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PRIME IDEALS IN STRONGLY GRADED RINGS BY POLYCYCLIC-BY-FINITE GROUPS II

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1. Introduction. Let G be any group with identity e and let $R = \sum\oplus_{x\in G} R_x$ be a strongly G-graded ring. In the case G is a finite group and R is a crossed product of G over its base ring R_e, G-prime, Lorenz and Passman proved that the number of the minimal prime ideals of R and the nilpotency of the prime radical of R are both less than or equal to $|G|$, the order of G. These results were extended to the case of graded rings by Cohen and Montgomery (cf. [P], Theorems 16.2 and 17.7). They also obtained the relationship between the prime ideals of R and of R_e which are the classical properties known as *Lying over, Going up, Going down and Incomparability*.

It is a natural question to see how these results are carried over to the case G is a polycyclic-by-finite group. In this paper, we will give the affirmative answer to the question above under an additional with R_e right Noetherian. More precisely; if R_e is G-prime, then the number of minimal prime ideals of R and the nilpotency of the prime radical is less than or equal to $|\Delta^{+}(G)|$, where $\Delta^{+}(G)$ is the unique maximal finite normal subgroup of G (cf. Theorem 3.6).

Let P and $\wp$ be prime ideals of R and R_e, respectively. Then we say that P lies over $\wp$ if $\bigcap_{x\in G}\wp^x = P\cap R_e$ and $\wp$ is minimal over $\bigcap_{x\in G}\wp^x$. If G is a finite group, then the second condition is superfluous, and this condition is equivalent to "*lying over*" in [P]. In Theorems 3.8 and 3.10, we will give the classical properties known as *Lying over, Going down, Going up and Incomparability*. These theorems will be proved in §3 after giving, in §2, some properties of the prime radicals and minimal prime ideals. If G is a finite group, then Passman and Lorenz proved two different types of Going up theorem and Going down theorem, respectively. But if G is infinite, then one of them does not hold, respectively, in general. An easy example will be offered immediately after Theorem 3.10.

2. Preliminaries. Let G be a group and let $R = \sum\oplus_{x\in G} R_x$ be a strongly G-graded ring. Then we defined an action of G on a subset of S of R via $S^x = R_{x^{-1}}SR_x$. A subset S is called *G-stable* if $S^x \subseteq S$ for any $x \in G$. The subgroup $\{x\in G \mid S^x = S\}$ is called the *stabilizer* of S in G. An ideal I of R_e is called *G-prime* if I is G-stable and if $AB \subseteq I$ implies that either $A\subseteq I$ or $B \subseteq I$ for any G-stable ideals A and B of R_e. R_e is called *G-prime* if 0 is a G-prime ideal.

For example, if $P \subseteq R$ is a prime ideal, then $P \cap R_e$ is G-prime. Note that if R_e is right Noetherian, then any G-stable ideal A of R_e is G-invariant, i.e., $A^x = A$ for all $x \in G$.

If R_e is a semi-prime right Goldie ring, then the set $C_e = C_{R_e}(0)$ is a regular right Ore set of R by Proposition 1.4 of [N.N.V], where $C_{R_e}(A) = \{c \in R_e \mid c$ is regular mod $A\}$ for any ideal of A of R_e. The right quotient ring $Q^g = Q^g(R)$ with respect to C_e is also a strongly G-graded ring and we can write $Q^g = \sum \oplus_{x \in G} R_x Q_e$, where Q_e is a right quotient ring of R_e with respect to C_e, and Q_e is a semi-simple Artinian ring. In this section, we shall give, more or less known, some relations between R and Q^g, which are needed to prove the main theorems in §3.

Lemma 2.1. *Let R be a strongly G-graded ring, and let R_e be a G-prime right Goldie ring. Then*

(1) *R_e is semi-prime.*

(2) *There exists a minimal prime ideal $\wp$ of R_e (unique up to G-conjugation) with $\bigcap_{x \in G} \wp^x = 0$, and $\{\wp^x \mid x \in G\}$ is the set of minimal primes of R_e. In particular, the stabilizer $\{x \in G \mid \wp^x = \wp\}$ of $\wp$ in G has a finite index in G.*

(3) *Q_e, the quotient ring of R_e, is G-simple, i.e., Q_e has no proper G-stable ideals.*

Proof. (1) By Theorem 1.35 of [C.H], the prime radical N_e of R_e is nilpontet. Since N_e is G-stable, it must be zero and so R_e is semi-prime.

