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GENERALIZED TILTING MODULES AND APPLICATIONS TO MODULE THEORY

Dedicated to Professor Takasi Nagahara on his 60th birthday

YOICHI MIYASHITA

In the representation theory of algebras, the notion of a tilting module of finite projective dimension (cf. [7]) is well known, and there are many papers concerning this. On the other hand, we know that Matlis [6] and Facchini [2] treat some divisible modules over a commutative integral domain, and present theorems which are analogous to the tilting theorems. However, in the latter theories, "tilting modules" are infinitely generated and are not projective, in contrast with the fact that usual tilting modules of finite projective dimensional, over a commutative ring are necessarily projective. In view of this and others, we aim to extend tilting theorems in [7] to more general ones which contain theorems of Matlis [6] and Facchini [2] as special cases. As a result, we obtain two tilting theorems which are dual to each other. One theorem holds under some conditions concerning projective modules, and the other theorem holds under some conditions concerning injective modules.

We now state definitions, notations, and main theorems. As to some definitions and notations we follow [7].

Let A and B be rings with 1. By A-Mod (resp. A-mod) we denote the category of left A-modules (resp. finitely generated left A-modules). Similarly, for right A-modules, we use Mod-A and mod-A. Let ${}_AT_B$ be a bimodule, and $e \geqq 0$ be an integer. We put

$$\mathrm{KT}({}_AT) = \{N'_A \mid N'_A \in \text{Mod-}A,\ \mathrm{Tor}_i^A(N', T) = 0 \quad (i \geqq 0)\},$$
$$\mathrm{KE}({}_AT) = \{{}_AN \mid {}_AN \in A\text{-Mod},\ \mathrm{Ext}_A^i(T, N) = 0 \quad (i \geqq 0)\},$$

where $\mathrm{Tor}_0^A(N', T) = N' \otimes_A T$ and $\mathrm{Ext}_A^0(T, N) = \mathrm{Hom}_A(T, N)$. Furthermore we put

$$\mathrm{kT}({}_AT) = \{X'_A \mid \text{if } f: N'_A \to X'_A \text{ and } N'_A \in \mathrm{KT}({}_AT) \text{ then } \mathrm{Cok}\, f \in \mathrm{KT}({}_AT)\},$$
$$\mathrm{kE}({}_AT) = \{{}_AX \mid \text{if } g: {}_AX \to {}_AN \text{ and } {}_AN \in \mathrm{KE}({}_AT) \text{ then } \mathrm{Ker}\, g \in \mathrm{KE}({}_AT)\}.$$

Similarly we define $\mathrm{KT}(T_B)$, $\mathrm{KE}(T_B)$, $\mathrm{kT}(T_B)$, and $\mathrm{kE}(T_B)$.

For any projective B-module ${}_BP$ we denote by h_P the canonical map

$$_BP \to {}_B\mathrm{Hom}({}_AT_B, {}_AT\otimes_B P),\ p \mapsto (t \to t\otimes p),$$

and we put $_BP^* = {}_B\mathrm{Hom}({}_AT_B, {}_AT\otimes_B P)$.

For any injective A-module $_AI$, we denote by k_I the canonical map

$$_AT\otimes_B \mathrm{Hom}({}_AT_B, {}_AI) \to {}_AI,\ t\otimes f \mapsto (t)f,$$

and we put $I^\dagger = {}_AT\otimes_B\mathrm{Hom}({}_AT_B, {}_AI)$.

Our main theorems hold under the following conditions.

Condition $_BP$. (1) For any projective B-module $_BP$, there hold Ker h_P, Cok $h_P \in \mathrm{KT}(T_B)$, and $_B\mathrm{Ext}^i({}_AT_B, {}_AT\otimes_B P) \in \mathrm{kT}(T_B)$ $(i \geqq 1)$.

(2) There is an integer $r \geqq 0$ such that $_B\mathrm{Ext}^i({}_AT_B, {}_AX) \in \mathrm{KT}(T_B)$ for any $i > r$ and any $_AX \in A$-Mod.

Condition $_AI$. (1) For any injective A-module $_AI$, there hold Ker k_I, Cok $k_I \in \mathrm{KE}({}_AT)$, and $_A\mathrm{Tor}_i({}_AT_B, {}_B\mathrm{Hom}({}_AT_B, {}_AI)) \in \mathrm{kE}({}_AT)$ $(i \geqq 1)$.

(2) There is an integer $r \geqq 0$ such that $_A\mathrm{Tor}_i({}_AT_B, {}_BY) \in \mathrm{KE}({}_AT)$ for any $i > r$ and any $_BY \in B$-Mod.

Then our main theorms are the following

Theorem 1.12. *Assume that $_AT_B$ satisfies Condition $_BP$, and $e \geqq 0$ be an integer. Let $_BY$ be a B-module such that* $\mathrm{Tor}_i(T_B, {}_BY) = 0$ $(0 \leqq i < e)$. $_A\mathrm{Tor}_i({}_AT_B, {}_BY) \in \mathrm{KE}({}_AT)$ $(i > e)$ *and such that* $\mathrm{Ext}^j({}_BN', {}_BY) = 0$ $(j = 0, 1)$ *for any* $_BN' \in \mathrm{KT}(T_B)$. *Put* $X = {}_A\mathrm{Tor}_e({}_AT_B, {}_BY)$. *Then* $\mathrm{Ext}^i({}_AT, {}_AX) = 0$ $(0 \leqq i < e)$, $_B\mathrm{Ext}^i({}_AT_B, {}_AX) \in \mathrm{KT}(T_B)$ $(i > e)$, *and* $\mathrm{Ext}^j({}_AX, {}_AN) = 0$ $(j = 0, 1)$ *for any* $_AN \in \mathrm{KE}({}_AT)$. *Furthermore there is an isomorphism*

