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ON DERIVATIONS IN NEAR-RINGS AND RINGS

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The major purpose of this paper is to study two kinds of derivations in near-rings. The first kind, called strong commutativity-preserving derivations, are motivated by recent studies of mappings f in rings having the property that $[f(x), f(y)] = 0$ whenever $[x, y] = 0$. (For references, see [3].) The second kind, called Daif derivations, are near-ring analogues of some derivations in rings which were intruduced recently by Daif and studied in [4]. Most of our results are in the context of near-rings; however, one of the principal theorems (Theorem 5) belongs entirely to ring theory.

All our near-rings N will be zero-symmetric left near-rings. The multiplicative center of N will be denoted by Z, and the commutator $xy-yx$ by $[x, y]$. By a derivation on N we mean a mapping $D: N \rightarrow N$ such that $D(x+y) = D(x)+D(y)$ and $D(xy) = xD(y)+D(x)y$ for all $x, y \in N$. As usual, an element $c \in N$ for which $D(c) = 0$ is called a constant; and as in [2], a derivation D is said to be commuting if $[x, D(x)] = 0$ for all $x \in N$.

1. Strong commutativity-preserving derivations: N without zero divisors. We define a strong commutativity-preserving derivation (scp-derivation) to be a derivation D such that $[x, y] = [D(x), D(y)]$ for all $x, y \in N$. Clearly such derivations preserve commutativity in the sense that $[D(x), D(y)] = 0$ whenever $[x, y] = 0$. If N is a commutative near-ring, then every derivation is an scp-derivation; and it can be expected that the existence of an scp-derivation on N will imply some measure of commutativity. We show that for certain well-behaved classes of near-rings, this is indeed the case.

Lemma 1. *If D is an scp-derivation on N, then constants are in Z. If N also has 1, then $(N, +)$ is abelian.*

Proof. For c constant, we have $[c, y] = [D(c), D(y)] = [0, D(y)] = 0$ for all $y \in N$. In particular, if N has 1, then $1+1 \in Z$; hence $[1+1, x+y] = 0$ for all $x, y \in N$, from which it follows that $(N, +)$ is abelian.

It will be useful to recall a result from [2]:

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Lemma 2. *If D is any derivation on N, the following partial distributive law is satisfied* :

$$(aD(b)+D(a)b)c = aD(b)c+D(a)bc \quad \text{for all } a, b, c \in N.$$

Theorem 1. *If N has right cancellation and D is a nonzero scp-derivation on N, then D is commuting and $(N, +)$ is abelian.*

Proof. For all $x \in N$, $[x, xD(x)] = [D(x), D(xD(x))]$; hence,

$$x[x, D(x)] = x[D(x), D^2(x)] \doteq [D(x), xD^2(x)+D(x)^2].$$

Now by Lemma 2, the right-hand side of this equality equals

$$D(x)xD^2(x)+D(x)^3-(xD^2(x)D(x)+D(x)^3) = D(x)xD^2(x)-xD^2(x)D(x) ;$$

hence

$$xD(x)D^2(x)-xD^2(x)D(x) = D(x)xD^2(x)-xD^2(x)D(x),$$

or

$$xD(x)D^2(x) = D(x)xD^2(x).$$

If $D^2(x) = 0$, then $D(x)$ is constant, hence central ; otherwise, $D^2(x)$ can be cancelled on the right. In either event, $[x, D(x)] = 0$. Finally, $(N, +)$ is abelian by [2, Theorem 1].

Note that if N admits a commuting scp-derivation, all idempotents e are central, for $D(e) = eD(e)+D(e)e = 2eD(e)$ gives $eD(e) = 2eD(e)$, hence $eD(e) = 0 = D(e)$. Centrality follows by Lemma 1.

Theorem 2. *If N has no zero divisors and admits a nonzero commuting scp-derivation, then N is a commutative ring with no idempotents except* 0 *or* 1.

Proof. For all $x, y \in N$ we have $[x, xy] = [D(x), D(xy)] = [D(x), xD(y) +D(x)y]$, so by using Lemma 2 we get $x[x, y] = D(x)xD(y)+D(x)^2y -D(x)yD(x)-xD(y)D(x)$. Since D is commuting, and therefore $(N, +)$ is abelian by [2, Theorem 1], we now have $x[x, y] = x[D(x), D(y)] = x[D(x), D(y)]+D(x)[D(x), y]$. Hence $D(x)[D(x), y] = 0$; and since N has no zero divisors, we conclude that $[D(x), y] = 0$ for all $x, y \in N$. In particular, $[D(x), D(y)] = 0$; and therefore $[x, y] = 0$ for all $x, y \in N$. Thus, N is a commutative ring.