(2) By Lemma 1.16 of [C.H], there are a finite number of minimal prime ideals $\wp_1, \wp_2, \cdots, \wp_n$ of R_e with $\bigcap_{i=1}^n \wp_i = 0$. It is clear that $\wp_i^x$ is also minimal prime for any $x \in G$. Put $\wp_i^* = \bigcap_x \wp_i^x$. Then $\wp_i^*$ are G-stable and $\wp_1^* \cdot \wp_2^* \cdots \wp_n^* = 0$. So we have $\wp_i^* = 0$ for some i. Hence for any $\wp_j$, there exists $x \in G$ with $\wp_i^x \subseteq \wp_j$ and so $\wp_i^x = \wp_j$. Thus $\{\wp_i^x \mid x \in G\}$ is the full set of minimal prime ideals of R_e. Further, the mapping $x \mapsto \bar{x} = \begin{pmatrix} \wp_1 & \wp_2 & \cdots & \wp_n \\ \wp_1^x & \wp_2^x & \cdots & \wp_n^x \end{pmatrix}$ induces a homomorphism from G into the symmetric group on n symbols and so the stabilizer of $\wp$ in G is of finite index in G.

(3) Let A' be a G-stable ideal of Q_e. Then the left annihilator $B' = \ell_{Q_e}(A')$ of A' in Q_e is also G-stable such that $0 = B'A' \supseteq BA$, where $B = B' \cap R_e$ and $A = A' \cap R_e$. Since B and A are also G-stable, either $B = 0$ of $A = 0$. Hence either $B' = BQ_e = 0$ or $A' = AQ_e = 0$ and so Q_e is G-simple.

In the following, we suppose that R_e *is a G-prime right Goldie ring and that G is a polycyclic-by-finite group*. Then, by Theorem 3.7 of [P] and Lemma 2.1, Q^g

is Noetherian. Moreover, R has an Artinian classical right quotient ring by Corollary 1.8 of [N.N.V]. Let N be the prime radical of R. Then there exist a finite number of minimal prime ideals $P_1, \cdots, P_n$ of R with $N = P_1 \cap \cdots \cap P_n$ (cf. Lemma 1.16 of [C.H]).

Lemma 2.2. (1) *If $P_1, \cdots, P_n$ are the minimal prime ideals of R, then $P_1Q^g, P_2Q^g, \cdots, P_nQ^g$ are the minimal prime ideals of Q^g.*
(2) *Let N be the prime radical of R. Then NQ^g is the prime radical of Q^g.*

Proof. (1) First we shall prove that $P_i = P_iQ^g \cap R$. Let r be any element in $P_iQ^g \cap R$. Then $rc \in P_i$ for some $c \in C_e \subseteq C_R(0)$. Hence $r \in P_i$, because of $C_R(0) = C_R(N) = \bigcap_{i=1}^n C_R(P_i)$ by Small's theorem. Hence $P_i = P_iQ^g \cap R$ and so P_iQ^g is also prime. Further, if $P_iQ^g \supseteq P'$, a minimal prime ideal of Q^g, then $P_i \supseteq P' \cap R$, a prime ideal of R, hence $P_i = P' \cap R$ and so $P_iQ^g = (P' \cap R)Q^g = P'$. Finally, if P' is a minimal prime ideal of Q^g, then $P' \cap R$ is a prime ideal of R so that it contains a P_i for some i, and so, $P' = (P' \cap R)Q^g \supseteq P_iQ^g$, hence $P' = P_iQ$.

(2) Let N' be the prime radical of Q^g. Then

$$N' \cap R = \bigcap_{i=1}^n P_iQ^g \cap R = \bigcap\{P_iQ^g \cap R\} = \bigcap P_i = N$$

shows that $N' = NQ^g$.

