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SIGNATURES ON A RING

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Introduction. The purpose of this paper is to define and to investigate a generalization of signatures. As is well known, an ordering of a field K is given by a signature $\sigma: K^* \to \{\pm 1\}$, or a ring homomorphism $\sigma: \mathrm{W}(K) \to \boldsymbol{Z}$. More generally, an ordering of higher level of K is given by a character $\chi: K^* \to \mathrm{S}^1$ which is called a signature of K in [4]. We give a generalization of such signatures, and a general theory of signatures of a ring. Let R be any ring with identity 1, and F a *field-like semigroup* (simply called *f-semigroup*) with zero element 0, unit element 1 and a unique element -1 of order 2, which is defined in §1. A signature of R over F is defined as a map $\sigma: R \to F$ satisfying conditions $\sigma(-1) = -1$, $\sigma(ab) = \sigma(a)\sigma(b)$, and $\sigma(a+b) = \sigma(b)$ providing $\sigma(a) = 0$ or $\sigma(a) = \sigma(b)$, which includes notions of orderings or higher level orderings of a field. Indeed, for a field R, if one takes $F = \mathrm{GF}(3) := \{0, 1, -1\}$, the signature σ gives an ordering of the field R. If one takes $F = \{0\} \cup \mathrm{S}^1$, the signature σ coincides with one in [4]. This definition is motivated by the results of Craven [5], [6], [7], and Becker [2], [3]. In §1, we introduce a topology on the set $\mathrm{X}(R, F)$ of all signatures of R over F, which is a generalization of the space of "real spectrum" in [8] and "space of orderings" in [16]. In §3, under the assumption that R is a commutative ring and F is a finite-f-semigroup, it is proved that for the quotient ring $S^{-1}R$ by a multiplicatively closed set S or R, the topological space $\mathrm{X}(S^{-1}R, F)$ is homeomorphic to a subspace $\mathrm{X}^S(R, F)$ of $\mathrm{X}(R, F)$, and for a semilocal commutative ring R, that there is a one to one correspondence between the set of infinite primes of level 1 and the set of ring homomorphism of the Witt ring $\mathrm{W}(R)$ onto the integers. Throughout this paper, we assume that every ring has identity 1, every ring homomorphism maps 1 to 1, and the unit group of the ring R is denoted by R^*. Furthermore, the number of elements of a finite set F is denoted by $|F|$, and for sets A and B, $A \backslash B := \{a \in A \mid a \notin B\}$.

1. Signatures over any f-semigroup. Let R be any (non-commutative) ring with identity 1.

Definition. A multiplicative abelian semigroup F with unit element 1 and zero element 0, i.e. $x1 = 1x = x$, $x0 = 0x = 0$ for $\forall x \in F$, is called an *f-semigroup* (*field-like semigroup*), if the subset $F^* = F \backslash \{0\}$ is a group with a unique element -1 of order 2.

Any field with characteristic not 2 is an f-semigroup.

Definition. Let F be an f-semigroup. A map $\sigma: R \rightarrow F$ is called a signature of R over F, if it satisfies the following conditions;

1) $\sigma(-1) = -1$,
2) $\sigma(ab) = \sigma(a)\sigma(b)$ for all $a, b \in R$,
3) either $\sigma(a) = 0$ or $\sigma(a) = \sigma(b)$ implies $\sigma(a+b) = \sigma(b)$.

By [10], a subset P of R which is closed under the addition and multiplication of R, and which does not contain -1, is called a preprime, and a maximal preprime is called a prime of R. A preprime containing 1 will be called an infinite preprime, and a maximal infinite preprime of R. Furthermore, an infinite preprime P will be called an infinite quasiprime, if $P \cap -P$ is a two sided ideal of R such that $R/(P \cap -P)$ is an integral domain.

Notation. For a signature $\sigma: R \rightarrow F$ of R over F and $\alpha \in F$, we denote by F^*, $\mathrm{G}(\sigma)$, $\wp_\alpha(\sigma)$ and $\mathrm{P}(\sigma)$ the following sets; $F^* = F \backslash \{0\}$, $\mathrm{G}(\sigma) = \mathrm{Im}\,\sigma \cap F^*$, $\wp_\alpha(\sigma) = \{r \in R \mid \sigma(r) = \alpha\}$ and $\mathrm{P}(\sigma) = \wp_0(\sigma) \cup \wp_1(\sigma)$.

Proposition 1.1. *Let $\{\wp_\alpha \mid \alpha \in F\}$ be a family of subsets of R. There exists a signature $\sigma: R \rightarrow F$ of R over F with $\wp_\alpha(\sigma) = \wp_\alpha$ for all $\alpha \in F$, if and only if the following conditions hold;*

(1) $R = \bigcup_{\alpha \in F} \wp_\alpha$, *and* $\wp_\alpha \cap \wp_\beta = \phi$ *for* $\alpha \neq \beta$ *in* F,
(2) $-1 \in \wp_{-1}$,
(3) $\wp_\alpha \wp_\beta \subseteq \wp_{\alpha\beta}$, *if* $\wp_\alpha \neq \phi$ *and* $\wp_\beta \neq \phi$,
(4) $\wp_\alpha + \wp_\alpha \subseteq \wp_\alpha$ *and* $\wp_0 + \wp_\alpha \subseteq \wp_\alpha$ *for* $\wp_\alpha \neq \phi$.

The proof is immediately from the definition of signature.

Corollary 1.2. *Let $\sigma: R \rightarrow F$ be a signature.*

(1) $\wp_0(\sigma) = \mathrm{P}(\sigma) \cap -\mathrm{P}(\sigma)$ *is a prime ideal, and* $\mathrm{P}(\sigma)$ *is an infinite quasiprime of* R. $\mathrm{G}(\sigma)$ *is a subsemigroup of* F^* *containing* -1.
(2) *If* $\mathrm{G}(\sigma)$ *is a finite set, then it is a group with even order.*

Lemma 1.3. *Let σ and τ be signatures of R over F, and assume that $\mathrm{G}(\sigma)$ is a subgroup of F^*.*

(1) *If* $\wp_0(\sigma) \subseteq \wp_0(\tau)$, *then the following conditions are equivalent;*
  1) $\wp_0(\sigma) = \wp_0(\tau)$,

2) $\wp_\alpha(\sigma)\cap\wp_0(\tau) = \phi$ *for some* $\alpha \in G(\sigma)$,

3) $\wp_\alpha(\sigma)\cap\wp_0(\tau) = \phi$ *for all* $\alpha \in G(\sigma)$.

(2) *Suppose* $P(\sigma) \subseteq P(\tau)$. *Then* $\wp_0(\sigma) = \wp_0(\tau)$ *if and only if* $\wp_1(\sigma) \subseteq \wp_1(\tau)$.

Proof. (1): 1) $\Longrightarrow$ 3) and 3) $\Longrightarrow$ 2) are easy.

2) $\Longrightarrow$ 1): Suppose $\wp_0(\sigma) \neq \wp_0(\tau)$, then $H = \{\alpha \in G(\sigma) \mid \wp_\alpha(\sigma)\cap\wp_0(\tau) \neq \phi\}$ is a non-empty subset of $G(\sigma)$. By (3) in (1.1), it follows that $\alpha \in H$ and $\beta \in G(\sigma)$ imply $\alpha\beta \in H$. Since $G(\sigma)$ is a group we get $H = G(\sigma)$.

(2) is easy from (1).

Definition. By $X(R, F)$, we denote the set of all signatures of R over F. On the f-semigroup F, we can define a topology such that $\{0\}$ is a closed subsets and $\{\alpha\}$ is an open subset of F for every $\alpha \in F^*$. For the discrete space R, the power space F^R has an open base consisting of $\{f \in F^R \mid f(a_i) \in U_i\,;\ i = 1, 2, \cdots, n\}$ for every finite subset $\{a_1, a_2, \cdots, a_n\}$ of R and any open subsets $U_1, U_2, \cdots, U_n$ of F. We introduce a topology on $X(R, F)$ as a subspace of F^R.