$$_B\mathrm{Ext}^e({}_AT_B, {}_AX) \xrightarrow{\sim} {}_BY.$$

Theorem 1.14. *Assume that $_AT_B$ satisfies Condition $_AI$, and let $e \geqq 0$ be an integer. Let $_AX$ be an A-module such that* $\mathrm{Ext}^i_A(T, X) = 0$ $(0 \leqq i < e)$, $_B\mathrm{Ext}^i_A(T_B, X) \in \mathrm{KT}(T_B)$ $(i > e)$, *and such that* $\mathrm{Ext}^j_A(X, N) = 0$ $(j = 0, 1)$ *for any* $_AN \in \mathrm{KE}({}_AT)$. *Put* $Y = {}_B\mathrm{Ext}^e({}_AT_B, {}_AX)$. *Then* $\mathrm{Tor}_i(T_B, {}_BY) = 0$ $(0 \leqq i < e)$. $_A\mathrm{Tor}_i({}_AT_B, {}_BY) \in \mathrm{KE}({}_AT)$ $(i > e)$, *and* $\mathrm{Ext}^j({}_BN', {}_BY) = 0$ $(j = 0, 1)$ *for any* $_BN' \in \mathrm{KT}(T_B)$. *Furthermore there is an isomorphism*

$$_A\mathrm{Tor}_e({}_AT_B, {}_BY) \xrightarrow{\sim} {}_AX.$$

To compare the above theorems with [7 ; Theorems 1.14 and 1.15] we recall definitions of a tilting module of finite projective dimension, and Conditions $(\mathrm{P})_r$, $(\mathrm{E})_r$, $(\mathrm{G})_r$ (for some integer $r \geqq 0$). (We refer to [4] for classical tilting modules of projective dimension $\leqq 1$.)

For a left A-module ${}_AT$ and an integer $r \geqq 0$, we say that ${}_AT$ satisfies $(\mathrm{P})_r$ if there is a projective resolution of ${}_AT$

$$0 \to P_r \to \cdots \to P_1 \to P_0 \to T \to 0$$

such that each ${}_AP_i$ is finitely generated.

We say that ${}_AT$ satisfies $(\mathrm{E})_r$ if $\mathrm{Ext}_A^i(T, T) = 0$ $(i = 1, \cdots, r)$.

We say that ${}_AT$ satisfies $(\mathrm{G})_r$ if there is an exact sequence

$$0 \to A \to T_0 \to T_1 \to \cdots \to T_r \to 0$$

such that each ${}_AT_i$ is a direct summand of a finite direct sum of copies of ${}_AT$.

If ${}_AT$ is satisfies $(\mathrm{P})_r$, $(\mathrm{E})_r$, and $(\mathrm{G})_r$ for some integer $r \geqq 0$, then we call ${}_AT$ a tilting module of finite projection dimension. In this case, if we put $\mathrm{End}({}_AT) = B$ then T_B is also a tilting module of finite projective dimension, and $A \cong \mathrm{End}(T_B)$.

If ${}_AT$ satisfies $(\mathrm{P})_r$, $(\mathrm{E})_r$, and $\mathrm{End}({}_AT) = B$, then ${}_AT_B$ satisfies Conditions ${}_BP$ and I_B, by [7 ; Lemma 1.7]. Note that, in this case, $\mathrm{KE}(T_B)$ and $\mathrm{KT}(T_B)$ are trivial, by [7 ; Proposition 1.17].

Hence the above theorems yield more detailed presentation of [7 ; Theorems 1.14 and 1.15].

Let ${}_AT'$ be a tilting module of $\mathrm{pdim}_A T' \leqq r < \infty$, ${}_A\bigoplus T' = {}_AT$ (infinite direct sum of copies of ${}_AT'$), and $\mathrm{End}({}_AT) = B$. Then ${}_AT_B$ satisfies Conditions ${}_BP$, P_A, and ${}_AI$ (cf. Proposition 3.1 and its proof).

If we assume that A and B are finite dimensional algebras over a field, and restrict modules within finitely generated modules we obtain similar results (Theorem 2.1). In this case, Theorems 1.12 and 1.14 are unified into one theorem, by virtue of the existence of duality.

There are two applications of main theorems in the case when $r = 1$. Firstly, we can extend Theorem of [2]. In fact, we can see that the bimodule ${}_E\partial_R$ in [2] satisfies Conditions ${}_RP$ and ${}_EI$. Therefore we get two equivalences between full subcategories of E-Mod and R-Mod, induced by ${}_R\mathrm{Ext}_E^e(\partial_R, -)$ and its inverse ${}_E\mathrm{Tor}_e^R({}_E\partial, -)$ $(e = 0, 1)$. On the other hand, Theorem of [2] corresponds to the fact that ${}_E\partial_R$ satisfies Conditions I_R and P_E.

Secondly, let $R \subseteq Q$ be rings with common identity, and suppose that $\mathrm{Tor}_i(Q_R, {}_RQ) = 0$ $(i \geqq 1)$ and that $Q \otimes_R Q \cong Q$, $x \otimes y \mapsto xy$. Then an R-R-bimodule Q/R satisfies Conditions ${}_RP$, P_R, ${}_RI$ and I_R (Proposition 3.4). Furthermore, if we put ${}_RK_R = Q/R$, $\mathrm{End}({}_RK) = H$, and $\mathrm{End}(K_R) = H'$, then we get a bimodule ${}_{H'}K_H$ which satisfies Conditions ${}_HP$, $P_{H'}$, ${}_{H'}I$, and I_H (Proposition

3.5).

1. Main Theorems. We begin with the following

Lemma 1.1. *Let* $s \geqq 0$ *be an integer,* ${}_BY' \in B$-Mod, *and* ${}_AN \in A$-Mod. *If* $\mathrm{Tor}_i(T_B, {}_BY') = 0$ $(1 \leqq i \leqq s)$ *and* $\mathrm{Ext}^j({}_AT, {}_AN) = 0$ $(0 \leqq j \leqq s+1)$ *then* $\mathrm{Ext}^j({}_AT\otimes_B Y', {}_AN) = 0$ $(0 \leqq j \leqq s+1)$.