Finally, if $e^2 = e \neq 0$, then e is central as noted above. Since $e(ex-x) = 0$ for all $x \in N$, e is a left identity element ; and since $e \in Z$, it follows that e

$= 1$.

Corollary 1. *A near-field with an scp-derivation is a field.*

Recall that Graves and Malone [5] define a near-ring to be a *near-domain* if it has right cancellation and satisfies the left Ore condition. (Pilz [8, p. 310] uses *near-integral domain* for this concept and reserves near-domain for another idea [8, §8. 41]). Invoking Theorems 1 and 2, we obtain

Corollary 2. *A near-domain admitting a nonzero scp-derivation is a commutative ring (and hence an ordinary integral domain).*

Corollary 3. *If N has no nonzero nilpotent elements and admits a commuting scp-derivation, then N is a commutative ring.*

Proof. By Lemma 4 of [2], there exists a family of completely prime ideals $\{P_\alpha \mid \alpha \in \Lambda\}$ such that N is a subdirect product of the near-rings N/P_α, and such that for each $\alpha \in \Lambda$, the definition $\tilde{D}_\alpha(x+P_\alpha) = D(x)+P_\alpha$ yields a derivation $\tilde{D}_\alpha$ on N/P_α. Let $\tilde{N}$ denote a typical N/P_α; and note that $\tilde{N}$ has no nonzero divisors of zero. If $\tilde{D}_\alpha$ is nonzero, then $\tilde{N}$ is a commutative ring by Theorem 2. On the other hand, if $\tilde{D}_\alpha$ is trivial, then the definition of scp-derivation shows that $\tilde{N}$ is commutative, hence distributive. But then $(\tilde{N}^2, +)$ is abelian, so that $\tilde{x}^2 + \tilde{x}\tilde{y} - \tilde{x}^2 - \tilde{x}\tilde{y} = 0$ for all $\tilde{x}, \tilde{y} \in \tilde{N}$; and cancelling $\tilde{x}$ shows that $(\tilde{N}, +)$ is abelian.

Hongan [6] has shown how some of the results in [2] can be generalized by assuming that the hypotheses apply to a nonzero ideal of N rather than to N itself. In the same spirit, we offer

Theorem 3. *Let A be a nonzero ideal of N which contains no zero divisors of N. If N admits a nonzero derivation D such that $[x, D(x)] = 0$ for all $x \in A$ and $[x, y] = [D(x), D(y)]$ for all $x, y \in A$, then N is a commutative ring.*

Proof. By Lemma 2 of [2], the additive-group commutator $(x, a) = x+a-x-a$ is constant for all $a \in A$ and $x \in N$. Since A is an ideal, $y(x, a) = (yx, ya)$ is also constant for arbitrary $y \in N$, hence $D(N)(x, a) = \{0\}$. Now $(x, a) \in A$, hence cannot be a nonzero divisor of zero; therefore $(x, a) = 0$, and $(A, +)$ is abelian. It follows that for arbitrary $a \in A\backslash\{0\}$ and $x, y \in N$, $(ax, ay) = a(x, y) = 0$; hence $(N, +)$ is abelian. We can now adapt the proof of Theorem 2 to show that $D(x)[D(x), y] = 0$ for all $x, y \in A$; and since $[D(x), y] \in A$, we conclude that $[D(x), y] = 0$ or $D(x) = 0$. Thus $[D(x), y] = 0$ for all $x, y \in A$; in particular, for all $x, y \in A$ we have $[D(x), yD(y)] = 0 = y[D(x), D(y)]$. We conclude that $0 = [D(x), D(y)] = [x, y]$ for all $x, y \in A$. If $a \in A \backslash \{0\}$ and $u, v \in N$, this gives $auav - avau = 0 = a^2uv - a^2vu = a^2[u, v]$, so $[u, v] = 0$. Therefore, N is a commutative ring.

For N with 1, the existence of an scp-derivation implies a distributivity principle:

Lemma 3. *If N has 1 and admits an scp-derivation, then*

$$(zx+z)y = zxy+zy \quad \textit{for all } x, y, z \in N.$$

Proof. Since $D(1) = 0$ and $[x+1, y] = [D(x+1), D(y)] = [D(x), D(y)] = [x, y]$, it follows that $(x+1)y = xy+y$ for all $x, y \in N$; and left-multiplying by z gives the result.

If N has 1 and $zN = N$ for all $z \in N \backslash \{0\}$, this lemma shows that N is distributive; thus, we have an alternative approch to Corollary 1. The lemma also enables us to prove a theorem related to Ligh's work in [7].