Proposition 1.4. *The topological space* $X(R, F)$ *has the following properties ;*

(1) *If* F *is a finite set, then* $X(R, F)$ *is a compact space.*

(2) *For any* $a \in R$ *and* $\alpha \in F^*$, $H_\alpha(a) = \{\sigma \in X(R, F) \mid \sigma(a) = \alpha\}$ *is an open subset of* $X(R, F)$. *The finite intersections of* $H_\alpha(a)$*'s for* $a \in R$ *and* $\alpha \in F^*$ *form an open basis of* $X(R, F)$, *so* $X(R, F)$ *is a* T_0*-space.*

(3) *For any* $a \in R$ *and* $\alpha \in F$, $H_0(a) = \{\sigma \in X(R, F) \mid \sigma(a) = 0\}$ *and* $H_\alpha^*(a) = H_0(a)\cup H_\alpha(a)$ *are closed subset of* $X(R, F)$.

Proof. (1): Suppose $|F| < \infty$. By F_d, we denote the discrete space on the set F in order to distinguish from the above topology on F. It is easy to see that $X(R, F)$ is a closed subset of $(F_d)^R$. Since $X(R, F)$ is a compact subspace of $(F_d)^R$ which is compact by Tychonoff's theorem, so is the subspace $X(R, F)$ of F_d^R which is the image of the continuous identity map $I: (F_d)^R \to F^R$.

(2): From the definitions of the topology on F and F^R, it follows that the subsets $H_\alpha(a)$ for $a \in R$ and $\alpha \in F^*$ form a subbasis of open sets in $X(R, F)$. Suppose $\sigma \neq \tau$ in $X(R, F)$. There is an $\alpha \in F^*$ with $\wp_\alpha(\sigma) \neq \wp_\alpha(\tau)$, and there exists an element a in R such that either $a \in \wp_\alpha(\sigma)$ with $a \notin \wp_\alpha(\tau)$ or $a \in \wp_\alpha(\tau)$ with $a \notin \wp_\alpha(\sigma)$, that is, $H_\alpha(a)$ is an open subset of $X(R, F)$ such that either $\sigma \in H_\alpha(a)$ with $\tau \notin H_\alpha(a)$ or $\tau \in H_\alpha(a)$ with $\sigma \notin H_\alpha(a)$. Hence, $X(R, F)$ is T_0-space.

(3): Since $X(R, F) = \bigcup_{\alpha\in F} H_\alpha(a)$ for any $a \in R$, it follows that $H_0(a)$ and

$H_a^*(a)$ are closed subsets in $X(R, F)$.

Notation. By $\boldsymbol{F}$, we denote the category of f-semigroups in which morphism $f: F_1 \to F_2$ satisfies $f(-1) = -1$, $f(0) = 0$ and $f(xy) = f(x)f(y)$ for any $x, y \in F_1$. By $\boldsymbol{R}$ and $\boldsymbol{T}$, we denote the category of rings with identity 1, and the category of topological spaces, respectively.

Proposition 1.5. *For any morphisms $f: F_1 \to F_2$ in $\boldsymbol{F}$ and $g: R_2 \to R_1$ in $\boldsymbol{R}$, map $X(g, f): X(R_1, F_1) \to X(R_2, F_2)$; $\sigma \leadsto f\cdot\sigma\cdot g$ is continuous, so $X(-, -): \boldsymbol{R}^\circ \times \boldsymbol{F} \to \boldsymbol{T}$ is a functor.*

Proof. Let $f: F_1 \to F_2$ and $g: R_2 \to R_1$ be morphisms in categories $\boldsymbol{F}$ and $\boldsymbol{R}$, respectively. To show that $X(g, f)$ is continuous, it is sufficient to show that for any $\sigma \in X(R_1, F_1)$ and an open subset $H_{\alpha'}(a')$ containing $X(g, f)(\sigma)$ $(= f\cdot\sigma\cdot g)$ of $X(R_2, F_2)$, $X(g, f)^{-1}(H_{\alpha'}(a'))$ is an open subset of $X(R_1, F_1)$. Since $f\cdot\sigma\cdot g(a') = \alpha' \neq 0$ and $f^{-1}(\alpha') \subseteq F_1^*$, $X(g, f)^{-1}(H_{\alpha'}(a')) = \bigcup_{\beta\in f^{-1}(\alpha')} H_\beta g((a'))$ is an open subset of $X(R_1, F_1)$.

Theorem 1.6. *Let $\mathfrak{a}$ be a two sided ideal of R and $\psi_{\mathfrak{a}}: R \to R/\mathfrak{a}$; $r \leadsto [r] = r + \mathfrak{a}$ the canonical ring homomorphism. Then, $X_{\mathfrak{a}}(R, F) := \{\tau \in X(R, F) \mid \mathfrak{a} \subset \wp_0(\tau)\}$ is a closed subset of $X(R, F)$, and the map $X(\psi_{\mathfrak{a}}, I)$ induces a homeomorphism $X(R/\mathfrak{a}, F) \rightarrowtail\!\!\!\to X_{\mathfrak{a}}(R, F)$.*

Proof. For any $\sigma \in X(R, F)\backslash X_{\mathfrak{a}}(R, F)$, $\mathfrak{a} \nsubseteq \wp_0(\sigma)$ and there is an $a \in \mathfrak{a}$ with $a \notin \wp_0(\sigma)$, hence $\sigma \in H_{\sigma(a)}(a)$ and $H_{\sigma(a)}(a) \cap X_{\mathfrak{a}}(R, F) = \phi$, so $X_{\mathfrak{a}}(R, F)$ is a closed subset of $X(R, F)$. For any $\sigma \in X_{\mathfrak{a}}(R, F)$, a signature $[\sigma]: R/\mathfrak{a} \to F$ is naturally defined by $[\sigma]([r]) = \sigma(r)$ for $[r] \in R/\mathfrak{a}$, because of $\sigma(a + r) = \sigma(r)$ for all $a \in \mathfrak{a}$ $(\subseteq \wp_0(\sigma))$. Hence, $X(\psi_{\mathfrak{a}}, I): X/(R/\mathfrak{a}, F) \to X_{\mathfrak{a}}(R, F)$ is a bijection, and is a homeomorphism, because of $X(\psi_{\mathfrak{a}}, I)(H_\alpha([r])) = X_{\mathfrak{a}}(R, F) \cap H_\alpha(r)$ for any $r \in R$ and $\alpha \in F^*$.

Notation. For any $\sigma \in X(R, F)$, we use notations ψ_σ and $X_\sigma(R, F)$ instead of $\psi_{\wp_0(\sigma)}$ and $X_{\wp_0(\sigma)}(R, F)$, i.e. $\psi_\sigma: R \to R/\wp_0(\sigma)$; $r \leadsto [r] = r + \wp_0(\sigma)$ and $X_\sigma(R, F) := \{\tau \in X(R, F) \mid \wp_0(\sigma) \subseteq \wp_0(\tau)\}$, respectively.

Corollary 1.7. *For any $\sigma \in X(R, F)$, $X_\sigma(R, F)$ is a closed subset of $X(R, F)$, and $X(\psi_\sigma, I): X(R/\wp_0(\sigma), F) \to X_\sigma(R, F)$ is a homeomorphism.*

Remark 1.8. (1) For $\sigma \in X(R, F)$, $\sigma(R^*)$ is a subgroup of F^*, and $\sigma(R^*)$

$\subseteq \mathrm{G}(\sigma)$. If $u \in R^*$ and $\sigma(u) = \alpha$, then $u^{-1} \in \wp_{\alpha^{-1}}(\sigma)$ and $\wp_\alpha(\sigma) = u\wp_1(\sigma) = \wp_1(\sigma)u$,

(2) For any $u \in R^*$, $\mathrm{H}_0(u) = \phi$ and $\mathrm{H}^*_\alpha(u) = \mathrm{H}_\alpha(u)$ is a open subset of $\mathrm{X}(R, F)$ for all $\alpha \in F^*$. If R is a division ring, then $\mathrm{X}(R, F)$ is Hausdorff and totally disconnected. Furthermore, if $|F| < \infty$, then $\mathrm{X}(R, F)$ is a Boolean space, i.e. a totally disconnected, compact and Hausdorff space.