Proof. Take a projective resolution of ${}_BY'$:

$$\cdots \to {}_BP_2 \to {}_BP_1 \to {}_BP_0 \to {}_BY' \to 0,$$

and an integer $i \geqq 0$. Then $\mathrm{Tor}_i(T_B, {}_BY') = 0$ if and only if

$${}_AT\otimes_B P_{i+1} \to {}_AT\otimes_B P_i \to {}_AT\otimes_B P_{i-1}$$

is exact where we put $P_{-1} = 0$, and the latter is equivalent to the fact that, for any injective module ${}_AI \in A$-Mod, $\mathrm{Ext}^i({}_BY', {}_B\mathrm{Hom}({}_AT_B, {}_AI)) = 0$ holds. From an injective coresolution

$$0 \to {}_AN \to {}_AI_0 \to {}_AI_1 \to {}_AI_2 \to {}_AI_3 \to \cdots$$

it follows an exact sequence

$$0 \to {}_B\mathrm{Hom}({}_AT_B, {}_AI_0) \to \mathrm{Hom}({}_AT, {}_AI_1) \to \mathrm{Hom}({}_AT, {}_AI_2) \to \\ \cdots \to \mathrm{Hom}({}_AT, {}_AI_{s+2}).$$

Then, by assumption, we have an exact sequence

$$0 \to \mathrm{Hom}({}_BY', {}_B\mathrm{Hom}({}_AT, {}_AI_0)) \to \mathrm{Hom}({}_BY', {}_B\mathrm{Hom}({}_AT, {}_AI_1)) \to \\ \cdots \to \mathrm{Hom}({}_BY', {}_B\mathrm{Hom}({}_AT, {}_AI_{s+2}))$$

and this yields the desired result by virtue of the canonical isomorphism

$$\mathrm{Hom}({}_BY', {}_B\mathrm{Hom}({}_AT_B, {}_AY'')) \simeq \mathrm{Hom}({}_AT\otimes_B Y', {}_AY'') \quad ({}_AY'' \in A\text{-Mod}).$$

Lemma 1.2. Let e, s be non-negative integers, and assume that

$$\mathrm{Tor}_i(T_B, {}_BY') = 0 \ (0 \leqq i \leqq e+s \text{ and } i \neq e) \text{ and } \mathrm{Ext}^j({}_AT, {}_AN) = 0 \\ (0 \leqq j \leqq e+s+1).$$

Then $\mathrm{Ext}^t({}_A\mathrm{Tor}_e({}_AT_B, {}_BY'), {}_AN) = 0$ $(0 \leqq t \leqq s+1)$.

Proof. If $e = 0$ then the result follows from Lemma 1.1. Therefore we may assume that $e \geqq 1$. Take a projective resolution of ${}_BY'$:

$$\begin{array}{ccccccccc} \cdots \to P_e & \longrightarrow & P_{e-1} \to \cdots \to P_1 & \longrightarrow & P_0 \to {}_BY' \to 0, \\ \searrow & \nearrow & & \searrow & \nearrow \\ & R^eY' & \cdots & RY' & \end{array}$$

where each $\overset{*}{\searrow\nearrow}$ denotes the standard factorization of $\overset{*}{\to}$. Then, $\mathrm{Tor}_i(T_B, {}_BR^eY') = 0 \ (1 \leqq i \leqq s)$ implies that $\mathrm{Ext}^j({}_AT\otimes_BR^eY', {}_AN) = 0$ $(0 \leqq j \leqq s+1)$, by Lemma 1.1. By assumption we have an exact sequence

$$0 \to {}_A\mathrm{Tor}_1({}_AT_B, {}_BR^{e-1}Y') \to T\otimes_BR^eY' \longrightarrow T\otimes_BP_{e-1} \to$$
$$\searrow \nearrow$$
$$W_{e-1}$$
$$\cdots \longrightarrow T\otimes_BP_1 \to T\otimes_BP_0 \to 0,$$
$$\searrow \nearrow$$
$$W_1$$

where ${}_A\mathrm{Tor}_1({}_AT_B, {}_BR^{e-1}Y') \simeq {}_A\mathrm{Tor}_e({}_AT_B, {}_BY')$. Then $\mathrm{Ext}^i({}_AW_1, {}_AN) = 0$ $(0 \leqq i \leqq s+e)$, and hence $\mathrm{Ext}^i({}_AW_2, {}_AN) = 0$ $(0 \leqq i \leqq s+e-1)$, and so on. Thus $\mathrm{Ext}^t({}_A\mathrm{Tor}_e({}_AT_B, {}_BY'), {}_AN) = 0$ $(0 \leqq t \leqq s+1)$.

Dualizing Lemma 1.2, we have the following

Lemma 1.3. *Let $e, s \geqq 0$ be non-negative integers, and assume that $\mathrm{Ext}^i({}_AT, {}_AY) = 0$ $(0 \leqq i \leqq e+s$ and $i \neq e)$ and $\mathrm{Tor}_j(T_B, {}_BN') = 0$ $(0 \leqq j \leqq e+s+1)$. Then $\mathrm{Ext}^t({}_BN', {}_B\mathrm{Ext}_e({}_AT_B, {}_AY)) = 0$ $(0 \leqq t \leqq s+1)$.*

By using the usual long exact sequences the following lemma is easily seen.

Lemma 1.4. (1) *Let $0 \to {}_AN' \to {}_AN \to {}_AN'' \to 0$ be an exact sequence. If two modules of the above three modules belong to $\mathrm{KE}({}_AT)$ then so does the third.*

(2) *Let $0 \to M'_A \to M_A \to M''_A \to 0$ be an exact sequence. If two modules of the above three modules belong to $\mathrm{KT}({}_AT)$ then so does the third.*

Now we define $\mathrm{kE}({}_AT)$ and $\mathrm{kT}({}_AT)$ as follows:

$\mathrm{kE}({}_AT)$
$= \{{}_AX \mid \text{if } g: {}_AX \to {}_AN \text{ and } N \in \mathrm{KE}({}_AT) \text{ then } \mathrm{Ker}\, g \in \mathrm{KE}({}_AT)\}$,
$\mathrm{kT}({}_AT)$
$= \{X'_A \mid \text{if } f: N'_A \to X'_A \text{ and } N' \in \mathrm{KT}({}_AT) \text{ then } \mathrm{Cok}\, f \in \mathrm{KT}({}_AT)\}$.

Then the following hold.