Theorem 4. *Let N be a nonzero near-ring such that $aN = N$ for all $a \in N \backslash \{0\}$. If N admits an scp-derivation, then N is a division ring.*

Proof. It is easily shown that N has no zero divisors. Moreover if $y \in N \backslash \{0\}$, there exists $e \in N$ such that $ye = y$, $ye^2 = ye$, and $y(e^2 - e) = 0$. Thus, e is a nonzero idempotent, which must be a left identity. For the scp-derivation D, we have $eD(e)+D(e)e = D(e)$; hence $D(e)+D(e)e = D(e)$ and $D(e)e = 0$. Thus $D(e)N = D(e)eN = \{0\}$, so $D(e) = 0$. Therefore $e \in Z$ by Lemma 1, hence N has 1. It follows by Lemmas 1 and 3 that N is a ring, which must of course be a division ring.

2. Scp-derivations: N distributively-generated and prime. Our definition of prime near-ring is as in [2]: N is prime if $aNb = \{0\}$ implies that $a = 0$ or $b = 0$. Since our N are zero-symmetric, $AN \subseteq A$ and $NA \subseteq A$ for all ideals A; and it follows that if N is prime and A is a nonzero ideal, then either of $xA = \{0\}$ or $Ax = \{0\}$ implies that $x = 0$.

Our first theorem of this section is a commutativity theorem for rings, reminiscent of those in [1].

Theorem 5. *Let R be a prime ring, and U a nonzero right ideal of R. If R admits a derivation D such that $[x, y] = [D(x), D(y)]$ for all $x, y \in U$, then R is commutative.*

Proof. We may assume that D is nonzero; otherwise U is commutative, and so is R. For all $x, y \in U$, we have $[x, xy] = [D(x), D(xy)]$, from which follow $x[x, y] = [D(x), xD(y)+D(x)y] = [D(x), xD(y)]+[D(x), D(x)y]$ and $x[x, y] = x[D(x), D(y)]+[D(x), x]D(y)+D(x)[D(x), y]$; hence

(1) $$[D(x), x]D(y)+D(x)[D(x), y] = 0 \quad \text{for all } x, y \in U.$$

Replacing y by yr yields

$$[D(x), x](yD(r)+D(y)r)+D(x)(y[D(x), r]+[D(x), y]r) = 0,$$

which when compared with (1) yields

(2) $$[D(x), x]yD(r)+D(x)y[D(x), r] = 0 \quad \text{for all } x, y \in U \text{ and } r \in R.$$

Taking $r = D(x)$ now yields

$$[D(x), x]UD^2(x) = \{0\} = [D(x), x]URD^2(x) \text{ for each } x \in U;$$

hence for each $x \in U$, either $D^2(x) = 0$ or $[D(x), x]U = \{0\}$.

Suppose that $D^2(x) = 0$. Then for each $y \in U$, $[x, yD(x)] = [D(x), D(yD(x))] = [D(x), D(y)D(x)]$; and it follows that $y[x, D(x)] = 0$. Therefore $U[x, D(x)] = \{0\}$, hence $[x, D(x)] = 0$. On the other hand, if $[D(x), x]U = \{0\}$, it follows from (2) that $D(x)U[D(x), r] = \{0\} = D(x)UR[D(x), r]$; hence either $D(x) \in Z$, in which case $[x, D(x)] = 0$, or $D(x)U = \{0\}$.

Assume for the moment that there exists $y \in U$ such that $D(y) \in Z\backslash\{0\}$. Then for each $x \in U$ for which $D(x)U = \{0\}$, (1) yields $[D(x), x]D(y) = 0$; and since $D(y)$ is not a zero divisor, $[D(x), x] = 0$. Hence in this case $[D(x), x] = 0$ for all $x \in U$, and R is commutative by Theorem 4 of [1].

It remains only to dispose of the case where for each $x \in U$, either $D^2(x) = 0$ or $D(x)U = \{0\}$. The sets of elements of U for which these two conditions hold are additive subgroups of U whose union is U; consequently, we must have either $D^2(U) = \{0\}$ or $D(U)U = \{0\}$. If the first of these holds, the computation above shows that $[x, D(x)] = 0$ for all $x \in U$, so that commutativity of R again follows from Theorem 4 of [1]. If $D(U)U = \{0\}$, then the condition that $[x, yz] = [D(x), D(yz)]$ yields $yD(x)D(z) = 0$ for all $x, y, z \in U$. Hence $U[D(x), D(z)] = \{0\} = U[x, z]$ for all $x, z \in U$. We conclude that U is commutative, hence R is commutative.