3. Prime ideals of strongly G-graded rings with G-prime base rings. Throughout this section, $R = \sum\oplus_{x\in G} R_x$ *be a strongly G-graded ring whose base ring R_e is a G-prime right Goldie ring and let G be a polycyclic-by-finite group.* Following [P], $\Delta^+(G) = \{x \in G \mid |G : C_G(x)| < \infty$ and x is finite order$\}$, is a finite normal subgroup of G (cf. Lemma 5.1 of [P]). Set $S = R(\Delta^+(G)) = \sum\oplus_{y\in\Delta^+(G)} R_y$. Then S is a G-stable subring of R. We denote by $\mathcal{N}$ the prime radical of S. Since S is also a right Goldie ring, $\mathcal{N}$ is a finite intersection of minimal prime ideals $\mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_m$ of S and is G-stable. Hence $\mathcal{N} = \bigcap_{i=1}^m \mathcal{P}_i^*$, where $\mathcal{P}_i^* = \bigcap_{x\in G} \mathcal{Q}_i^x$ are G-stable ideals of S. Renumbering these $\mathcal{P}_i^*$'s, if necessary, we write the distinct $\mathcal{P}_i^*$'s as $\mathcal{P}_1^*, \mathcal{P}_2^*, \cdots, \mathcal{P}_k^*$.

Lemma 3.1. *$\mathcal{P}_1^*, \cdots, \mathcal{P}_k^*$ are the minimal G-prime ideals of S and $\mathcal{P}_i^* \cap R_e$* $= 0$.

Proof. Let A and B be G-stable ideals of S whth $AB \subseteq \mathcal{P}_i^*$. Then $AB \subseteq \mathcal{Q}_i$, and so, either $A \subseteq \mathcal{Q}_i$ or $B \subseteq \mathcal{Q}_i$, hence either $A \subseteq \bigcap_{x\in G} \mathcal{Q}_i^x = \mathcal{P}_i^*$ or $B \subseteq \bigcap_{x\in G} \mathcal{Q}_i^x = \mathcal{P}_i^*$. Thus $\mathcal{P}_i^*$ are G-prime. If I is a G-prime ideal of S with $\mathcal{P}_i^* \supseteq$

I, then I contains some $\mathscr{P}_j^*$, because $0 = (\mathscr{N})^\ell \supseteq (\mathscr{P}_1^* \cdots P_k^*)^\ell$, and so $\mathscr{P}_i^* = I = \mathscr{P}_j^*$, showing $\mathscr{P}_i^* = I$. Hence $\mathscr{P}_i^*$ is a minimal G-prime ideal of S. Conversely, let I be a minimal G-prime ideal of S. Then, as above, $I \supseteq 0 = (\mathscr{P}_1^* \cdots \mathscr{P}_k^*)^\ell$. Hence $I \supseteq \mathscr{P}_i^*$ for some i and so $I = \mathscr{P}_i^*$. Further, let $J = \mathscr{P}_i^* \cap R_e$. Then J is a G-stable ideal of R_e. Hence JQ_e is also G-stable and we get either $JQ_e = 0$ or $JQ_e = Q_e$ since Q_e is G-simple. If $JQ_e = Q_e$, then J contains a regular element c in R_e. It is clear that c is a regular element in S and so it belongs to $C_S(\mathscr{P}_i^*)$ by Small's Theorem, because $\mathscr{N} = \bigcap_{i=1}^k \mathscr{P}_i^*$, hence $c = c \cdot 1 \in \mathscr{P}_i^*$ implies that $1 \in \mathscr{P}_i^*$, a contradiction. Thus we get that $J = 0$.

Conversely,

Lemma 3.2. *If $\mathscr{P}$ is a G-prime ideal of S with $\mathscr{P} \cap R_e = 0$, then $\mathscr{P}$ is minimal G-prime.*

Proof. We will show that $\mathscr{P} = \mathscr{P}_i^*$ for some i. On the contrary, we suppose that $\mathscr{P} \nsubseteq \mathscr{P}_i^*$ for all i, $1 \leq i \leq k$. Then $\mathscr{P} \nsubseteq \mathscr{Q}_i$ for all i. In fact, if $\mathscr{P} \subseteq \mathscr{Q}_i$ for some i, then $\mathscr{P}^x = \mathscr{P} \subseteq \mathscr{Q}_i^x$ for all $x \in G$, and so, $\mathscr{P} \subseteq \bigcap_x \mathscr{Q}_i^x = \mathscr{P}_i^*$, a contradiction. Hence it follows $\mathscr{P}Q^g(\Delta^+(G)) \nsubseteq \mathscr{Q}_i Q^g(\Delta^+(G))$ for all i by Lemma 2.2, and $\mathscr{Q}_i Q^g(\Delta^+(G))$ are the maximal ideals of $Q^g(\Delta^+(G))$ since $Q^g(\Delta^+(G))$ is an Artinian ring. Hence $\mathscr{P}Q^g(\Delta^+(G)) = Q^g(\Delta^+(G))$ and so $\mathscr{P}$ contains a regular element, which implies that $\mathscr{P} \cap R_e \neq 0$, a contradiction. Hence $\mathscr{P} \subseteq \mathscr{P}_i^*$ for some i and so $\mathscr{P} = \mathscr{P}_i^*$.