Lemma 1.5. *Let $0 \to {}_AW' \overset{f}{\to} {}_AW \overset{g}{\to} {}_AW'' \to 0$ be an exact sequence.*

(1) *If two of the above three modules belong to $\mathrm{kE}({}_AT)$ then so does the third.*

(2) *Let* ${}_AW \in \mathrm{kE}({}_AT)$. *In this case,* ${}_AW' \in \mathrm{KE}({}_AT)$, ${}_AW' \in \mathrm{kE}({}_AT)$, ${}_AW'' \in \mathrm{KE}({}_AT)$, *and* ${}_AW'' \in \mathrm{kE}({}_AT)$ *are equivalent conditions.*

(3) *Let* ${}_AW_1, {}_AW_2 \in \mathrm{kE}({}_AT)$, *and* $h : {}_AW_1 \to {}_AW_2$. *Then* Ker h, Im h, Cok $h \in \mathrm{kE}({}_AT)$.

(4) *Every direct summand of a module of* $\mathrm{kE}({}_AT)$ *belongs to* $\mathrm{kE}({}_AT)$.

Proof. (1) and (2). Assume that $W \in \mathrm{kE}({}_AT)$ and $W' \in \mathrm{KE}({}_AT)$. Let $k'' : {}_AW'' \to {}_AN''$, and ${}_AN'' \in \mathrm{KE}({}_AT)$. Then $\mathrm{Cok}\, k'' = \mathrm{Cok}\, gk'' \in \mathrm{KE}({}_AT)$. Hence $W'' \in \mathrm{kE}({}_AT)$. On the other hand, let $k' : {}_AW' \to {}_AN'$, and ${}_AN' \in \mathrm{KE}({}_AT)$. Then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & W' & \xrightarrow{f} & W & \xrightarrow{g} & W'' & \to & 0 \\ & & {\scriptstyle k'}\downarrow & & {\scriptstyle s}\downarrow & & \| & & \\ 0 & \to & N' & \to & X & \to & W'' & \to & 0 \end{array}$$

where $X \in \mathrm{KE}({}_AT)$. Thus $\mathrm{Cok}\, k' \cong \mathrm{Cok}\, s \in \mathrm{KE}({}_AT)$. Hence $W' \in \mathrm{kE}({}_AT)$. Therefore $W \in \mathrm{kE}({}_AT)$ and $W'' \in \mathrm{KE}({}_AT)$ mean $W' \in \mathrm{kE}({}_AT)$. Next, let $W', W'' \in \mathrm{kE}({}_AT)$, and let $k : {}_AW \to {}_AN$, where ${}_AN \in \mathrm{KE}({}_AT)$. Then we have a commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & W' & \xrightarrow{f} & W & \xrightarrow{g} & W'' & \to & 0 \\ & & {\scriptstyle fk}\downarrow & & {\scriptstyle k}\downarrow & & \downarrow & & \\ 0 & \to & N & = & N & \to & 0 & \to & 0. \end{array}$$

Therefore we have an exact sequence

$$0 \to \mathrm{Ker}\, fk \to \mathrm{Ker}\, k \longrightarrow W'' \to \mathrm{Cok}\, fk \to \mathrm{Cok}\, k \to 0.$$
$$\searrow \quad \nearrow$$
$$X'$$

Then, as Ker fk, Cok $fk \in \mathrm{KE}({}_AT)$, we have $X' \in \mathrm{KE}({}_AT)$, and so Ker $k \in \mathrm{KE}({}_AT)$. Thus $W \in \mathrm{kE}({}_AT)$.

(3) is evident.

(4) Every direct summand of a member of $\mathrm{KE}({}_AT)$ belongs to $\mathrm{KE}({}_AT)$. Therefore this follows from (2) above.

The dual version of the above is the following

Lemma 1.6. *Let* $0 \to G'_A \to G_A \to G''_A \to 0$ *be an exact sequence.*

(1) *If two of the above three modules belong to* $\mathrm{kT}({}_AT)$, *then so does the third.*

(2) *Let* $G_A \in \mathrm{kT}({}_AT)$. *In this case,* $G' \in \mathrm{KT}({}_AT)$, $G' \in \mathrm{kT}({}_AT)$, $G'' \in \mathrm{KT}({}_AT)$, *and* $G'' \in \mathrm{kT}({}_AT)$ *are equivalent conditions.*

(3) *Let* $G_1, G_2 \in \mathrm{kT}({}_AT)$, *and* $k : G_{1A} \to G_{2A}$. *Then* Ker k, Im k, Cok $k \in$

$\mathrm{kE}({}_AT)$.

(4) *Every direct summand of a member of* $\mathrm{kT}({}_AT)$ *is a member of* $\mathrm{kT}({}_AT)$.

For any projective module ${}_BP$, we consider the canonical map

$$h_P:\ {}_BP \to {}_B\mathrm{Hom}({}_AT_B, {}_AT\otimes_B P),\ p \mapsto (t \to t\otimes p).$$

We put ${}_BP^* = {}_B\mathrm{Hom}({}_AT_B, {}_AT\otimes_B P)$. Then we have an exact sequence

$$(*)\quad 0 \to \mathrm{Ker}\, h_P \to P \to P^* \to \mathrm{Cok}\, h_P \to 0.$$

Note that if ${}_BP$ is finitely generated and $\mathrm{End}({}_AT) \simeq B$ then h_P is an isomorphism.

Lemma 1.7. *Assume that, for any projective module* ${}_BP$, $\mathrm{Ker}\, h_P, \mathrm{Cok}\, h_P \in \mathrm{KT}(T_B)$, *and let* ${}_BY$ *be a left B-module such that* $\mathrm{Ext}^i({}_BN, {}_BY) = 0\ (i = 0, 1)$ *for any* ${}_BN' \in \mathrm{KT}(T_B)$. *Then the following hold.*

(1) *There is an exact sequence*

$$\cdots \to {}_BP_3^* \to {}_BP_2^* \to {}_BP_1^* \to {}_BP_0^* \to {}_BY \to 0,$$

where each ${}_BP_i$ *is projective.*

(2) *For any projective module* ${}_BP$, *there hold* $T\otimes_B P \simeq T\otimes_B P^*$ *canonically,* $\mathrm{Tor}_i(T_B, {}_BP^*) = 0\ (i \geqq 1)$, *and* $h_{P^*}:\ P^* \simeq \mathrm{Hom}({}_AT, {}_AT\otimes_B P^*)$. *Furthermore,* $\mathrm{Hom}({}_BP^*, {}_BY) \simeq \mathrm{Hom}({}_BP, {}_BY)$ *canonically.*

(3) *If* $\mathrm{Ext}^i({}_BN', {}_BY) = 0\ (i \geqq 0)$ *for any* ${}_BN' \in \mathrm{KT}(T_B)$ *then* $\mathrm{Ext}^i({}_BP^*, {}_BY) = 0\ (i \geqq 1)$ *for any projective module* ${}_BP$.