We remark that Theorem 5 implies that the division rings in Theorem 4 are in fact fields.

Returning to near-rings, we have

Theorem 6. *Let N be a prime near-ring and A a nonzero ideal of N which is a distributively-generated near-ring with identity. If N admits a derivation D such that $[x, y] = [D(x), D(y)]$ for all $x, y \in A$, then N is a commutative ring.*

Proof. Let e be the identity element of A. Since $ex = x$ for all $x \in A$, we have $eD(x)+D(e)x = D(x)$, hence $eD(e)A = \{0\}$ and $eD(e) = 0$. Thus, for each $x \in A$ we have $xD(e) = xeD(e) = 0$, so that $AD(e) = \{0\}$ and $D(e) = 0$. Of course $D(e+e) = 0$ as well, so by a modification of Lemma 1, we see that both e and $e+e$ commute with elements of A and hence $(A, +)$ is abelian. Thus, for all $a \in A$ and all $x, y \in N$, we have $a(x+y-x-y) = 0$; consequently $(N, +)$ is abelian.

Now since A is d-g with identity and $(A, +)$ is abelian, it follows that A is distributive. Let $x, y \in N$ and $a, b \in A$. Then $(ax+ay)b = axb+ayb$, hence $a((x+y)b-(xb+yb)) = 0 = (x+y)b-(xb+yb)$ —— i.e. elements of A are distributive in N. Replacing b by zb for arbitrary $z \in N$ gives $(x+y)zb = xzb+yzb$; and using the distributivity of b, we get $((x+y)z-(xz+yz))A = \{0\}$, so that N is distributive. We have now shown that N is a ring, and commutativity follows by Theorem 5.

Corollary 4. *Let N be a prime distributively-generated near-ring with* 1. *If N admits an scp-derivation, then N is a commutative ring.*

3. Daif derivations: the first kind. A Daif derivation of the first kind (a Daif 1-derivation) on a near-ring N is defined to be a derivation with the property that $-xy+D(xy) = -yx+D(yx)$ for all $x, y \in N$; a Daif derivation of the second kind (a Daif 2-derivation) is one for which $xy+D(xy) = yx+D(yx)$ for all $x, y \in N$. In [4] it is shown that a prime ring R must be commutative if it admits one of these kinds of derivation; and the two kinds are treated together. For near-rings the two kinds require quite different treatment. In this section we consider the first kind, which is somewhat simpler.

We shall make use of the following lemma, due to Wang [9]. For the sake of completeness, we reproduce the proof.

Lemma 4. *Let D be a derivation on the near-ring N. Then $D(xy) = D(x)y$*

$+xD(y)$ *for all* $x, y \in N$.

Proof. For all $x, y \in N$, we have

$$D(x(y+y)) = xD(y+y)+D(x)(y+y) = xD(y)+xD(y)+D(x)y+D(x)y$$

and

$$D(xy+xy) = D(xy)+D(xy) = xD(y)+D(x)y+xD(y)+D(x)y.$$

Comparing these two expressions gives $xD(y)+D(x)y = D(x)y+xD(y)$.

Our aim in this section is to prove

Theorem 7. *If* N *is a prime near-ring admitting a nonzero Daif* 1-*derivation, then* $(N, +)$ *is abelian. Moreover, if* N *is* 2-*torsion-free, then* N *is a commutative ring.*

The following two lemmas will be needed for its proof.

Lemma 5. *Let* D *be a Daif* 1-*derivation on the near-ring* N. *Then*

(i) $D(c) = c$ *for each commutator* $c = [x, y]$;

(ii) $D(z)[x, y] = [x, y]D(z)$ *for all* $x, y, z \in N$.

Proof. The first statement is clear from the definition. To arrive at the second, we note that $-[x, y]z+D([x, y]z) = -z[x, y]+D(z[x, y])$; and using Lemma 4, we rewrite this as $-[x, y]z+D([x, y])z+[x, y]D(z) = -z[x, y]+zD([x, y])+D(z)[x, y]$. In view of Lemma 5(i), we now get $[x, y]D(z) = D(z)[x, y]$.

Lemma 6. *Let* N *be a prime near-ring admitting a Daif* 1-*derivation* D.

(i) *If* c *is a commutator and* $uc = vc$, *then* $cD(u-v) = 0$.

(ii) *If* c_1 *and* c_2 *are commutators with* $c_1c_2 = 0$, *then* $c_1 = 0$ *or* $c_2 = 0$.