(3) For any $a, b \in R$ and $\alpha, \beta \in F$, the following equalities and inequalities hold:

1) $\mathrm{H}_\alpha(a) = \mathrm{H}_{-\alpha}(-a)$ and $\mathrm{H}^*_\alpha(a) \cap \mathrm{H}^*_\alpha(-a) = \mathrm{H}_0(a)$,

2) $\mathrm{H}^*_\alpha(a) \cap \mathrm{H}^*_\beta(b) \subseteq \mathrm{H}^*_{\alpha\beta}(ab)$ and $\mathrm{H}^*_\alpha(a) \cap \mathrm{H}^*_\alpha(b) \subseteq \mathrm{H}^*_\alpha(a+b)$,

3) $\mathrm{H}_0(a) \cup \mathrm{H}_0(b) = \mathrm{H}_0(ab)$, $\mathrm{H}_0(0) = \mathrm{H}_1(1) = \mathrm{X}(R, F)$ and $\mathrm{H}_0(1) = \mathrm{H}^*_\alpha(1) = \phi$ for all $\alpha \in F^*$ with $\alpha \neq 1$.

From (1.8), the following proposition immediately follows:

Proposition 1.9. *For any $\sigma \in \mathrm{X}(R, F)$, the conditions* (1) *and* (2) *are equivalent, and* (1) $\Longrightarrow$ (3). *If R is commutative, then the converse* (3) $\Longrightarrow$ (1) *holds.*

(1) $\sigma(R^*) = \mathrm{G}(\sigma)$,

(2) *For any $\alpha \in \mathrm{G}(\sigma)$, there is a $u \in R^*$ with $\wp_\alpha(\sigma) = u\wp_1(\sigma) = \wp_1(\sigma)u$.*

(3) *$R^*/(R^* \cap \wp_1(\sigma)) \cong \mathrm{G}(\sigma)$ and $\wp_\alpha(\sigma)\wp_\beta(\sigma) = \wp_{\alpha\beta}(\sigma)$ for every $\alpha, \beta \in \mathrm{G}(\sigma)$.*

2. Signature over a finite f-semigroup. In this section, we assume that F is a finite set, and deal with signatures $\sigma: R \to F$ of R over a finite f-semigroup F so that $\mathrm{G}(\sigma)$ is a finite group with an even order.

Lemma 2.1. *For any $\sigma \in \mathrm{X}(R, F)$ and a prime ideal $\wp$ of R with an integral residue domain $R/\wp$, it follows that*

(1) *$\mathrm{G}_\wp(\sigma) := \{\alpha \in \mathrm{G}(\sigma) \mid \wp_\alpha(\sigma) \nsubseteq \wp\}$ is a subgroup of $\mathrm{G}(\sigma)$, so is $\mathrm{G}_\tau(\sigma) := \{\alpha \in \mathrm{G}(\sigma) \mid \wp_\alpha(\sigma) \nsubseteq \wp_0(\tau)\}$ for any $\tau \in \mathrm{X}(R, F)$, and*

(2) *$\wp_0(\sigma) \nsubseteq \wp$ (resp. $\wp_0(\sigma) \nsubseteq \wp_0(\tau)$) implies $G_\wp(\sigma) = \mathrm{G}(\sigma)$ (resp. $\mathrm{G}_\tau(\sigma) = \mathrm{G}(\sigma)$).*

Proof. (1) follows from that $|F| < \infty$ and $R/\wp$ is an integral domain.

(2): For any $\alpha \in \mathrm{G}(\sigma)$, $\wp_\alpha(\sigma) \subseteq \wp$ implies that $\wp_0(\sigma) + \wp_\alpha(\sigma) \subseteq \wp_\alpha(\sigma) \subseteq \wp$ and $\wp_0(\sigma) \subseteq \wp$, so (2) follows.

Proposition 2.2. *For $\sigma, \tau \in \mathrm{X}(R, F)$ with $\mathrm{P}(\sigma) \subseteq \mathrm{P}(\tau)$, there is a group* *epimorphism* $f: \mathrm{G}_\tau(\sigma) \to \mathrm{G}(\sigma)$ *such that*

$$\wp_\beta(\tau) \subseteq \bigcup_{\alpha\in f^{-1}(\beta)}\wp_\alpha(\sigma) \subseteq \wp_0(\tau)\cup\wp_\beta(\tau)$$

for all $\beta \in \mathrm{G}(\tau)$.

Proof. A map $f: \mathrm{G}_\tau(\sigma) \to \mathrm{G}(\sigma)$ can be defined by making the value $f(\alpha) = \tau(a)$ with $a \in \wp_\alpha(\sigma)\backslash\wp_0(\tau)$ for $\alpha \in \mathrm{G}_\tau(\sigma)$. Because, for $\alpha \in \mathrm{G}_\tau(\sigma)$ and any $a, a' \in \wp_\alpha(\sigma)\backslash\wp_0(\tau)$, we have $1 = \tau(ba) = \tau(b)\tau(a)$ and $1 = \tau(ba') = \tau(b)\tau(a')$ for any $b \in \wp_{\alpha^{-1}}(\sigma)\backslash\wp_0(\tau)$ $(\neq \phi)$, since $\alpha^{-1} \in \mathrm{G}_\tau(\sigma)$ and $ba, ba' \in \wp_1(\sigma)\backslash\wp_0(\tau)$ $(\subseteq \mathrm{P}(\sigma)\backslash\wp_0(\tau) \subseteq \mathrm{P}(\tau)\backslash\wp_0(\tau) = \wp_1(\tau))$. Since for $\alpha, \alpha' \in \mathrm{G}_\tau(\sigma)$, $a \in \wp_\alpha(\sigma)\backslash\wp_0(\tau)$ and $a' \in \wp_{\alpha'}(\sigma)\backslash\wp_0(\tau)$ imply $aa' \in \wp_{\alpha\alpha'}(\sigma)\backslash\wp_0(\tau)$, we get $f(\alpha, \alpha') = \tau(aa') = \tau(a)\tau(a') = f(\alpha)f(\alpha')$, so f is a homomorphism. For any $\beta \in \mathrm{G}(\tau)$, there is an $\alpha \in \mathrm{G}(\sigma)$ with $\wp_\alpha(\sigma)\cap\wp_\beta(\tau) \neq \phi$, because of $\wp_0(\sigma)\cap\wp_\beta(\tau) \subseteq \wp_0(\tau)\cap\wp_\beta(\tau) = \phi$. Hence, there is a $b \in \wp_\alpha(\sigma)\backslash\wp_0(\tau)$ with $f(\alpha) = \tau(b) = \beta$, so f is surjective. Suppose $\beta \in \mathrm{G}(\tau)$ and $x \in \wp_\beta(\tau)$. Since $\wp_0(\sigma)\cap\wp_\beta(\tau) = \phi$, there exists an $\alpha \in \mathrm{G}(\sigma)$ with $x \in \wp_\alpha(\sigma)$, so we get $f(\alpha) = \beta$ and $\wp_\beta(\tau) \subseteq {}_{\alpha\in f^{-1}(\beta)}\wp_\alpha(\sigma)$. It is easy to see that $\wp_\alpha(\sigma) \subseteq \wp_0(\tau)\cup\wp_\beta(\tau)$ for all $\alpha \in \mathrm{G}_\tau(\sigma)$ with $f(\alpha) = \beta$.

Definition. Let G and H be groups. A partial map $f: G \to H$ which is a homomorphism of a subgroup G_1 onto H, will be called a *partial epimorphism*, and for $b \in H$, $f^{-1}(H)$ and $f^{-1}(b)$ denote subsets $f^{-1}(H) := G_1$ and $f^{-1}(b) := \{x \in G_1 \mid f(x) = b\}$ of G.

Theorem 2.3. *Let* σ *and* τ *be elements of* $\mathrm{X}(R, F)$. $\mathrm{P}(\sigma) \subseteq \mathrm{P}(\tau)$ *holds if and only if there is a partial epimorphism* $f: \mathrm{G}(\sigma) \to \mathrm{G}(\tau)$ *with* $\wp_\beta(\tau) \subseteq \bigcup_{\alpha\in f^{-1}(\beta)}\wp_\alpha(\sigma) \subseteq \wp_0(\tau)\cup\wp_\beta(\tau)$ *for every* $\beta \in \mathrm{G}(\tau)$.