Proof. (1) It is easily seen that $\mathrm{Hom}({}_BP^*, {}_BY) \simeq \mathrm{Hom}({}_BP, {}_BY)$ canonically. Therefore every diagram

$$\begin{array}{ccccc} P & \to & Y & \to & 0 \\ {\scriptstyle h_P}\downarrow & & & & \\ P^* & & & & \end{array}$$

can be embeded in a commutative diagram

$$\begin{array}{ccccc} P & \to & Y & \to & 0. \\ {\scriptstyle h_P}\downarrow & \nearrow & & & \\ P^* & & & & \end{array}$$

Let $0 \to {}_BY' \to {}_BP^* \to {}_BY \to 0$ be an exact sequence. Then ${}_BY'$ satisfies the same conditions as ${}_BY$ does. Therefore we can complete the proof by induction.

(2) From the exact sequence $(*)$, it follows that

$$\mathrm{Tor}_i(T_B, {}_BP) \simeq \mathrm{Tor}_i(T_B, {}_B\mathrm{Im}\, h_P) \simeq \mathrm{Tor}_i(T_B, {}_BP^*)$$

for all $i \geqq 0$. In particular $T \otimes_B P \xrightarrow{\sim} T \otimes_B P^*$. Then it is easily seen that h_{P^*} is an isomorphism. The remainder may be omitted.

For any injective module ${}_AI$, we consider the canonical map

$$k_I : {}_AT \otimes_B \mathrm{Hom}({}_AT_B, {}_AI) \to {}_AI,\ t \otimes f \mapsto (t)f.$$

We put ${}_AI^\dagger = {}_AT \otimes_B \mathrm{Hom}_A(T, I)$. Then we have an exact sequence

$$(\dagger) \quad 0 \to \mathrm{Ker}\, k_I \to I^\dagger \to I \to \mathrm{Cok}\, k_I \to 0.$$

Then we have the dual version of Lemma 1.7.

Lemma 1.8. *Assume that, for any injective A-module ${}_AI$, ${}_A\mathrm{Ker}\, k_I$, $\mathrm{Cok}\, k_I \in \mathrm{KE}({}_AT)$, and let ${}_AX$ be a left A-module such that $\mathrm{Ext}^i({}_AX, {}_AN) = 0$ ($i = 0, 1$) for any ${}_AN \in \mathrm{KE}({}_AT)$. Then the following hold.*

(1) *There is an exact sequence*

$$0 \to {}_AX \to {}_AI_0^\dagger \to {}_AI_1^\dagger \to {}_AI_2^\dagger \to {}_AI_3^\dagger \to \cdots$$

where each ${}_AI_i$ is injective.

(2) *For any injective module ${}_AI$, there hold $\mathrm{Hom}({}_AT, {}_AI^\dagger) \xrightarrow{\sim} \mathrm{Hom}({}_AT, {}_AI)$ canonically, $\mathrm{Ext}^i({}_AT, {}_AI^\dagger) = 0$ ($i \geqq 1$), and $T \otimes_B \mathrm{Hom}({}_AT_B, {}_AI^\dagger) \xrightarrow{\sim} I^\dagger$, $t \otimes g \mapsto (t)g$. Furthermore $\mathrm{Hom}({}_AX, {}_AI^\dagger) \xrightarrow{\sim} \mathrm{Hom}({}_AX, {}_AI)$ canonically.*

(3) *If $\mathrm{Ext}^i({}_AX, {}_AN) = 0$ ($i \geqq 0$) for any ${}_AN \in \mathrm{KE}({}_AT)$, then $\mathrm{Ext}^i({}_AX, {}_AI^\dagger) = 0$ ($i \geqq 1$) for any injective module ${}_AI$.*

The next is used in the proof of Lemma 1.13.

Lemma 1.9. *Assume that the following diagram*

$$\begin{array}{ccccccc} {}_BW & \xrightarrow{g} & {}_BV & \xrightarrow{f} & {}_BF & \to & {}_B\mathrm{Cok}\, f \to 0 \\ & & h \downarrow & \swarrow s & & & \\ & & X & & & & \end{array}$$

has an exact row, and that ${}_BW$, ${}_B\mathrm{Cok}\, f \in \mathrm{KE}(T_B)$, and $\mathrm{Ext}^j({}_BN', {}_BX) = 0$ ($j = 0, 1$) for any ${}_BN' \in \mathrm{KT}(T_B)$. Then there exists a unique homomorphism $s : {}_BF \to {}_BX$ such that $fs = h$. If h is an epimorphism, and $\mathrm{Ker}\, h$, $\mathrm{Cok}\, f \in \mathrm{kT}(T_B)$ then $\mathrm{Ker}\, s \in \mathrm{kT}(T_B)$.

Proof. Take the standard factorization $\begin{array}{ccc} V & \xrightarrow{f} & F \\ & \searrow \quad \nearrow & \\ & W' & \end{array}$. Then it is easily seen that $\mathrm{Hom}_B(F, X) \xrightarrow{\sim} \mathrm{Hom}_B(W', X) \xrightarrow{\sim} \mathrm{Hom}_B(V, X)$ canonically. This implies the first assertion. To see the remainder, we consider the exact sequence

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} h \to \operatorname{Ker} s \to \operatorname{Cok} f \to \operatorname{Cok} h \to \operatorname{Cok} s \to 0.$$

Then, as $\operatorname{Ker} h \in \mathrm{kT}(T_B)$, we have $\operatorname{Ker} f$ $(= \operatorname{Im} g)$, $\operatorname{Ker} h/\operatorname{Im} g \in \mathrm{kT}(T_B)$, and the latter is isomorphic to $(\operatorname{Ker} h)f$. Then, from the exact sequence

$$0 \to (\operatorname{Ker} h)f \to \operatorname{Ker} s \to \operatorname{Cok} f \to 0,$$

it follows that $\operatorname{Ker} s \in \mathrm{kT}(T_B)$, by Lemma 1.6.