Let H be a normal subgroup of G and let π be the canonical mapping from G to G/H. Then R is considered as a strongly G/H-graded ring whose base ring is $R(H) = \Sigma\oplus_{h \in H} R_h$; $R = R(H)(G/H) = \Sigma\oplus_{\pi(x) \in G/H} R(H)_{\pi(x)}$, where $R(H)_{\pi(x)} = \Sigma\oplus_{h \in H} R_{hx}$. Under this notation, if $\mathscr{P}$ is a G-stable ideal of $R(H)$, then $R/\mathscr{P}R = \{R(H)/\mathscr{P}\}(G/H)$, i.e., a strongly G/H-graded ring whose base ring is $R(H)/\mathscr{P}$.

The following was obtained in [R] in the case of the group rings (cf. Corollary 22 of [R]).

Proposition 3.3. *Let P be an ideal of R. Then P is minimal prime if and only if*

(1) $P = (P \cap S)R$ *with* $P \cap S$, *G-prime,*

and

(2) $P \cap R_e = 0$.

Proof. Let P be a minimal prime ideal of R. Then $\mathscr{P} = P \cap S$ is a G-prime ideal of S. Hence$R/\mathscr{P}R = (S/\mathscr{P})(G/\Delta^{+}(G))$ is a prime ring by the Proposition 8.3 of [P]. Hence $\mathscr{P}R$ is a prime ideal $\subseteq P$, and so, $\mathscr{P}R = P$. Further, if $P\cap R_e \neq 0$, then $P\cap R_e$ is a G-stable ideal of R_e, and so $(P\cap R_e)Q_e = Q_e$. Thus $PQ^g = Q^g$, which contradicts to Lemma 2.2. Hence $P\cap R_e = 0$. Conversely, if $\mathscr{P} = P\cap S$ is G-prime, then by the above proof $\mathscr{P}R$ is a prime ideal of R. Hence P is prime. If $P \supseteq Q$, a prime ideal, then $Q\cap S$ is G-prime with $Q\cap R_e = 0$. So $Q\cap S$ and $P\cap S$ are both minimal G-prime by Lemma 3.2. Hence $Q\cap S = P\cap S$ and so $P = Q$.

Lemma 3.4. *Let N and $\mathscr{N}$ be the prime radical of R and S respectively. Then $N = \mathscr{N}R$.*

Proof. Since $\mathscr{N}$ is G-stable, $\mathscr{N}R$ is a nilpotent ideal of R. Further, since $S/\mathscr{N}$ is semi-prime, $R/\mathscr{N}R = (S/\mathscr{N})\{G/\Delta^{+}(G)\}$ is also semi-prime by Proposition 8.3 of [P], and so, $\mathscr{N}R$ is a semi-prime ideal, hence $\mathscr{N}R = N$.

The following Lemma is the graded-ring version of Passman's Theorem 14.7 of [P]. The proof is quite similar to that of the Theorem.

Lemma 3.5. *Let G be a polycyclic-by-finite group and let $R = R(G)$ be a strongly G-graded ring with R_e a right Noetherian G-prime ring. Suppose that $\wp$ is a minimal prime ideal of R_e, $N = Ann(\wp)$, and H is the stabilizer of $\wp$ in G. Then the mappings*

$$P \mapsto P_{|H}$$

and

$$L \mapsto L^{|G}$$

determine a one-to-one correspondence between prime ideals P of R with $P\cap R_e = 0$ and prime ideals L of $R(H)$ with $L\cap R_e = \wp$. Here

$$P_{|H} = \{r \in R(H) \mid Nr \subseteq P\}$$

and

$$L^{|G} = \bigcap_{x\in G}\{LR(G)\}^x.$$

Theorem 3.6. *Let R be a strongly G-graded ring whose base ring R_e is G-prime right Noetherian, G be a polycyclic-by-finite group and $\{P_1, P_2, \cdots, P_k\}$ be the set of minimal prime ideals of R. Then*

(1) $k \leq |\Delta^{+}(G)|$.