Proof. By (2.2), the "only if" part is proved. Suppose that $f: \mathrm{G}(\sigma) \to \mathrm{G}(\tau)$ is a partial epimorphism and $\wp_\beta(\tau) \subseteq \bigcup_{\alpha\in f^{-1}(\beta)}\wp_\alpha(\sigma) \subseteq \wp_0(\tau)\cup\wp_\beta(\tau)$ for every $\beta \in \mathrm{G}(\tau)$. Then we have $\bigcup_{\beta\in G(\tau)}\wp_\beta(\tau) \subseteq \bigcup_{\beta\in G(\tau)}(\bigcup_{\alpha\in f^{-1}(\beta)}\wp_\alpha(\sigma)) \subseteq \bigcup_{\alpha\in G(\sigma)}\wp_\alpha(\sigma)$. and so $\wp_0(\sigma) \subseteq \wp_0(\tau)$. Since $\wp_1(\sigma) \subseteq \bigcup_{\alpha\in f^{-1}(1)}\wp_\alpha(\sigma) \subseteq \wp_0(\tau)\cup\wp_1(\tau)$, we get $\mathrm{P}(\sigma) = \wp_0(\sigma)\cup\wp_1(\sigma) \subseteq \mathrm{P}(\tau)$.

Corollary 2.4. *Let* σ *and* τ *be elements of* $\mathrm{X}(R, F)$.

(1) $\mathrm{P}(\sigma) \subseteq \mathrm{P}(\tau)$ *and* $\wp_0(\sigma) = \wp_0(\tau)$ *hold, if and only if there is an epimorphism* $f: \mathrm{G}(\sigma) \to \mathrm{G}(\tau)$ *with* $\wp_\beta(\tau) = \bigcup_{\alpha\in f^{-1}(\beta)}\wp_\alpha(\sigma)$ *for every* $\beta \in \mathrm{G}(\tau)$.

(2) $\mathrm{P}(\sigma) \subseteq \mathrm{P}(\tau)$ *and* $|\mathrm{G}(\sigma)| = |\mathrm{G}(\tau)|$ *hold, if and only if there is an isomorphism* $f: \mathrm{G}(\sigma) \rightarrowtail\!\!\!\!\!\twoheadrightarrow \mathrm{G}(\tau)$ *with* $\wp_{f(\alpha)}(\tau) \subseteq \wp_\alpha(\sigma) \subseteq \wp_0(\tau)\cup\wp_{f(\alpha)}(\tau)$ *for every*

$\alpha \in G(\sigma)$.

(3) $P(\sigma) = P(\tau)$ *holds, if and only if there is an isomorphism* $f : G(\sigma) \rightarrowtail\!\!\!\twoheadrightarrow G(\tau)$ *with* $\tau = f \cdot \sigma$ *(i.e.* $\wp_\alpha(\sigma) = \wp_{f(\alpha)}(\tau)$ *for every* $\alpha \in G(\sigma)$*).*

Proof. (1): Suppose $P(\sigma) \subseteq P(\tau)$ and $\wp_0(\sigma) = \wp_0(\tau)$. By (2.2), there is an epimorphism $f : G_\tau(\sigma) \to G(\tau)$ with $\wp_\beta(\tau) \subseteq \bigcup_{\alpha \in f^{-1}(\beta)} \wp_\alpha(\sigma) \subseteq \wp_0(\tau) \cup \wp_\beta(\tau)$ for all $\beta \in G(\tau)$. The identity means that $\wp_\beta(\tau) = \bigcup_{\alpha \in f^{-1}(\beta)} \wp_\alpha(\sigma)$ and $G_\tau(\sigma) = G(\sigma)$. The converse is easy by (2.3).

(2) follows from that a partial epimorphism $f : G(\sigma) \to G(\tau)$ with $|G(\sigma)| = |G(\tau)|$ is an isomorphism.

(3) is easy from (1) and (2).

Definition. On $X(R, F)$, we can define an equivalent relation $\sim$ as follows: For $\sigma, \tau \in X(R, F)$, $\sigma \sim \tau$ if and only if there is an isomorphism $f : G(\sigma) \rightarrowtail\!\!\!\twoheadrightarrow G(\tau)$ with $\tau = f \cdot \sigma$, i.e. $P(\sigma) = P(\tau)$. By $X^*(R, F)$, we denote the quotient set $X(R, F)/\sim$. Then, we can identify $X^*(R, F)$ with the sets $P(\sigma)$ for all $\sigma \in X(R, F)$, and introduce the Zarisky topology on $X^*(R, F)$, that is, the finite intersection of $D(a) := \{P(\sigma) \mid a \notin P(\sigma), \sigma \in X(R, F)\}$ for all $a \in R$.

Proposition 2.5. *The map* $P(-) : X(R, F) \to X^*(R, F)$; $\sigma \leadsto P(\sigma)$ *is a continuous map, so* $X^*(R, F)$ *is a compact space.*

Proof. For any $a \in R$, $P^{-1}(D(a)) = \{\sigma \in X(R, F) \mid \sigma(a) \neq 0, 1\} = \bigcup_{\alpha \in G(\sigma) - \{1\}} H_\alpha(a)$, so $P^{-1}(D(a))$ is a open subset of $X(R, F)$.

Definition. A subset Y of $X(R, F)$ is said to be *irreducible*, if for any closed subset A and B of $X(R, F)$, $Y \subseteq A \cup B$ implies either $Y \subseteq A$ or $Y \subseteq B$.

The following lemma is immediately obtained from the above definition:

Lemma 2.6. *A subset* Y *of* $X(R, F)$ *is irreducible, if and only if for any* $H^*_{\alpha_1}(a_1), H^*_{\alpha_2}(a_2), \cdots, H^*_{\alpha_n}(a_n)$, $Y \subseteq \bigcup_{i=1}^{n} H^*_{\alpha_i}(a_i)$ *implies* $Y \subseteq H^*_{\alpha_i}(a_i)$ *for some* i.

Theorem 2.7. *If* Y *is a non-empty irreducible subset of* $X(R, F)$, *there exists a* $\sigma \in X(R, F)$ *such that the closure* $Cl(\{\sigma\})$ *of* $\{\sigma\}$ *coincides with the closure* $Cl(Y)$ *of* Y, *and the following identities hold :* $P(\sigma) = \bigcap_{\tau \in Y} P(\tau)$ *and* $\wp_\alpha(\sigma) = (\bigcap_{\tau \in Y} \wp_0(\tau) \cup \wp_\alpha(\tau)) \backslash \wp_0(\sigma)$ *for* $\alpha \in F^*$.

Proof. Let Y be a non-empty irreducible subset of $\mathrm{X}(R, F)$. We set $\wp_0(Y) := \bigcap_{\tau\in Y}\wp_0(\tau)$ $(= \{a \in R \mid Y \subseteq \mathrm{H}_0(a)\})$ and $\wp_\alpha(Y) := (\bigcap_{\tau\in Y}\wp_0(\tau)\cup \wp_\alpha(\tau))\backslash\wp_0(Y)$ $(= \{a \in R \mid Y \subseteq \mathrm{H}^*_\alpha(a), Y \nsubseteq \mathrm{H}_0(a)\})$ for $\alpha \in F^*$. Then, we have $\wp_0(Y)\cup\wp_\alpha(Y) = \{a \in R \mid Y \subseteq \mathrm{H}^*_\alpha(a)\}$ for every $\alpha \in F^*$. It is easy to check the conditions of (1.1) for the set $\{\wp_\alpha(Y) \mid \alpha \in F\}$. But, we try only to check for conditions 1) and 3).

1): To show $R = \bigcup_{\alpha\in F}\wp_\alpha(Y)$, suppose $a \in R\backslash\wp_0(Y)$. Put $\{\alpha_1, \alpha_2, \cdots, \alpha_r\} = \{\tau(a) \mid \tau \in Y\}\cap F^*$ $(\neq \phi)$, then it means $Y \subseteq \mathrm{H}^*_{\alpha_1}(a)\cup\mathrm{H}^*_{\alpha_2}(a)\cup \cdots \cup\mathrm{H}^*_{\alpha_r}(a)$, so $Y \subseteq \mathrm{H}^*_{\alpha_i}(a)$ for some α_i, since Y is irreducible. Hence, we get $a \in \wp_{\alpha_i}(Y)$.

