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ON COMMUTATIVITY OF A CERTAIN CLASS OF RINGS

HIROAKI KOMATSU, TSUNEKAZU NISHINAKA and ADIL YAQUB

Throughout, R will represent a ring with center $C = C(R)$, $N = N(R)$ the set of nilpotents in R, and $E = E(R)$ the set of idempotents in R. Given $x \in R$, we denote by $C_R(x)$ the centralizer of x in R. We consider the following conditions:

(*) For each $x, y \in R$, either $x \in C_R(y)$ or $x^n - x^{n+1}f(x) \in C_R(y) \cap N$ for some positive integer n and $f(X) \in \boldsymbol{Z}[X]$ with $f(\pm 1) = 1$.

(S) For each $x, y \in R$, there exists $f(X, Y) \in \boldsymbol{Z}\langle X, Y\rangle[X, Y]\boldsymbol{Z}\langle X, Y\rangle$ each of whose monomial terms is of length ≥ 3 such that $[x, y] = f(x, y)$.

(In [3], the condition (S) is cited as (SC).)

Our present objective is to prove the following theorem.

Theorem 1. *Let R be a ring satisfying the conditions* (*) *and* (S).

(1) *The following conditions are equivalent*:

1) *R is commutative.*

2) *R is normal, namely $E \subseteq C$.*

3) *R contains no subring isomorphic to*

$$\begin{pmatrix} \boldsymbol{Z}/2^n\boldsymbol{Z} & 2^{n-1}\boldsymbol{Z}/2^n\boldsymbol{Z} \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 2^{n-1}\boldsymbol{Z}/2^n\boldsymbol{Z} \\ 0 & \boldsymbol{Z}/2^n\boldsymbol{Z} \end{pmatrix}.$$

(2) *If R is s-unital, namely $x \in xR \cap Rx$ for each $x \in R$, then R is commutative.*

In preparation for proving our theorem, we state the next

Lemma 1. *Let R be a ring satisfying the conditions* (*) *and* (S).

(1) *Every factorsubring of R satisfies* (*) *and* (S).

(2) *If e is in $E \backslash C$, then $2e \in N \cap C$.*

(3) *If R contains* 1, *then R is normal.*

Proof. (1) This is obvious.

(2) Choose an arbitrary $x \in R$ with $[e, x] \neq 0$. Since $-e \notin C_R(x)$, there exists a positive integer n and $f(X) \in \boldsymbol{Z}[X]$ with $f(\pm 1) = 1$ such that $(-e)^n - (-e)^{n+1}f(-e) \in C_R(x) \cap N$. Noting here that $(-e)^n - (-e)^{n+1}f(-e) = (-1)^n e - (-1)^{n+1}f(-1)e = (-1)^n 2e$, we obtain $2e \in C_R(x) \cap N$. Needless to say, $2e \in C_R(x)$ for any $x \in R$ with $[e, x] = 0$, and so we have seen that $2e \in$

$N \cap C$.

(3) Let $e \in E$, $x \in R$, and put $a = ex - exe$. If $a \neq 0$, then $ea = a \neq 0 = ae$, and so $2e \in C$ by (2). Hence $2a = 0$. Since $1+a \notin C_R(e)$, there exists a positive integer n and $f(X) = (X^2-1)g(X)+1 \in \boldsymbol{Z}[X]$ such that $(1+a)^n - (1+a)^{n+1}f(1+a) \in C_R(e) \cap N$. Noting that $f(1+a) = f(1)+f'(1)a = 1 + 2g(1)a = 1$, we obtain $-a = (1+a)^n - (1+a)^{n+1}f(1+a) \in C_R(e)$. This contradiction shows that $ex = exe$; similarly, $xe = exe$. We have thus seen that $E \subseteq C$.

Proof of Theorem 1. (1) Obviously, 1) implise 3).