Further, let N be the prime radical of R. Then

(2) $N^{|\Delta^+(G)|}=0$.

Proof. First, we assume that R_e is prime. Then Q_e is simple Artinian. So $Q^g(\Delta^+(G))=Q_e*\Delta^+(G)$, a crossed product of $\Delta^+(G)$ over Q_e by Corollary I.3.25 of [N.V]. Hence the number of minimal prime ideals of S is at most $|\Delta^+(G)|$ by Theorem 16.2 of [P] and Lemma 2.2. Hence $k\leq|\Delta^+(G)|$ by Lemmas 3.1, 3.2 and Proposition 3.3. As in Lemma 3.4, let $\mathcal{N}$ be the prime radical of S. Then $\mathcal{N}Q^g(\Delta^+(G))$ is the prime radical of $Q^g(\Delta^+(G))$ by Lemma 2.2. Hence $(\mathcal{N}Q^g)^{|\Delta^+(G)|}=0$ by Theorem 16.2 of [P] and so $\mathcal{N}^{|\Delta^+(G)|}=0$. Thus $N^{|\Delta^+(G)|}=0$, because of $N=\mathcal{N}R$ by Lemma 3.4.

Suppose that R_e is G-prime but not prime. Then there exists a minimal prime ideal $\wp$ of R_e such that $\bigcap_{x\in G}\wp^x=0$ and the stabilizer H of $\wp$ is a subgroup of finite index by Lemma 2.1. Hence $\Delta^+(H)\subseteq\Delta^+(G)$ and H is also polycyclic-by-finite. We consider the ring $\bar{R}=\bar{R}(H)=R(H)/\wp R(H)$. Since $\wp$ is H-stable, $\bar{R}$ is a strongly H-graded ring with $\bar{R}_e=R_e/\wp$ prime right Noetherian. Hence by the above, there exist $m\leq|\Delta^+(H)|$ ($\leq|\Delta^+(G)|$) minimal prime ideals $\bar{L}_1,\cdots,\bar{L}_m$ of $\bar{R}$ with $\bar{L}_i\cap\bar{R}_e=\bar{0}$. We write the ideals $L_i\supseteq\wp$ of $R(H)$ such that the canonical image of L_i is $\bar{L}_i$ for any i, $1\leq i\leq m$. Then L_i are prime ideals with $L_i\cap R_e=\wp$. Hence $(L_1)^{|G},\cdots,(L_m)^{|G}$ are prime ideals of $R(G)$ with $(L_i)^{|G}\cap R_e=0$ by Lemma 3.5. So it suffices to prove that they are the minimal prime ideals of R. To prove that $(L_i)^{|G}$ is minimal, let P be a prime ideal of R with $P\subseteq(L_i)^{|G}$, then $P\cap R_e=0$ and so, $P_{|H}\subseteq\{(L_i)^{|G}\}_{|H}=L_i$ by Lemma 3.5. Hence $\overline{P_{|H}}\subseteq\bar{L}_i$. Since $P_{|H}$ is prime, $\overline{P_{|H}}=\bar{L}_i$, and so, $P_{|H}=L_i$. Again by Lemma 3.5, we have $P=L_i^{|G}$. Conversely, let P be a minimal prime ideal of R, then $P\cap R_e=0$ by Proposition 3.3. By Lemma 3.5, there exists a prime ideal of $R(H)$ with $P_{|H}=L$. It follows that $\bar{L}\supseteq\bar{L}_i$ for some i and so $P_{|H}\supseteq L_i$. Thus we have $P\supseteq(L_i)^{|G}$ and so $P=(L_i)^{|G}$.

Finally, we show that $N^{|\Delta^+(H)|}=0$. Let $J=L_1\cap\cdots\cap L_m$. Then $\bar{J}=$ the prime radical of $\bar{R}$ and so, by first case, $\bar{J}^{|\Delta^+(H)|}=\bar{0}$, i.e., $J^{|\Delta^+(H)|}\subseteq\wp R(H)$. Since the prime radical $N=\bigcap_i(L_i)^{|G}=(\bigcap_i L_i)^{|G}=J^{|G}$, we have $N^{|\Delta^+(H)|}=(J^{|G})^{|\Delta^+(H)|}\subseteq(J^{|\Delta^+(H)|})^{|G}\subseteq(\wp R(H))^{|G}=0$ (note that $A^{|G}\cap B^{|G}=(A\cap B)^{|G}$ and $A^{|G}B^{|G}\subseteq(AB)^{|G}$ for any ideals A and B of $R(H)$. Cf. Lemma 16.1 of [P]).