3): To show $\wp_\alpha(Y)\wp_\beta(Y) \subseteq \wp_{\alpha\beta}(Y)$, suppose $\wp_\alpha(Y) \neq \phi$ and $\wp_\beta(Y) \neq \phi$. If $\alpha\beta = 0$, either $\alpha = 0$ or $\beta = 0$, so $\wp_\alpha(Y)\wp_\beta(Y) \subseteq \wp_0(Y)$ $(= \wp_{\alpha\beta}(Y))$ holds. Suppose $\alpha\beta \neq 0$. If $a \in \wp_\alpha(Y)$ and $b \in \wp_\beta(Y)$, $\tau(ab) = \tau(a)\tau(b)$ is either $\alpha\beta$ or 0 for every $\tau \in Y$, that is, $ab \in \wp_0(Y)\cup\wp_{\alpha\beta}(Y)$. If $ab \in \wp_0(Y)$, i.e. $\tau(ab) = 0$ for all $\tau \in Y$, then $Y \subseteq \mathrm{H}_0(ab) = \mathrm{H}_0(a)\cup\mathrm{H}_0(b)$, so we get either $Y \subseteq \mathrm{H}_0(a)$ or $Y \subseteq \mathrm{H}_0(b)$, i.e. either $a \in \wp_0(Y)$ or $b \in \wp_0(Y)$, which contradicts to $\wp_0(Y)\cap\wp_\alpha(Y) = \wp_0(Y)\cap\wp_\beta(Y) = \phi$. Hence, we get $ab \notin \wp_0(Y)$ and $ab \in \wp_{\alpha\beta}(Y)$. Thus, by (1.1) there is a signature $\sigma \in \mathrm{X}(R, F)$ which satisfies $\wp_\alpha(\sigma) = \wp_\alpha(Y)$ for all $\alpha \in F$. Since $\{a \in R \mid \sigma \in \mathrm{H}_\alpha{}^*(a)\} = \{a \in R \mid Y \subseteq \mathrm{H}^*_\alpha(a)\}$ holds for all $\alpha \in F$, we get $\mathrm{Cl}(\{\sigma\}) = \mathrm{Cl}(Y)$ and $\mathrm{P}(\sigma) = \bigcap_{\tau\in Y}\mathrm{P}(\tau)$.

Definition. By $\mathrm{X}_2(R, F)$, or simply $\mathrm{X}_2(R)$, we denote a subspace $\{\sigma \in \mathrm{X}(R, F) \mid |\mathrm{G}(\sigma)| = 2\}$ of $\mathrm{X}(R, F)$, and by $\mathrm{X}_2(R)$ a subspace $\{\mathrm{P}(\sigma) \mid \sigma \in \mathrm{X}_2(R)\}$ of $\mathrm{X}^*(R, F)$.

Proposition 2.8. (1) $\mathrm{X}_2(R)$ *is a closed subset of* $\mathrm{X}(R, F)$, *so it is compact.*

(2) *The map* $\mathrm{P}(-): \mathrm{X}_2(R) \to \mathrm{X}^*_2(R)$; $\sigma \leadsto \mathrm{P}(\sigma)$ *is homeomorphism, so we may regard as* $\mathrm{X}_2(R) = \mathrm{X}^*_2(R)$.

(3) *The finite intersections of* $\mathrm{H}_1(a)\cap\mathrm{X}_2(R)$ *for* $a \in R$ *form an open basis of* $\mathrm{X}_2(R)$.

Proof. If $\sigma \in \mathrm{X}(R, F)\backslash\mathrm{X}_2(R)$, there is an $a \in R$ with $\sigma(a) \notin \{1, -1\}$, which means $\sigma \in \mathrm{H}_\alpha(a)$ and $\mathrm{H}_\alpha(a)\cap\mathrm{X}_2(R) = \phi$. Thus, we get (1).

(2): It is easy to see that $\mathrm{P}(-): \mathrm{X}_2(R) \to \mathrm{X}^*_2(R)$ is a bijection. By (2.5), $\mathrm{P}(-)$ is continuous and the image of $\mathrm{H}_1(a)\cap\mathrm{X}_2(R)$ by $\mathrm{P}(-)$ is $\{\mathrm{P}(\sigma) \in \mathrm{X}^*_2(R) \mid \sigma(a) = 1, \sigma \in \mathrm{X}_2(R)\} = \{\mathrm{P}(\sigma) \in \mathrm{X}^*_2(R) \mid -a \notin \mathrm{P}(\sigma)\} = \mathrm{D}(-a)\cap\mathrm{X}^*_2(R)$ which is an open subset of $\mathrm{X}^*_2(R)$, so $\mathrm{P}(-): \mathrm{X}_2(R) \to \mathrm{X}^*_2(R)$ is a homeomorphism.

(3) is obvious.

Remark 2.9. Let σ and τ be elements in $X_2(R)$.

(1) $P(\sigma) \subseteq P(\tau)$ if and only if $\wp_1(\sigma) \supseteq \wp_1(\tau)$.

(2) $P(\sigma) \subseteq P(\tau)$ and $\wp_0(\sigma) = \wp_0(\tau)$ imply $\sigma = \tau$.

(3) Suppose $P(\sigma) \subseteq P(\tau)$. Then, for any $a \in R$, we have that $\sigma \in H_1^*(a) \Rightarrow \tau \in H_1^*(a)$, $\sigma \in H_0(a) \Rightarrow \tau \in H_0(a)$ and $\tau \in H_1(a) \Rightarrow \sigma \in H_1(a)$. Furthermore, if $P(\sigma) \neq P(\tau)$, there exists an $r \in R$ with $\sigma \in H_1(r)$ and $\tau \notin H_1(r)$.

Definition. An infinite preprime P with $P \cup -P = R$ will be said to be of *level* 1. For an infinite preprime P, we denote by $P^\dagger$ a subset $P \backslash (P \cap -P)$ $(= P \backslash -P)$ of P. If P is an infinite preprime of level 1, then $(P \cap -P)$ is a two sided ideal of R.

The following lemma is easy:

Lemma 2.10. *Let P be an infinite preprime of level* 1. *Then the following conditions are equivalent :*

(1) *P is an infinite quasiprime.*

(2) $P^\dagger \cdot P^\dagger \subseteq P^\dagger$.

(3) *For $x \in R$ and $y \in P^\dagger$, either $xy \in P$ or $yx \in P$ implies $x \in P$.*

Proposition 2.11. (1) *Let P be an infinite quasiprime of level* 1. *Then, $P^\dagger + P \subseteq P^\dagger$ holds. Suppose $x \in R$ and $y \in P^\dagger$. If $xy \in P^\dagger$ or $yx \in P^\dagger$ (resp. $xy \in (P \cap -P)$ or $yx \in (P \cap -P)$), then $x \in P^\dagger$ (resp. $x \in (P \cap -P)$).*

(2) *$X_2^*(R)$ is the set of all infinite quasiprimes of level* 1.

Proof. (1): Suppose that P is infinite quasiprime of level 1, $y \in P^\dagger$ $(= R \backslash -P)$ and $z \in P$. $-(y+z) \in P$ implies $-(y+z)+z \in P$, but $-y \notin P$, hence $y+z \notin -P$, i.e. $y+z \in P^\dagger$. We get $P^\dagger + P \subseteq P^\dagger$. If for $x \in R$, $xy \in P^\dagger$ (resp. $xy \in P \cap -P$), then by (2.10), $x \in P$ (resp. $x \in P \cap -P$). Since $x \in P \cap -P$, implies $xy \in P \cap -P$, i.e. $xy \notin P^\dagger$, we get that $xy \in P^\dagger$ imples $x \in P \backslash (P \cap -P) = P^\dagger$.

(2): For any $P(\sigma) \in X_2^*(R)$, by (1.2), $P(\sigma)$ is an infinite quasiprime of level 1. Conversely, suppose that P is any infinite quasiprime of level 1. (2.10), (1) in (2.11) and (1.1) mean that there is a $\sigma \in X_2(R)$ such that $\wp_0(\sigma) = P \cap -P$, $\wp_1(\sigma) = P^\dagger$ and $\wp_{-1}(\sigma) = -P^\dagger$. Hence, we get $P = P(\sigma) \in X_2^*(R)$.

Proposition 2.12. *Let Y be any totally ordered non-empty subset of* $(X_2^*(R), \subseteq)$.

(1) *Regarding as* $Y \subseteq X_2^*(R) = X_2(R) \subseteq X(R, F)$, *$Y$ is irreducible subset of* $X(R, F)$, *and there is a* $\sigma \in X_2(R)$ *such that* $Cl(\{\sigma\}) = Cl(Y)$ *and* $P(\sigma) = \bigcap_{\tau \in Y} P(\tau) = Inf(Y)$ *in* $(X_2^*(R), \subseteq)$.

