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# ON JORDAN LEFT DERIVATIONS

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Througout the present paper, $R(\neq 0)$ will represent an associative ring with center $Z$ and $X$ a nonzero left $R$-module. Following [1], $X$ is called prime if $aRx = 0$ for $a \in R$ and $x \in X$ implies that either $x = 0$ or $aX = 0$. As is well known, $R$ is a prime ring if and only if there exists a nonzero faithful prime left $R$-module. Following [2], an additive mapping $D: R \to X$ is called a Jordan left derivation if $D(a^2) = 2aD(a)$ for all $a \in R$.

Now, let $X$ be a faithful prime left $R$-module and $p$ a prime number. Suppose that $R$ is of characteristic $p$. Then for any nonzero $a \in R$, $aR(pX) = (pa)RX = 0$, and so $pX = 0$. Conversely, suppose that $px = 0$ for some nonzero $x \in X$. Then, for any $a \in R$, $0 = aR(px) = (pa)Rx$, and so $pa = 0$. Consequently, we see that $X$ is not $p$-torsionfree, or what is the same, $pX = 0$, if and only if $R$ is of characteristic $p$.

Our present objective is to improve [2, Theorem 1.2] as follows.

**Theorem 1.** *Let $R$ be a prime ring of characteristic $\neq 2$, and $X$ a nonzero left $R$-module. Suppose that $X$ is faithful and prime. If there exists a nonzero Jordan left derivation $D: R \to X$, then $R$ is commutative.*

In preparation for proving our theorem, we state several lemmas.

**Lemma 1.** *Suppose that $X$ is faithful and prime. Let $a, b \in R$, and $x \in X$. If (the prime ring) $R$ is of characteristic $\neq 2$ and $arbrx = 0$ for all $r \in R$, then $a = 0$ or $b = 0$ or $x = 0$.*

*Proof.* Obviously, $0 = a(u+v)b(u+v)x = aubvx + avbux$ for all $u, v \in R$. Replacing $v$ by $rarbr$, we have $0 = aubrarbrx + ararbrbux = ararbrbux$ for all $u, r \in R$. Suppose that $x \neq 0$. Noting that $X$ is faithful and prime, we obtain $ararbrb = 0$. Since $R$ is prime, [5, Theorem] shows that either $a = 0$ or $b = 0$.

**Lemma 2.** *Let $R$ be a ring of characteristic 3. If $D: R \to X$ is a Jordan left derivation, then for all $a, b, c \in R$, there holds the following:*

(1) $D(ab+ba) = 2aD(b) + 2bD(a)$.

(2) $D(aba) = a^2D(b) - baD(a)$.

(3) $D(abc+cba) = (ac+ca)D(b) - baD(c) - bcD(a)$.

*Proof.* See the proof of [2, Proposition 1.1].

**Lemma 3.** *Let $R$ be a ring of characteristic 3, and $D: R \to X$ a Jordan left derivation. Suppose that $X$ is faithful and prime. If $D(a) \neq 0$ for some $a \in R$, then $[a, [a, b]]^2 = 0$ for all $b \in R$.*

*Proof.* Note that $X$ is 2-torsionfree, and see the proof of [2, Lemma A].

The next will play an essential role in the proof of Theorem 1.

**Lemma 4.** *Let $R$ be a prime ring of characteristic 3. Suppose that $X$ is faithful and prime. If there exists a nonzero Jordan left derivation $D: R \to X$, then $R$ has no nonzero nilpotent elements (more precisely, $R$ has no nonzero divisors of zero).*

*Proof.* Suppose, to the contrary, that $R$ contains a nonzero element $a$ with $a^2 = 0$. Then $aD(a) = 0$. Now, by Lemma 2(1) and (2), we obtain $2aD(ba) = 2aD(ba) + 2baD(a) = D(aba) = a^2D(b) - baD(a) = 0$, and so $aD(ba) = 0$. Next by Lemma 2(3),

$$\begin{aligned} D(ab^2a) + D(baba) &= D(ab^2a + baba) \\ &= (aba + ba^2)D(b) - baD(ba) - b^2aD(a) = abaD(b). \end{aligned}$$

Combining this with $D(ab^2a) = 0$ (Lemma 2(2)) and $D(baba) = 2baD(ba) = 0$, we obtain $abaD(b) = 0$. So, linearizing this on $b$, we obtain

$$(\#) \qquad abaD(c) + acaD(b) = 0 \quad \text{for all } b, c \in R.$$

Replacing $c$ by $ac+ca$ in (#), we obtain $abaD(ac+ca) = 0$, and so $abacD(a) = 0$ by Lemma 2(1). Hence $D(a) = 0$ by Lemma 1. Further, $acaD(bab) = aca(b^2D(a) - abD(b)) = 0$ by Lemma 2(2). Now, replacing $b$ by $bab$ in (#), we get $ababaD(c) = ababaD(c) + acaD(bab) = 0$. Hence $aD(c) = 0$ by Lemma 1. Replace $c$ by $c^2$ to get $acD(c) = 0$. Linearizing this on $c$, we obtain $abD(c) + acD(b) = 0$. Furthermore, replacing $c$ by $ac$, we have $abD(ac) = 0$ for all $b \in R$, and so $D(ac) = 0$ by the faithfulness and the primeness of $X$. Recalling that $D(a) = 0$ and $aD(c) = 0$, we obtain $D(ca) = D(ac+ca) = 0$ by Lemma 2(1), and so $D(cba) = 0$. Combining this with $D(a(bc)) = 0$, we have $acD(b) = D(a(bc) + (cb)a) = 0$ by Lemma 2(1). Hence $D(b) = 0$ for all $b \in R$. But this is a contradiction.

We are now ready to complete the proof of Theorem 1.

*Proof of Theorem 1.* In view of [2, Theorem 1.2], it suffices to consider the case that $R$ is of characteristic 3. Choose $a \in R$ such that $D(a) \neq 0$. Then $[a, [a, b]] = 0$ for all $b \in R$, by Lemmas 3 and 4, and so [4, Theorem 1] shows that $a$ is in $Z$. Thus $R = Z \cup \{a \in R \mid D(a) = 0\}$. Since $D$ is nonzero, we conclude that $R = Z$, by Brauer's trick.

Finally in connection with Theorem 1, we shall improve [6, Theorem 2] as follows:

**Theorem 2.** *Let $R$ be a prime ring of characteristic $\neq 2$. If there exists a nonzero derivation $D: R \to R$ such that $[a, [a, D(a)]] \in Z$ for all $a \in R$, then $R$ is commutative.*

*Proof.* In view of [6, Theorem 2], it suffices to consider the case that $R$ is of characteristic 3. Then, for any $a \in R$,

$$\begin{aligned} D(a^3) &= a^2D(a)+aD(a)a+D(a)a^2 = a^2D(a)-2aD(a)a+D(a)a^2 \\ &= [a, [a, D(a)]] \in Z, \end{aligned}$$

and so $D(a^{3\cdot 3}) = 3a^{3\cdot 2}D(a^3) = 0$. Hence $R$ is commutative by [3, Theorem 2].

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