Corollary 3.7. *Let $\Delta^+(G)$ be a p-group and R_e be a G-prime ring of characteristic $p>0$. Then there exists a unique minimal prime ideal P of R which is nilpotent.*

Proof. Suppose first that R_e is prime. Then $Q^g(\Delta^+(G))=Q_e*\Delta^+(G)$ and Q_e is of characteristic p. Hence by Proposition 16.4 of [P], there exists a unique minimal prime ideal $\mathscr{P}'$ of $Q^g(\Delta^+(G))$ which is nilpotent. Then $\mathscr{P}=\mathscr{P}'\cap R(\Delta^+(G))$ is a unique minimal prime ideal which is nilpotent and G-stable by Lemma 2.2. Hence, by Lemma 3.4, $R(G)$ is a unique minimal prime ideal of $R(G)$ which is nilpotent. Finally, we suppose that R_e is G-prime but not prime. Then there exists a minimal prime ideal $\wp \subseteq R_e$ and $\bar{R}_e = R/\wp$ is prime and the stabilizer H of $\wp$ in G has a finite index in G and so $\Delta^+(H) \subseteq \Delta^+(G)$ is a p-group. Hence there exists a unique minimal prime ideal $\bar{L}$ of $\bar{R}$ which is nilpotent. Then by the argument in the proof of Theorem 3.6 $L^{|G}$ is a unique minimal prime ideal of $R(G)$ which is nilpotent.

Next we will investigate the relationship between the prime ideals of R_e and R which are the classical properties known as *Lying over*, *Going up*, *Going down and Incoparability*. In the case G is a finite group, these were obtained by Lorenz and Passman (cf. Theorems 16.6, 16.9 and 17.9 of [P]). Let P and $\wp$ be prime ideals of R and R_e, respectively. Then we say that P *lies over* $\wp$ if $\bigcap_{x\in G}\wp^x = P\cap R_e$ and $\wp$ is a minimal prime ideal over $\bigcap_{x\in G}\wp^x$. If G is a finite group, then the second condition is superfluous. Note that if P lies over $\wp$, then $P=\mathscr{P}R$, where $\mathscr{P} = P\cap S$, because $\mathscr{P}R$ is a prime ideal of R by Proposition 8.3 of [P].

Theorem 3.8. *Let R be a strongly G-graded ring whose base ring R_e is right Noetherian and let G be a polycyclic-by-finite group. Then*

(1) (*Cutting down*) *Let P be a prime ideal of R. Then there exists a prime ideal $\wp$ of R_e, unique up to G-conjugation, such that $\wp$ is minimal over $P\cap R_e$ and $\bigcap\wp^x = P\cap R_e$.*

(2) (*Lying over*) *Let $\wp$ be a prime ideal of R_e, then there exist prime ideals P_1, $P_2, \cdots, P_n$ of R with $n \leq |\Delta^+(G)|$ such that P_i lies over $\wp$.*

(3) (*Incomparability*) *Let $\wp_1 \subseteq \wp_2$ be prime ideals of R_e, and let $P_1 \subseteq P_2$ be prime ideals lying over $\wp_1$ and $\wp_2$, respectively. If $P_1 \neq P_2$, then $\wp_1 \neq \wp_2$.*

Proof. (1) Since $P\cap R_e$ is G-prime, $\bar{R}_e = R_e/(P\cap R_e)$ is a G-prime ring, and so, $\bar{R}_e$ has a minimal prime ideal $\bar{\wp}$ with $\bigcap_{x\in G}\bar{\wp}^x = \bar{0}$, by Lemma 2.1. Then $\wp$, the ideal of R_e whose canonical image in $\bar{R}_e$ equals to $\bar{\wp}$, is a prime ideal which is minimal over $P\cap R_e$ and $\bigcap_{x\in G}\wp^x = P\cap R_e$.