(2) *There exists the* $Sup(Y)$ *in* $(X_2^*(R), \subseteq)$.

(3) *For any* $\sigma \in X_2(R)$, *there is a maximal element in* $\{P(\tau) \in X_2^*(R) \mid P(\sigma) \subseteq P(\tau)\}$, *and there is a minimal element in* $\{P(\rho) \in X_2^*(R) \mid P(\rho) \subseteq P(\sigma)\}$.

Proof. Let Y be a totally ordered subset of $(X_2^*(R), \subseteq)$. Suppose $Y \subseteq H_1^*(a_1) \cup \cdots \cup H_1^*(a_r) \cup H_0(b_1) \cup \cdots \cup H_0(b_s)$. If $Y \nsubseteq H_1^*(a_i)$ and $Y \nsubseteq H_0(b_j)$ for every i and j, then there exist elements σ_i and τ_j of Y such that $\sigma_i \notin H_1^*(a_i)$ and $\tau_j \notin H_0(b_j)$ for $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$. Since Y is totally ordered, there is a unique minimal element $P(\rho)$ in $P(\sigma_1), \cdots, P(\sigma_r), P(\tau_1), \cdots, P(\tau_s)$. By (2.9), it follows that $\rho \notin H_1^*(a_i)$ and $\rho \notin H_0(b_j)$ for every $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$, which is a contradiction. Hence, Y is included in some $H_1^*(a_i)$ or $H_0(b_j)$, so Y is irreducible.

(2): Since Y is totally ordered, it follows that P: $\bigcup_{\tau \in Y} P(\tau)$ is an infinite preprime of level 1, $P \cap -P = \bigcup_{\tau \in Y}(P(\tau) \cap -P(\tau)) = \bigcup_{\tau \in Y} \wp_0(\tau)$ and $R/(P \cap -P)$ is an integral domain. Hence, P is an infinite quasiprime of level 1 which is contained in $X_2^*(R)$ by (2.11) and coincides with $Sup(Y)$.

(3) is obtained by Zorn's lemma.

Lemma 2.13. *(cf.* [8], (2.1)*). Let σ and τ be elements in* $X_2(R)$.

(1) $P(\sigma) \nsubseteq P(\tau)$ *and* $P(\tau) \nsubseteq P(\sigma)$ *imply* $\wp_1(\sigma) \cap \wp_{-1}(\tau) \neq \phi$.

(2) *If there exists a* $\rho \in X_2(R)$ *with* $P(\rho) \subseteq P(\sigma) \cap P(\tau)$, *either* $P(\sigma) \subseteq P(\tau)$ *or* $P(\tau) \subseteq P(\sigma)$ *holds.*

(3) *A set* $\{P(\rho) \in X_2^*(R) \mid P(\sigma) \subseteq P(\rho)\}$ *has a unique maximal element with respect to the ordering "$\subseteq$".*

Proof. (1) Suppose $P(\sigma) \nsubseteq P(\tau)$ and $P(\tau) \nsubseteq P(\sigma)$, then there are $a \in P(\sigma) \backslash P(\tau)$ and $b \in P(\tau) \backslash P(\sigma)$ which mean $a \in \wp_1(\tau) \cap P(\sigma)$ and $b \in \wp_1(\sigma) \cap P(\tau)$. Accordingly, we get that $a - b \in (P(\sigma) + \wp_1(\sigma)) \cap (\wp_{-1}(\tau) - P(\tau)) \subseteq \wp_1(\sigma) \cap \wp_{-1}(\tau)$, so $\wp_1(\sigma) \cap \wp_{-1}(\tau) \neq \phi$.

(2): Suppose that $P(\rho) \subseteq P(\sigma) \cap P(\tau)$, $P(\sigma) \nsubseteq P(\tau)$ and $P(\tau) \nsubseteq P(\sigma)$ hold for some $\rho, \sigma, \tau \in X_2^*(R)$. By (1), we get $\wp_1(\sigma) \cap \wp_{-1}(\tau) \neq \phi$. However, it is contrary to $\wp_1(\sigma) \cap \wp_{-1}(\tau) \subseteq \wp_1(\rho) \cap \wp_{-1}(\rho) = \phi$.

(3) is immediate from (2) and (2.12).

Notation. By $X_2^M(R)$ and $X_2^m(R)$, we denote $X_2^M(R) := \{\sigma \in X_2(R) \mid P(\sigma)$ is maximal in $(X_2^*(R), \subseteq)\}$ and $X_2^m(R) := \{\sigma \in X_2(R) \mid P(\sigma)$ is minimal in $(X_2^*(R), \subseteq)\}$.

Remark 2.14. If R is commutative, then $X_2^M(R)$ coincides with the set of infinite primes of level 1 in R.

Proposition 2.15. (1) *$X_2^M(R)$ and $X_2^m(R)$ are Hausdorff spaces as subspaces of $X_2(R)$.*
(2) *$X_2^m(R)$ is dense in $X_2(R)$, i.e. $\mathrm{Cl}(X_2^m(R)) = X_2(R)$.*
(3) *For any $\sigma \in X_2(R)$, a subset $\{\tau \in X_2(R) \mid P(\sigma) \subseteq P(\tau)\}$ is a closed subset of $X_2(R)$, so is a subset $\{\tau\}$ for every element $\tau \in X_2^M(R)$.*

Proof. (1): If σ and τ are distinct elements in $X_2^M(R)$ (resp. $X_2^m(R)$), then $P(\sigma) \nsubseteq P(\tau)$ and $P(\tau) \nsubseteq P(\sigma)$. By (2.13), there is an $a \in \wp_1(\sigma) \cap \wp_{-1}(\tau)$ which satisfies $\sigma \in H_1(a)$, $\tau \in H_{-1}(a)$ and $H_1(a) \cap H_{-1}(a) = \phi$.

(2): For any $\sigma \in X_2(R)$, by (2.12) there is a $\rho \in X_2^m(R)$ with $P(\sigma) \subseteq P(\sigma)$. (2.9) means that for any $a \in R$, $\sigma \in H_1(a)$ implies $\rho \in H_1(a)$, that is, $X_2^m(R)$ is dense in $X_2(R)$.

(3): For a $\sigma \in X_2(R)$, we put $Y = \{P(\tau) \in X_2(R) \mid P(\sigma) \subseteq P(\tau)\}$. If $\rho \in X_2(R) \backslash Y$, then by (2.9) we have $P(\sigma) \nsubseteq P(\rho)$ and $\wp_1(\rho) \nsubseteq \wp_1(\sigma)$, so there is an $a \in \wp_1(\rho) \backslash \wp_1(\sigma)$. $H_1(a)$ is an open subset with $\rho \in H_1(a)$ and $H_1(a) \cap Y = \phi$, that is, Y is closed in $X_2(R)$.

3. Signatures of a commutative ring. In this section, we assume that R is a commutative ring with identity 1, and F is a finite f-semigroup.

Notation. Let S be a multiplicatively closed subset of R such that $1 \in S$ and $0 \notin S$. By $S^{-1}R$, we denote the quotient ring by S, and by $\psi^S : R \to S^{-1}R$, the canonical ring homomorphism. By $X^S(R, F)$, we denote a subset $X^S(R, F) := \{\sigma \in X(R, F) \mid \wp_0(\sigma) \cap S = \phi\}$. If $\lambda \in X(R, F)$ and $S = R \backslash \wp_0(\lambda)$ for the prime ideal, we denote by $X^\lambda(R, F)$, $R^{(\lambda)}$ and ψ^λ instead of $X^S(R, F)$, $S^{-1}R$ and ψ^S that is, $X^\lambda(R, F) = \{\sigma \in X(R, F) \mid \wp_0(\sigma) \subseteq \wp_0(\lambda)\}$, $R^{(\lambda)} = (R \backslash \wp_0(\lambda))^{-1}R$ and $\psi^\lambda : R \to R^{(\lambda)}$.

Theorem 3.1. *Let S be a multiplicatively closed subset of R with $1 \in S$ and $0 \notin S$. The map $X(\psi^S, I)$ induces a homeomorphism of $X(S^{-1}R, F)$ onto the subspace $X^S(R, F)$ of $X(R, F)$, and $G(\pi) = G(X(\psi^S, I)(\pi))$ holds for every $\pi \in$*

$X(S^{-1}R, F)$.