Dualizing the above we have the following

Lemma 1.10. *Assume that the following diagram*

$$\begin{array}{ccccccc} & & & & {}_AX & & \\ & & & {}^{s}\swarrow & \downarrow h & & \\ 0 \to {}_A\operatorname{Ker} f & \to & {}_AF & \xrightarrow{f} & {}_AV & \xrightarrow{g} & {}_AW \end{array}$$

has an exact row, and that $\operatorname{Ker} f$, $W \in \mathrm{KE}({}_AT)$,*and* $\operatorname{Ext}^i({}_AX, {}_AN) = 0$ $(i = 0, 1)$ *for any* ${}_AN \in \mathrm{KE}({}_AT)$. *Then there exists a unique homomorphism* s: ${}_AX \to {}_AF$ *such that* $sf = h$. *If* h *is a monomorphism, and* $\operatorname{Cok} h, \operatorname{Ker} f \in \mathrm{kE}({}_AT)$ *then* $\operatorname{Cok} s \in \mathrm{kE}({}_AT)$.

The following lemma follows from the usual exact sequence

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} fg \to \operatorname{Ker} g \to \operatorname{Cok} f \to \operatorname{Cok} fg \to \operatorname{Cok} g \to 0,$$

where f: ${}_AX' \to {}_AX$ and g: ${}_AX \to {}_AX''$.

Lemma 1.11. (1) *Let* f: ${}_AX' \to {}_AX$, *and* g: ${}_AX \to {}_AX''$. *If* $\operatorname{Cok} f \in \mathrm{kE}({}_AT)$ *and* $\operatorname{Cok} fg \in \mathrm{KE}({}_AT)$ (*resp.* $\operatorname{Cok} fg \in \mathrm{kE}({}_AT)$) *then* $\operatorname{Cok} g \in \mathrm{KE}({}_AT)$ (*resp.* $\operatorname{Cok} g \in \mathrm{kE}({}_AT)$).

(2) *Let* h: ${}_BY' \to {}_BY$ *and* k: ${}_BY \to {}_BY''$. *If* $\operatorname{Ker} k \in \mathrm{kT}(T_B)$ *and* $\operatorname{Ker} hk \in \mathrm{KT}(T_B)$ (*resp.* $\operatorname{Ker} hk \in \mathrm{kT}(T_B)$) *then* $\operatorname{Ker} h \in \mathrm{KT}(T_B)$ (*resp.* $\operatorname{Ker} h \in \mathrm{kT}(T_B)$).

We now explain two conditions under which our main theorems hold.

Condition ${}_BP$. (1) For any projective module ${}_BP$, $\operatorname{Ker} h_P$ and $\operatorname{Cok} h_P$ belong to $\mathrm{KT}(T_B)$, and ${}_B\operatorname{Ext}^i({}_AT_B, {}_AT \otimes_B P) \in \mathrm{kT}(T_B)$ $(i \geqq 1)$.

(2) There is an integer $r \geqq 0$ such that ${}_B\operatorname{Ext}^i({}_AT_B, {}_AX) \in \mathrm{KT}(T_B)$ for any $i > r$ and any ${}_AX \in A$-Mod.

If ${}_AT$ satisfies $(\mathrm{P})_r$, $(\mathrm{E})_r$, and $\operatorname{End}({}_AT) = B$ then ${}_AT_B$ satisfies Condition ${}_BP$ above (cf. [7]).

Dualizing the above we consider another condition for ${}_AT_B$.

Condition ${}_AI$. (1) For any injective module ${}_AI$, Ker k_I and Cok k_I belong to KE$({}_AT)$, and ${}_A\mathrm{Tor}_i({}_AT_B, {}_B\mathrm{Hom}({}_AT_B, {}_AI)) \in$ kE$({}_AT)$ for any $i \geqq 1$.

(2) There is an integer $r \geqq 0$ such that ${}_A\mathrm{Tor}_i({}_AT_B, {}_BY) \in \mathrm{KE}({}_AT)$ for any $i > r$ and any ${}_BY \in B$-Mod.

If T_B satisfies $(\mathrm{P})_r$, $(\mathrm{E})_r$, and End$(T_B) = A$ then ${}_AT_B$ satisfies Condition ${}_AI$ above (cf. [7]).

The following theorem holds under the Condition ${}_BP$.

Theorem 1.12. *Assume that ${}_AT_B$ satisfies Condition ${}_BP$. Let $e \geqq 0$ be an integer, and let ${}_BY$ be a B-module such that* $\mathrm{Tor}_i(T_B, {}_BY) = 0$ $(i < e)$, ${}_A\mathrm{Tor}_i({}_AT_B, {}_BY) \in \mathrm{KE}({}_AT)$ $(i > e)$ *and such that* $\mathrm{Ext}^j({}_BN', {}_BY) = 0$ $(j = 0, 1)$ *for any* ${}_BN' \in \mathrm{KT}(T_B)$. *Put* $X = \mathrm{Tor}_e({}_AT_B, {}_BY)$. *Then* $\mathrm{Ext}^i({}_AT, {}_AX) = 0$ $(i < e)$, ${}_B\mathrm{Ext}^i({}_AT_B, {}_AX) \in \mathrm{KT}(T_B)$ $(i > e)$, ${}_B\mathrm{Ext}^e({}_AT_B, {}_AX) \xrightarrow{\sim} {}_BY$, *and* $\mathrm{Ext}^j({}_AX, {}_AN) = 0$ $(j = 0, 1)$ *for any* ${}_AN \in \mathrm{KE}({}_AT)$.