(2) Since $\bigcap_{x\in G}\wp^x$ is a G-prime ideal of R_e,

$$\bar{R} = R/(\textstyle\bigcap_{x\in G}\wp^x)R$$

satisfies the condition in Theorem 3.6, hence there exist the minimal prime ideals $\bar{P}_1, \cdots, \bar{P}_n$ with $n \leq |\Delta^+(G)|$ and $\bar{P}_i \cap \bar{R}_e = \bar{0}$ for all i. Hence P_i, the ideal of R whose canonical image in $\bar{R}$ equals $\bar{P}_i$ clearly lies over $\wp$ for each i.

Furthermore, $P_1, \cdots, P_n$ are incomparable since $\bar{P}_1, \cdots, \bar{P}_n$, are minimal primes. Hence (3) follows immediately.

Lemma 3.9. *Let G be a finite group and let R be a ring such that R is the sum $\sum_{x\in G} R_x$ of (R_e, R_e)-bisubmodules R_x with $R_x \cdot R_y = R_{xy}$ for all $x, y \in G$. If I is an essential ideal of R_e, i.e., I intersects nontrivially all nonzero ideals of R_e then there exists a nonzero ideal J of R with $0 \neq J \cap R_e \subseteq I$.*

Proof. Let M be a maximal complement of R_e in R so that $R_e \oplus M$ is essential in R, hence $I \oplus M$ is essential in R. For each $x, y \in G$, we set $L_{x,y} = R_x(I\oplus M)R_y$. Then $L_{x,y}$ is essential in R. In fact, let U be a nonzero (R_e, R_e)-bisubmodule of R, then $R_{x^{-1}}UR_{y^{-1}} \neq 0$, and so, $(I\oplus M)\cap R_{x^{-1}}UR_{y^{-1}} \neq 0$, hence $L_{x,y} \cap U \neq 0$. We write $J = \bigcap_{x,y\in G} L_{x,y}$. Then it is clear that J is an ideal of R and that J is an essential as (R_e, R_e)-bimodule, because G is finite. Hence $J \cap R_e \neq 0$. Further, if $r \in J \cap R_e \subseteq I\oplus M$, then $r = i+m$, $i \in I$, $m \in M$, but $r-i = m \in R_e \cap M = 0$ and so, $J \cap R_e \subseteq I$.

Theorem 3.10. *Let R be a strongly G-graded ring whose base ring R_e is right Noetherian and let G be a polycyclic-by-finite group. Then*

(1) (*Going up*) *Let $\mathfrak{P}_1$ and $\mathfrak{P}$ be prime ideals of R_e with $\mathfrak{P}_1 \supseteq \mathfrak{P}$ and let P be a prime ideal of R lying over $\mathfrak{P}$. Then there exists a prime ideal P_1 of R such that P_1 lies over $\mathfrak{P}_1$ and $P_1 \supseteq P$.*

(2) (*Going down*) *Let $\mathfrak{P}_1$ and $\mathfrak{P}$ be prime ideals of R_e with $\mathfrak{P} \supseteq \mathfrak{P}_1$ and let P be a prime ideal of R lying over $\mathfrak{P}$. Then there exists a prime ideal P_1 of R such that P_1 lies over $\mathfrak{P}_1$ and $P \supseteq P_1$.*