Proof. First, we shall show that $\mathrm{Im}\, X(\psi^S, I) = X^S(R, F)$ and $X(\psi^S, I)$ is a bijection. For any $\pi \in X(S^{-1}R, F)$, it is easy that $\psi^S(\wp_0(\pi\cdot\psi^S)\cap S) \subseteq \wp_0(\pi)\cap \psi^S(S) = \phi$, so $\wp_0(\pi\cdot\psi^S)\cap S = \phi$ and $X(\psi^S, I)(\pi) = \pi\cdot\psi^S \in X^S(R, F)$. Conversely, for a $\sigma \in X^S(R, F)$, we can define a map $\pi: S^{-1}R \to F$ as follows: For any $x \in S^{-1}R$, there are $s \in S$ and $r \in R$ with $\psi^S(s)x = \psi^S(r)$. Then, we put $\pi(x) = \sigma(s)^{-1}\sigma(r)$, so the map π is well defined because of $\wp_0(\sigma)\cap S = \phi$. It is easy to see that π is a unique signature satisfying $X(\psi^S, I)(\pi) = \pi\cdot\psi^S = \sigma$. Hence, we get that $X(\psi^S, I): X(S^{-1}R, F) \to X^S(R, F)$ is a bijection. Since $|F| < \infty$, it follows that $G(\pi) = G(\pi\cdot\psi^S)$ is a finite subgroup of F^*. Let $|G(\pi)| = n$. For any $a \in S^{-1}R$ and $\alpha \in G(\pi)$, there are $s \in S$ and $r \in R$ with $\psi^S(s)a = \psi^S(r)$, and $H_\alpha(a) = H_\alpha(\psi^S(s^n)a) = H_\alpha(\psi^S(s^{n-1}r))$ hold. We get that $X(\psi^S, I)(H_\alpha(a)) = \{\pi\cdot\psi^S \in X^S(R, F) \mid \pi(\psi^S(s^{n-1}r) = \alpha\} = H_\alpha(s^{n-1}r)\cap X^S(R, F)$ is an open subset of $X^S(R, F)$, hence $X(\psi^S, I): X(S^{-1}R, F) \to X^S(R, F)$ is a homeomorphism. If $X(\psi^S, I)(\pi) = \pi\cdot\psi^S = \sigma$ for $\pi \in X(S^{-1}R, F)$, it is easy to see that $G(\pi) = G(\sigma)$.

Corollary 3.2. (1) *Let $\wp$ be a prime ideal of R, and let $S = R\backslash\wp$. The map $X(\psi^S, I)$ induces a homeomorphism $X(S^{-1}R, F) \to X^S(R, F)$.*
(2) *For any $\lambda \in X(R, F)$, $X(\psi^\lambda, I): X(R^{(\lambda)}, F) \to X^\lambda(R, F)$ is a homeomorphism.*

Notation 3.3. For any $\lambda \in X(R, F)$, λ belongs to $X^\lambda(R, F)$. By λ^*, we denote the signature $\lambda^*: R^{(\lambda)} \to F$ with $\lambda = \lambda^*\cdot\psi^\lambda$ determined by λ in (3.2). Then, $\lambda^*((R^{(\lambda)})^*) = G(\lambda^*) = G(\lambda)$ hold, (cf. (1.9)).

Remark 3.4. Let R be a semilocal ring with the maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_r$ and $\sigma \in X(R, F)$. If $\alpha \in \bigcap_{i=1}^r G_i(\sigma)$, then $\wp_\alpha(\sigma)\cap R^* \neq \phi$, that is, there is a $u \in R^*$ with $\wp_\alpha(\sigma) = u\wp_1(\sigma)$, where $G_i(\sigma) = G_{\mathfrak{m}_i}(\sigma) = \{\alpha \in G(\sigma) \mid \wp_\alpha(\sigma) \not\subseteq \mathfrak{m}_i\}$. Hence, if $G_i(\sigma) = G(\sigma)$ for every $i = 1, 2, \cdots, r$, then the conditions in (1.9) hold.

Proof. Suppose $\alpha \in \bigcap_{i=1}^r G_i(\sigma)$. Using the induction on the umber k of maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_k$, we show that there is a $u \in \wp_\alpha(\sigma)$ with $u \notin \mathfrak{m}_i$ for $i = 1, 2, \cdots, k$. Put $|G(\sigma)| = n$. If $\wp_\alpha(\sigma)\cap\mathfrak{m}_i = \phi$, then we may exclude such a maximal ideal $\mathfrak{m}_i$. Hence we may suppose $\wp_\alpha(\sigma)\cap\mathfrak{m}_i \neq \phi$ for $i = 1, 2, \cdots, k$. Using the assumption on induction for $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_{i-1}, \mathfrak{m}_{i+1}, \cdots, \mathfrak{m}_k$, we can find $u_i \in \wp_\alpha(\sigma)$ with $u_i \notin \mathfrak{m}_j$ for $j = 1, 2, \cdots, i-1, i+1, \cdots, k$. If u_i

$\notin \mathfrak{m}_i$ for some i, then we can take $u = u_i$. If $u_i = \mathfrak{m}_i$ for all i, then we put $v_i = u_2(u_3 \cdots u_k)^n$ and $v_i = u_1(u_2 \cdots u_{i-1}u_{i+1} \cdots u_k)^n$ for $i = 2, \cdots, k$. Since $v_i \in \mathfrak{m}_j \backslash \mathfrak{m}_i$ for every $i \neq j$ and $v_i \in \wp_a(\sigma)$ for all i, we get $u = v_1 + v_2 + \cdots + v_k \in \wp_a(\sigma)$ and $u \notin \mathfrak{m}_i$ for all i. Thus, there exists a $u \in R^*$ with $\wp_a(\sigma) = u\wp_1(\sigma)$.

In the last of this scetion, we note a relation between $X_2(R)$ and the set of ring homomorphisms of the Witt ring $W(R)$ on to the integers $\boldsymbol{Z}$. Let $W(R)$ be the Witt ring of bilinear spaces over R (cf. [1], p. 19). By $\mathrm{Sig}(R)$, we denote the set of ring-homomorphisms of $W(R)$ on to $\boldsymbol{Z}$. For $a \in R^*$, $[a]$ denotes the element of $W(R)$ with its representative $\langle a \rangle$, where $\langle a \rangle$ denotes a bilinear space of rank one with value a modulo R^{*2}.

Lemma 3.5. *For any $\lambda \in X_2(R)$, the signature λ^*, defined in* (3.3), *determines a ring homomorphism $\lambda^* : W(R^{(\lambda)}) \to \boldsymbol{Z}$, which is denoted by λ^* using the same notation. Therefore, a map $\theta : X_2(R) \to \mathrm{Sig}(R)\,;\ \lambda \rightsquigarrow \lambda^* \cdot W(\psi^\lambda)$ is defined.*

Proof. For $\lambda \in X_2(R)$, $\lambda^* : R^{(\lambda)} \to F$ is a signature with $\lambda^* \cdot \psi^\lambda = \lambda$ and $G(\lambda^*) = G(\lambda) = \{1, -1\}$. $\wp_1(\lambda^*)$ satisfies the following conditions; $\wp_1(\lambda^*) + \wp_1(\lambda^*) \subseteq \wp_1(\lambda^*)$, $\wp_1(\lambda^*) \cdot \wp_1(\lambda^*) \subseteq \wp_1(\lambda^*)$, $\wp_1(\lambda^*) \cap -\wp_1(\lambda^*) = \phi$ and $\wp_1(\lambda^*) \cup -\wp_1(\lambda^*) = R^{(\lambda)}$. Hence, we can define an ordering on the local ring $R^{(\lambda)}$ which determines a ring homomorphism $\lambda^* : W(R^{(\lambda)}) \to \boldsymbol{Z}$ with $\lambda^*([a]) = \lambda^*(a)$ ($\in G(\lambda^*) = \{1, -1\} \subseteq \boldsymbol{Z}$) for every $a \in (R^{(\lambda)})^*$ (cf. (2.2) and (2.5) in [12]).