Proof. The last assertion follows from Lemma 1.2. By Lemma 1.7, there is an exact sequence

$$\cdots \to P_e^* \longrightarrow P_{e-1}^* \longrightarrow \cdots \to P_2^* \longrightarrow P_1^* \longrightarrow P_0^* \to {}_BY \to 0,$$
$$\searrow\ \nearrow \quad \searrow\ \nearrow \qquad\qquad \searrow\ \nearrow \quad \searrow\ \nearrow$$
$$Y_e \qquad Y_{e-1} \ \cdots \qquad Y_2 \qquad Y_1$$

where each ${}_BP_i$ is projective, and $P_i^* = {}_B\mathrm{Hom}_A(T_B, T\otimes_B P_i)$. We put $Y = Y_0$. Then, for each $i \geqq 1$, we have an exact sequence of left A-modules

$$0 \to \mathrm{Tor}_1(T_B, {}_BY_{i-1}) \to T\otimes_B Y_i \to T\otimes_B P_{i-1}^* \to T\otimes_B Y_{i-1} \to 0,$$

and isomorphisms

$${}_A\mathrm{Tor}_j({}_AT_B, {}_BY_i) \xrightarrow{\sim} {}_A\mathrm{Tor}_{j+1}({}_AT_B, {}_BY_{i-1}) \quad (i, j \geqq 1).$$

First we assume that $e = 0$. In this case, each Y_i satisfies the same condition as $(Y_0 =)$ Y does. Then we have a long exact sequence of left B-modules:

$$\begin{aligned} 0 \to {}_B\mathrm{Hom}({}_AT_B, {}_AT\otimes_B Y_i) &\to {}_B\mathrm{Hom}({}_AT_B, T\otimes_B P_{i-1}^*) \to {}_B\mathrm{Hom}({}_AT_B, {}_AT\otimes_B Y_{i-1}) \\ \to \mathrm{Ext}^1({}_AT, {}_AT\otimes_B Y_i) &\to \mathrm{Ext}^1({}_AT, {}_AT\otimes_B P_{i-1}^*) \to \mathrm{Ext}^1({}_AT, {}_AT\otimes_B Y_{i-1}) \\ \to \mathrm{Ext}^2({}_AT, {}_AT\otimes_B Y_i) &\to \mathrm{Ext}^2({}_AT, {}_AT\otimes_B P_{i-1}^*) \to \mathrm{Ext}^2({}_AT, {}_AT\otimes_B Y_{i-1}) \\ \cdots. \end{aligned}$$

Since ${}_BP_{i-1}^* \xrightarrow{\sim} {}_B\mathrm{Hom}({}_AT_B, {}_AT\otimes_B P_{i-1}^*)$ canonically, we see that if $i \geqq 1$ then

$$0 \to Y_i \to \mathrm{Hom}({}_AT, {}_AT\otimes_B Y_i)$$

is exact. Therefore, if $i \geqq 2$ then

$$Y_i \xrightarrow{\sim} \mathrm{Hom}({}_AT, {}_AT\otimes_B Y_i),$$

and hence, for each $i \geqq 3$,

$$0 \to \mathrm{Ext}^1_A(T, T\otimes_B Y_i) \to \mathrm{Ext}^1_A(T, T\otimes_B P^*_{i-1})$$

is exact. Therefore, if $i \geqq 4$ then

$$\mathrm{Ext}^1_A(T, T\otimes_B Y_i) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Ext}^1_A(T, T\otimes_B P^*_{i-1}) \to \mathrm{Ext}^1_A(T, T\otimes_B P^*_{i-2})),$$

and the latter lies in $\mathrm{kT}(T_B)$. Then

$$\begin{aligned}&\mathrm{Cok}(\mathrm{Ext}^1_A(T, T\otimes_B P^*_{i-1}) \to \mathrm{Ext}^1_A(T, T\otimes_B Y_{i-1}))\\ &\quad\xrightarrow{\sim} \mathrm{Ker}(\mathrm{Ext}^2_A(T, T\otimes_B Y_i) \to \mathrm{Ext}^2_A(T, T\otimes_B P^*_{i-1})),\end{aligned}$$

and the former lies in $\mathrm{kT}(T_B)$ when $i \geqq 5$. Thus, if $i \geqq 6$ then ${}_B\mathrm{Ext}^2({}_AT_B, {}_AT\otimes_B Y_i) \in \mathrm{KT}(T_B)$. Similarly we can show that ${}_B\mathrm{Ext}^3({}_AT_B, {}_AT\otimes_B Y_i) \in \mathrm{kT}(T_B)$ ($i \geqq 8$), and so on. Using (2) of Condition ${}_BP$, we see that ${}_B\mathrm{Ext}^j({}_AT_B, {}_AT\otimes_B Y_i) \in \mathrm{KT}(T_B)$ for all $j \geqq 1$, if i is large. Then, for any $i, j \geqq 1$, we have ${}_B\mathrm{Ext}^j_A(T_B, T\otimes_B Y_{i-1}) \in \mathrm{KT}(T_B)$. In particular, for any $j \geqq 1$, ${}_B\mathrm{Ext}^j_A(T_B, T\otimes_B Y_0) \in \mathrm{KT}(T_B)$. Using two commutative diagram with exact rows

$$\begin{array}{ccccccc} 0\to & Y_1 & \to & P_0^* & \to & Y_0 & \to 0\\ & \downarrow & & \downarrow\wr & & \downarrow & \\ 0\to & \mathrm{Hom}_A(T, T\otimes_B Y_1) & \to & \mathrm{Hom}_A(T, T\otimes_B P_0^*) & \to & \mathrm{Hom}_A(T, T\otimes_B Y_0) & -\end{array}$$
$$\to \mathrm{Ext}^1_A(T, T\otimes_B Y_1) \to \mathrm{Ext}^1_A(T, T\otimes_B P_0^*)$$

and

$$\begin{array}{ccccccc} 0\to & Y_2 & \to & P_1^* & \to & Y_1 & \to 0\\ & \downarrow\wr & & \downarrow\wr & & \downarrow & \\ 0\to & \mathrm{Hom}_A(T, T\otimes_B Y_2) & \to & \mathrm{Hom}_A(T, T\otimes_B P_1^*) & \to & \mathrm{Hom}_A(T, T\otimes_B Y_1) & -\end{array}$$
$$\to \mathrm{Ext}^1_A(T, T\otimes_B Y_2) \to \mathrm{Ext}^1_A(T, T\otimes_B P_1^*),$$

we see that ${}_B\mathrm{Ker}\, h$, ${}_B\mathrm{Cok}\, h \in \mathrm{KT}(T_B)$, where $h: Y_0 \to \mathrm{Hom}_A(T, T\otimes_B Y_0)$ is the canonical map. Then, by assumption, $\mathrm{Ker}\, h = 0$, and the exact sequence

$$0 \to Y_0 \to \mathrm{Hom}_A(T, T\otimes_B Y_0) \to \mathrm{Cok}\, h \to 0$$

split. Therefore $\mathrm{Cok}\, h = 0$ by Lemma 1.3. Hence $h: Y_0 \xrightarrow{\sim} \mathrm{Hom}_A(T, T\otimes_B Y_0)$. Next we assume that $e \geqq 1$. Then we have an exact sequence of A-modules