3) $\Rightarrow$ 2). Suppose, to the contrary, that there exists an element e in $E \backslash C$. Then, either $ex - exe \neq 0$ or $xe - exe \neq 0$ for some $x \in R$. Assume without loss that $a = ex - exe \neq 0$. Then $2e \in N \cap C$, by Lemma 1 (2). Combining this with $ea = a$, $ae = 0$, $a^2 = 0$ and $2a = 0$, we can easily see that $\langle e, a\rangle$ is a subring of R isomorphic to

$$\begin{pmatrix} \boldsymbol{Z}/2^n\boldsymbol{Z} & 2^{n-1}\boldsymbol{Z}/2^n\boldsymbol{Z} \\ 0 & 0 \end{pmatrix}$$

for some positive integer n.

2) $\Rightarrow$ 1). By Lemma 1 (1), every factorsubring of R satisfies the conditions (*) and (S). In view of [3, Lemma 8], we can easily see that every factorsubring of R is normal. Hence R has no factorsubring of type a) in [3, Theorem S]. Next, if a factorsubring S of R has no non-zero nilpotent element, then S is commutative, by a theorem of Herstein [1]. Hence R has no factorsubring of type c) or d) in [3, Theorem S]. Now, let

$$\mathrm{M}_\sigma(K) = \left\{\begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \middle| \, \alpha, \beta \in K\right\},$$

where K is a finite field with a non-trivial automorphism σ, and suppose that $\mathrm{M}_\sigma(K)$ satisfies the conditions (*) and (S). Let

$$x = -\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; y = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}, \; \alpha \neq \sigma(\alpha).$$

Since $[x, y] = (\alpha - \sigma(\alpha))e_{12} \neq 0$, there exists a positive integer n and $f(X) = (X^2-1)g(X)+1 \in \boldsymbol{Z}[X]$ such that $x^n - x^{n+1}f(x) \in C_R(y) \cap N(\mathrm{M}_\sigma(K)) = 0$. Noting that

$$f(x) = \begin{pmatrix} f(-1) & -f'(-1) \\ 0 & f(-1) \end{pmatrix} = \begin{pmatrix} 1 & -f'(-1) \\ 0 & 1 \end{pmatrix},$$

we obtain

$$0 = x^n - x^{n+1}f(x) = (-1)^n \begin{pmatrix} 2 & 2n - f'(-1) + 1 \\ 0 & 2 \end{pmatrix},$$

whence $2 = 0$ and $f'(-1) = 1$ follows. But this contradicts $f'(-1) = 2(-1)g(-1) = 0$. This contradiction shows that R has no factorsubring of type b) in [3, Theorem S]. Therefore R is commutative, by [3, Corollary S.1].

(2) In view of [2, Proposition 1], we may assume that R contains 1. Then R is normal by Lemma 1 (3), and therefore R is commutative by (1).

Corollary 1. *Suppose that N is commutative and for each $x \in R$ there exists a positive integer n and a positive odd integer k such that $x^n - x^{n+k} \in N \cap C$.*

(1) *The following conditions are equivalent*:

1) *R is commutative.*

2) *R is normal.*

3) *R contains no subring isomorphic to*

$$\begin{pmatrix} \boldsymbol{Z}/2^n\boldsymbol{Z} & 2^{n-1}\boldsymbol{Z}/2^n\boldsymbol{Z} \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 2^{n-1}\boldsymbol{Z}/2^n\boldsymbol{Z} \\ 0 & \boldsymbol{Z}/2^n\boldsymbol{Z} \end{pmatrix}.$$

(2) *If R is s-unital, then R is commutative.*

Proof. One can easily see that R satisfies the conditions (*) and (S). (By the way, the proof of Lemma 1 (2) shows that $2E \subseteq N \cap C$.) Hence the statements are clear by Theorem 1.

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H. KOMATSU AND T. NISHINAKA
DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY
OKAYAMA, 700 JAPAN

A. YAQUB
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA 93106, U.S.A.

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