Proof. (1) Let $\mathscr{P} = P \cap S$, G-prime. Then $\mathscr{P} \cap R_e = \bigcap_{x\in G} \mathfrak{P}^x$. Let $\mathscr{P}_1$ be the maximal element of the set $\{A : \text{ideal of } S \mid A \cap R_e \subseteq \mathfrak{P}_1 \text{ and } A \supseteq \mathscr{P}\}$. Then it is easily checked that $\mathscr{P}_1$ is a prime ideal of S since $\mathfrak{P}_1$ is prime. First, we will prove that $\mathfrak{P}_1$ is a minimal prime ideal over $\mathscr{P}_1 \cap R_e$. Let $\bar{S} = S/\mathscr{P}_1$ and $\pi : S \to \bar{S}$ be the canonical mapping. Then the set $\{\bar{R}_x = \pi(R_x) \mid x \in \Delta^+(G)\}$ satisfies the condition of Lemma 3.9 with $\bar{R}_e = R_e/(\mathscr{P}_1 \cap R_e)$. If $\bar{I}$ is an ideal of $\bar{S}$ with $\bar{I} \cap \bar{R}_e \subseteq \bar{\mathfrak{P}}_1$, then $I \cap R_e \subseteq \mathfrak{P}_1 + \mathscr{P}_1$, where I is inverse image of $\bar{I}$ in S. Let $r = z+p$ be any element in $I \cap R_e$, where $z \in \mathfrak{P}_1$ and $p \in \mathscr{P}_1$. Then $p = r-z \in R_e \cap \mathscr{P}_1 \subseteq \mathfrak{P}_1$. Hence $I \cap R_e \subseteq \mathfrak{P}_1$ and so, by the choice of $\mathscr{P}_1$, $I = \mathscr{P}_1$, i.e., $\bar{I} = \bar{0}$. Thus, by Lemma 3.9, $\bar{\mathfrak{P}}_1$ is not essential. So there exists a nonzero ideal $\bar{J}$ of $\bar{R}_e$ with $\bar{\mathfrak{P}}_1 \cap \bar{J} = \bar{0}$. Since $\bar{R}_e$ is semi-prime, there exists a minimal prime ideal $\overline{\mathfrak{P}'}$ with $\overline{\mathfrak{P}'} \not\supseteq \bar{J}$. Thus $\overline{\mathfrak{P}'} \supseteq \bar{\mathfrak{P}}_1$ implies that $\overline{\mathfrak{P}'} = \bar{\mathfrak{P}}_1$. Hence $\mathfrak{P}_1$ is a minimal prime ideal over $\mathscr{P}_1 \cap R_e$ and so $\bigcap_{y\in\Delta^+(G)} \mathfrak{P}_1^y = \mathscr{P}_1 \cap R_e$. Put $P_1 =$

$(\bigcap_{x\in G}\mathscr{P}_1^x)R$. Then P_1 is a prime ideal of R by Proposition 8.3 of [P] and $P_1 \supseteq P$, because $\bigcap\mathscr{P}_1^x \supseteq \mathscr{P}$ and $P = \mathscr{P}R$. Furthermore, $P_1\cap S = \bigcap_{x\in G}\mathscr{P}_1^x$, G-prime with $P_1 = (P_1\cap S)R$ and $P_1\cap R_e = \bigcap\mathscr{P}_1^x\cap R_e = \bigcap(\mathscr{P}_1\cap R_e)^x = \bigcap(\bigcap\wp_1^y)^x = \bigcap\wp_1^x$. Hence P_1 is a minimal prime ideal over $(\bigcap\wp_1^x)R$ by Proposition 3.3 and therefore P_1 lies over $\wp_1$ with $P_1 \supseteq P$.

(2) By Theorem 3.8 there exist a finite number of prime ideals $P_1, P_2, \cdots, P_r$ of R lying over $\wp_1$. Then, it is clear from Proposition 3.3 that P_i's are the full set of minimal prime ideals over $(\bigcap\wp_1^x)R$. Therefore for some integer n,

$$(P_1\cap P_2 \cdots \cap P_r)^n \subseteq (\bigcap_{x\in G}\wp_1^x)R \subseteq (\bigcap_{x\in G}\wp^x)R \subseteq P,$$

and so $P_i \subseteq P$ for some i.

Remark 3.11. Another types of Going up and Going down Theorems of prime ideals do not hold; for example, let K be a field, G be an infinite cyclic group $\langle x\rangle$, and R be the group ring $K[G]$. Then consider two prime ideals $P = (x-1) \supsetneqq 0$. Obviously 0 is a prime ideal over the ideal 0 of $K = R_e$ but P does not lie over any ideal of R_e.

To give a counter example for another *Going down* theorem, let D be a commutative unique factorization domain and let $S = D[x]$, the polynomial ring over D in an indeterminate x. For any prime element p of D, put $\wp = pS+xS$, a prime ideal of S. Let $f(y) = xy+p \in S[y]$, the polynomial ring over S in an indeterminate y. Then by Eisenstein's theorem, $f(y)$ is a prime element in $S[y]$ with $f(y)S\cap S = 0$. Now let G be the infinite cyclic group generated by y and let R be the group ring $S[G]$ with $R_e = S$. Then $P = \wp[G]$ and $P_1 = f(y)R$ are both prime ideals of R satisfying; $P \supsetneqq P_1$, $P\cap R_e = \wp$ and $P_1\cap R_e = 0$. Hence P lies over $\wp$ but P_1 does not lie over 0, because 0 is a prime ideal of R.

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