Proposition 3.6. *Let $\wp$ be a prime ideal of R, and let $S = R \backslash \wp$.*

(1) *If P is an infinite quasiprime of level* 1 *in $S^{-1}R$, then so is the inverse image $(\psi^S)^{-1}(P)$ in R.*

(2) *Let $\lambda \in X_2(R)$ and $\mu \in \mathrm{Sig}(R)$, and $Q = \{a \in R \mid a \in S, \mu([\psi^S(a)]) = 1\}$. If $Q \subseteq \wp_1(\lambda)$, then there is an R-algebra homomorphism $f : S^{-1}R \to R^{(\lambda)}$ with $\mu = \lambda^* \cdot W(f)$.*

Proof. (1): For any infinite quasiprime P in $S^{-1}R$, it is easy to see that $(\psi^S)^{-1}(P)$ is an infinite preprime and $(\psi^S)^{-1}(P \cap -P) = (\psi^S)^{-1}(P) \cap -(\psi^S)^{-1}(P)$ is a prime ideal of R.

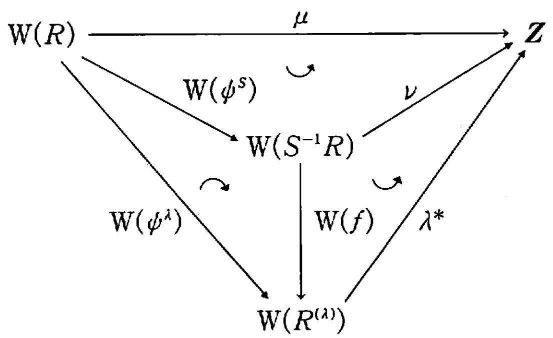
(2): Since Q is multiplicatively closed and $Q \cup -Q = S$, it follows that $Q^{-1}Q = \{\psi^S(a)^{-1} \cdot \psi^S(b) \in S^{-1}R \mid a, b \in Q\}$ is a positive cone of an ordering on $S^{-1}R$ defined by μ. Since $Q \subseteq \wp_1(\lambda)$, there is a natural R-algebra homomorphism $f : S^{-1}R \to \wp_1(\lambda)^{-1}R = R^{(\lambda)}$. Since f carries the positive cone $Q^{-1}Q$ of ordering on $S^{-1}R$ into the positive cone $\wp_1(\lambda^*)$ of ordering on $R^{(\lambda)}$, so we get $\mu = \lambda^* \cdot \mathrm{W}(f)$.

Corollary 3.7. *If* $\lambda, \lambda' \in \mathrm{X}_2(R)$ *with* $\mathrm{P}(\lambda) \subseteq \mathrm{P}(\lambda')$, *then there is an* R*-algebra homomorphism* $g: R^{(\lambda')} \to R^{(\lambda)}$ *with* $\lambda'^* = \lambda^* \cdot \mathrm{W}(g)$.

Proof. Since $\mathrm{P}(\lambda) \subseteq \mathrm{P}(\lambda')$ implies $\wp_1(\lambda') \subseteq \wp_1(\lambda)$, the proof is immediately from (3.6).

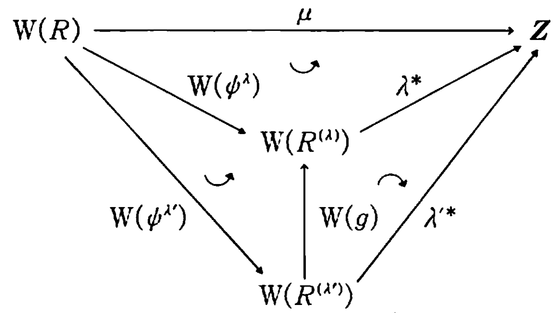
Theorem 3.8. *The map* $\theta: \mathrm{X}_2(R) \to \mathrm{Sig}(R)$ *is surjective, and* $\theta(\mathrm{X}_2^M(R)) = \mathrm{Sig}(R)$, *that is, for any ring homomorphism* $\mu: \mathrm{W}(R) \to \boldsymbol{Z}$, *there is an infinite prime* P *of level* 1 *in* R *such that* $\mu([a]) = 1$ *for all* $a \in P \cap R^*$.

Proof. To show that $\theta: \mathrm{X}_2(R) \to \mathrm{Sig}(R)\,;\ \lambda \rightsquigarrow \lambda^* \cdot \mathrm{W}(\psi^\lambda)$ is surjective, suppose $\mu \in \mathrm{Sig}(R)$. By Lemma 1 and Proposition 1 in [9], there is a maximal ideal $\mathfrak{m}$ of R, and for $S = R \backslash \mathfrak{m}$ there is a ring homomorphism $\nu: \mathrm{W}(S^{-1}R) \to \boldsymbol{Z}$ with $\mu = \nu \cdot \mathrm{W}(\psi^S)$. We put $Q' = \{\sum_i a_i \in S^{-1}R \mid a_i \in (S^{-1}R)^*, \nu([a_i]) = 1\}$. By Theorem 4.1 in [14], it follows that $Q' + Q' \subseteq Q'$, $Q' \cdot Q' \subseteq Q'$, $1 \in Q'$ and $\wp = (S^{-1}R) \backslash (Q' \cup Q')$ is a prime ideal of $S^{-1}R$ with $\wp + Q' \subseteq Q'$. Hence, $P = \wp \cup Q'$ is an infinite quasiprime of level 1 in $S^{-1}R$, and so is also $(\psi^S)^{-1}(P)$ in R by (3.6). Hence, there is a $\lambda \in \mathrm{X}_2(R)$ with $\mathrm{P}(\lambda) = (\psi^S)^{-1}(P)$. We shall show $\theta(\lambda) = \mu$. We put $Q = \{a \in S \mid \nu([\psi^S(a)]) = 1\}$. From the fact that $\psi^S(Q) = Q'$ and $(\psi^S)^{-1}(Q') = \wp_1(\lambda)$, it follows that $Q \subseteq \wp_1(\lambda)$, and using (3.6), there in an R-algebra homomorphism $f: S^{-1}R \to R^{(\lambda)}$ making the following diagram commute;



Thus, we get $\mu = \nu \cdot \mathrm{W}(\psi^S) = \lambda^* \cdot \mathrm{W}(\psi^S) = \theta(\lambda)$. In the second place, we shall show $\theta(\mathrm{X}_2^M(R)) = \mathrm{Sig}(R)$. For any $\mu \in \mathrm{Sig}(R)$, we can find a $\lambda \in \mathrm{X}_2(R)$ with $\theta(\lambda) = \mu$. By (2.13), we can also find a $\lambda' \in \mathrm{X}_2^M(R)$ with $\mathrm{P}(\lambda) \subseteq \mathrm{P}(\lambda')$. By (3.7), there is an R-algebra homomorphism $g: R^{(\lambda')} \to R^{(\lambda)}$ with $\lambda'^* = \lambda^* \cdot \mathrm{W}(g)$.

Therefore, we get $\mu = \theta(\lambda')$ by the following commutative diagram;



Corollary 3.9. *If R is a semilocal ring, then the map θ induces a bijection* $X_2^M(R) \to \mathrm{Sig}(R)$.

Proof. To show that $\theta : X_2^M(R) \to \mathrm{Sig}(R)$ is a bijection, we suppose that $\mu \in \mathrm{Sig}(R)$ and $\sigma, \tau \in X_2^M(R)$ with $\theta(\sigma) = \theta(\tau) = \mu$. By Appendix B in [14], a subset $Q = \{\sum_{i=1}^n a_i b_i^2 \in R \mid a_1, a_2, \cdots, a_n \in R^*, b_1, b_2, \cdots, b_n \in R\,;\ \mu([a_i]) = 1, \sum_{i=1}^n b_i R = R\}$ of R satisfies that $Q + Q \subseteq Q$, $QQ \subseteq Q$, $1 \in Q$, and $\wp = R \backslash (Q \cup -Q)$ is a prime ideal of R with $\wp + Q \subseteq Q$. Hence, $P = \wp \cup Q$ is an infinite quasiprime of R, and there is a $\lambda \in X_2(R)$ with $P(\lambda) = P$. We can easily check that $\wp_1(\lambda) = Q$, $Q \subseteq \wp_1(\sigma)$ and $Q \subseteq \wp_1(\tau)$. Accordingly, by (2.9), we get $P(\sigma) \subseteq P(\lambda)$ and $P(\tau) \subseteq P(\lambda)$, so $P(\sigma) = P(\tau) = P(\lambda)$, that is, $\sigma = \tau$.

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