Proof. (i) Apply D to $uc = vc$, and use both parts of Lemma 5.

(ii) If $c_1c_2 = 0 = 0c_2$, (i) yields

$$c_2D(c_1) = 0; \tag{3}$$

thus $c_2c_1 = 0$. Now since (3) depended only on the fact that c_2 is a commutator, we can replace c_1 by yc_1, thereby obtaining $c_2D(yc_1) = 0 = c_2yD(c_1)+c_2D(y)c_1$. Since $D(y)$ commutes with c_2 by Lemma 5(ii), we get $c_2D(y)c_1 = 0$ and hence $c_2ND(c_1) = \{0\} = c_2Nc_1$. Thus, $c_1 = 0$ or $c_2 = 0$.

Proof of Theorem 7. Since $[x, xy] = x[x, y]$ for all $x, y \in N$, Lemma 5

(ii) gives $D(z)x[x, y] = x[x, y]D(z) = xD(z)[x, y]$ for all $x, y, z \in N$. By Lemma 6(i), we get $[x, y]D(D(z)x - xD(z)) = 0$, hence $[x, y][D(z), x] = 0$. By Lemma 6(ii), we see that either $x \in Z$ or $[D(z), x] = 0$. Thus $D(N) \subseteq Z$, and our theorem follows from Theorem 2 of [2].

4. Daif derivations: the second kind. For rings, Daif 2-derivations have the useful property that $D(c) = -c$ for each commutator c. This will be true for a near-ring N only if $xy + yx = yx + xy$ for all $x, y \in N$. A near-ring with this property we shall call *pseudo-abelian.*

A fundamental tool in the study of Daif 2-derivations is the following lemma, the proof of which is similar to that of Lemma 5(ii).

Lemma 7. *Let N be pseudo-abelian and D a Daif 2-derivation on N. If $z \in N$ has the property that $cz + D(c)z = 0$ for all commutators c, then $[D(z), c] = 0$ for all commutators c. In particular, if z is distributive or if N has 1 and $[z, -1] = 0$, then $[D(z), c] = 0$ for all commutators c.*

Theorem 8. *If N has no nonzero divisors of zero and admits a nonzero Daif 2-derivation, then N is a commutative ring.*

Proof. The defining condition can be written as

$$-yx + xy = D([y, x]) \quad \text{for all } x, y \in N. \tag{4}$$

Replacing x by yx yields $-y^2x + yxy = D([y, yx]) = D(y[y, x])$, which may be rewritten as $yD([y, x]) = yD([y, x]) + D(y)[y, x]$. Thus

$$D(y)[y, x] = 0 \quad \text{for all } x, y \in N. \tag{5}$$

Replacing x by $D(y)$ and using the fact that N has no zero divisors shows that D is commuting, hence $(N, +)$ is abelian by Theorem 1 of [2]. It also follows from (5) that either $D(y) = 0$ or $y \in Z$ —— that is, nonconstants are central. But substituting into (4) shows that constants commute with each other, hence N is commutative and is therefore a ring.

It is easy to conjecture that this result can be extended to arbitrary prime near-rings. The best result we can prove, however, is

Theorem 9. *Let N be a 2-torsion-free pseudo-abelian prime d-g near-ring with 1. If N admits a nonzero Daif 2-derivation, then N is a commutative ring.*

Proof. Let x be an element of N which commutes with -1. Then by Lemma 7, both $D(x)$ and $D((-1)x) = (-1)D(x)$ commute with all commutators c. Since $[x, -1] = 0$ implies $[D(x), -1] = 0$ and since $c(-1) = -c$ is also a commutator, we have $D(x)(-1)c = (-1)D(x)c = c(-1)D(x) = D(x)c(-1)$, from which it follows that $D(x)[-1, c] = 0 = [-1, c]D(x)$. In particular, $[-1, c]D(u) = 0$ for all distributive u, so that $[-1, c]D(N) = \{0\}$. Thus, by Lemma 3(iii) of [2], $[-1, c] = 0$ for all commutators c.

A little calculation shows that $[-1, [-1, x]] = 0$ implies

$$x+x = (-1)(-x)+(-1)(-x) \quad \text{for all } x \in N. \tag{6}$$

Since N is pseudo-abelian, $(-1)x-x(-1) = -x(-1)+(-1)x$, so that $(-1)x+x = x+(-1)x$ for all $x \in N$. It now follows from (6) that $x+(-1)x$ has additive order 2, hence is 0. Thus, $-1 \in Z,(N, +)$ is abelian, and therefore N is a ring. Commutativity now follows from Theorem 1 of [4].

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