_{k=1}^{\infty}$ in R there in a positive integer k such that $r_k \neq 0$ in S. Since S is simple, $Sr_kS = S$. Thus $(Rx)^2 \neq 0$ and hence $xRx \neq 0$. Therefore R is semiprime.

In order to prove that R is 2-regular , let $x = (r_k)_{k=1}^{\infty}$ be any element in R. Then there exists a positive integer n such that for $k \geq n$, r_k is of the form

$$r_k = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

where $a, b \in F$. For $k \leq n-1$, since S is von Neumann regular, $r_k^2 s_k r_k^2 = r_k^2$ for some s_k in S. Also for $k \geq n$, let

$$s_k = \begin{pmatrix} 1/a^2 & -2b/a^3 \\ 0 & 1/a^2 \end{pmatrix} \text{ if } a \neq 0 \quad \text{and } s_k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ if } a = 0.$$

Now let y $= (s_k)_{k=1}^{\infty}$, where s_k is chosen as above. Then a direct computation shows that $x^2yx^2 = x^2$ in R, and this proves that R is 2-regular.

By Cayley-Hamilton Theorem on 2×2 matrices, S is a ring of bounded index 2, and thus R is of bounded index 2. Therefore by Azumaya [1, Theorem 5], R is strongly π-regular.

Now if we take $x = (r_k)_{k=1}^{\infty}$ with

$$r_k = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for each $k \geq 1$, there does not exist an element $y = (s_k)_{k=1}^{\infty}$ in R such that $xyx = x$. Actually, $xyx = (r_k s_k r_k)_{k=1}^{\infty}$, with

$$r_k s_k r_k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

eventually, for all element $y = (s_k)_{k=1}^{\infty}$ in R, and hence R is not von Neumann regular.

Finally we will show that R satisfies the condition (CS). Let ${}_RM$ be a nonzero, non-irreducible left R-module. It will suffice to show that the endomorphism ring $\mathrm{End}({}_RM)$ is not a division ring. For each $n \geq 1$, let

$$e_n = (\underbrace{1, 1, \cdots, 1}_{n\text{-times}}, 0, 0, \cdots) = (r_k)_{k=1}^{\infty}$$

in R, i.e.,

$$r_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } k \leq n \quad \text{and} \quad r_k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for } k \geq n+1$$

Then e_n is a central idempotent in R for each $n \geq 1$. Let $I_n = Re_n$, $J_n = R(1-e_n)$ and $I = \sum_{n=1}^{\infty} I_n$. Then I_n, J_n and I are ideals of R. The proof splits into three cases.

Case 1: If $I_nM \neq 0$ and $J_nM \neq 0$ for some $n \geq 1$, then ${}_RM = I_nM + J_nM = I_nM \oplus J_nM$ and thus $\mathrm{End}({}_RM)$ is not a division ring.

Case 2: If $I_nM \neq 0$ and $J_nM = 0$ for some $n \geq 1$, then M is an R/J_n-module. Moreover ${}_RM$ and ${}_{R/J_n}M$ have the same module structure and $\mathrm{End}({}_RM) = \mathrm{End}({}_{R/J_n}M)$. Since $R/J_n \cong I_n$ is semisimple Artinian and ${}_RM$ is not irreducible $\mathrm{End}({}_RM) = \mathrm{End}({}_{R/J_n}M)$ is not a division ring.

Case 3: If $I_nM = 0$ for all $n \geq 1$, then $IM = 0$ and hence M is an R/I-module. Since R/I is commutative and 2-regular (because R is 2-regular), by Theorem 1, R/I satisfies the condition (CS). But ${}_RM$ and ${}_{R/I}M$ have the same module structure and hence ${}_{R/I}M$ is not irreducible. Thus $\mathrm{End}({}_RM) = \mathrm{End}({}_{R/I}M)$ is not a division ring.

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Department of Mathematics
Pusan National University
Pusan 609-735, Korea

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