$$0 \to \mathrm{Tor}_1({}_AT_B, {}_BY_{e-1}) \to T\otimes_B Y_e \to T\otimes_B P^*_{e-1} \to \cdots \to T\otimes_B P_0^* \to 0,$$

where ${}_A\mathrm{Tor}_1({}_AT_B, {}_BY_{e-1}) \xrightarrow{\sim} {}_A\mathrm{Tor}_e({}_AT_B, {}_BY_0)$, and ${}_BY_e$ satisfies the condition of the case when $e = 0$. Furthermore we have a commutative diagram with exact rows:

$$\begin{array}{ccccc} 0\to & Y_e & \to & P^*_{e-1} & \to\cdots \\ & \downarrow\wr & & \downarrow\wr & \\ 0\to & \mathrm{Hom}_A(T,\ T\otimes_B Y_e) & \to & \mathrm{Hom}_A(T,\ T\otimes_B P^*_{e-1}) & \to\cdots \end{array}$$

$$\begin{array}{ccc} \to & P^*_0 & \to {}_BY\to 0 \\ & \downarrow\wr & \\ \to & {}_B\mathrm{Hom}_A({}_AT_B,\ T\otimes_B P^*_0). & \end{array}$$

Note that $T\otimes_B P_i \xrightarrow{\sim} T\otimes_B P_i^*$ $(i=0,\cdots,e-1)$. Thus we can complete the proof by using the following lemma.

Lemma 1.13. *Assume that ${}_AT_B$ satisfies Condition ${}_BP$, and let $e\geqq 1$ be an integer. Let the sequence of left A-modules*

$$0\to {}_AX\to V_e \longrightarrow V_{e-1}\to\cdots\to V_2\longrightarrow V_1\to V_0\to 0$$
$$\searrow\ \nearrow \qquad\qquad\qquad \searrow\ \nearrow$$
$$W_{e-1}\qquad\quad \cdots \qquad\quad W_1$$

be an exact sequence such that ${}_B\mathrm{Ext}^j({}_AT_B, {}_AV_i)\in \mathrm{kT}(T_B)$ $(j\geqq 1)$ $(i=0,\cdots,e-1)$, ${}_B\mathrm{Ext}^j({}_AT_B, {}_AV_e)\in \mathrm{KT}(T_B)$ $(j\geqq 1)$, and $\mathrm{Ext}^j({}_AV_i, {}_AN)=0$ $(j=0,1)$ $(i=0,\cdots,e)$ for any ${}_AN\in \mathrm{KE}({}_AT)$. Furthermore, assume that the sequence

$$0\to {}_B\mathrm{Hom}({}_AT, {}_AV_e)\to\cdots\to {}_B\mathrm{Hom}({}_AT, {}_AV_0)\to {}_BY\to 0$$

is exact, and that $\mathrm{Ext}^j({}_BN', {}_BY)=0$ $(j=0,1)$ for any ${}_BN'\in \mathrm{KT}(T_B)$. Then $\mathrm{Ext}^j({}_AT, {}_AX)=0$ $(j<e)$, ${}_B\mathrm{Ext}^j({}_AT_B, {}_AX)\in \mathrm{KT}(T_B)$ $(j>e)$, and

$${}_BY \xrightarrow{\sim} {}_B\mathrm{Ext}^e({}_AT_B, {}_AX).$$

Proof. If $e=1$ then the exact sequence

$$0\to {}_AX\to {}_AV_1\to {}_AV_0\to 0$$

yields a long exact sequence

$$\begin{array}{l} 0\to \mathrm{Hom}_A(T,X)\to \mathrm{Hom}_A(T,\ V_1)\to \mathrm{Hom}_A(T,\ V_0) \\ \to \mathrm{Ext}^1_A(T,X)\ \to \mathrm{Ext}^1_A(T,\ V_1)\ \to \mathrm{Ext}^1_A(T,\ V_0) \\ \to \mathrm{Ext}^2_A(T,X)\ \to \mathrm{Ext}^2_A(T,\ V_1)\ \to \mathrm{Ext}^2_A(T,\ V_0) \\ \to\cdots. \end{array}$$

Then, for the exact sequence

$$0\to {}_BY \xrightarrow{\alpha} \mathrm{Ext}^1_A(T,X) \longrightarrow \mathrm{Ext}^1_A(T,\ V_1)$$
$$\searrow\ \nearrow$$
$$G$$

there exists a unique homomorphism $\beta:\ {}_B\mathrm{Ext}^1_A(T,X)\to {}_BY$ such that $\alpha\beta=$

id_Y. By assumption, $\mathrm{Hom}_A(T, X) = 0$, and hence Lemma 1.13 implies $G = 0$, because $G \in \mathrm{KT}(T_B)$. Thus α is an isomorphism. It is easily seen that ${}_B\mathrm{Ext}_A^j(T_B, X) \in \mathrm{KT}(T_B)$ $(j \geqq 2)$. Assume that $e \geqq 2$. Then it is easily seen that, for each $i = 1, \cdots, e-1$,

$$\mathrm{Hom}_A(T, V_{i+1}) \to \mathrm{Hom}_A(T, W_i) \to 0$$

is exact. Then $\mathrm{Hom}_A(T, X) = 0$, and

$$0 \to \mathrm{Ext}_A^1(T, X) \to \mathrm{Ext}_A^1(T, V_e)$$

is exact. By using Lemma 1.6, we see that, for each $i = 1, \cdots, e-1$,

$${}_B\mathrm{Ext}^j({}_AT_B, {}_AW_i) \in \mathrm{kT}(T_B) \quad (j \geqq i+1)$$

and

$${}_B\mathrm{Ext}^j({}_AT_B, {}_AX) \in \mathrm{KT}(T_B) \quad (j \geqq e+1).$$

Consider the exact sequence

$$\begin{array}{ccccc} 0 \to {}_BY \xrightarrow{\alpha} {}_B\mathrm{Ext}_A^1(T_B, W_1) & & \longrightarrow & & \mathrm{Ext}_A^1(T, V_1) \\ & \searrow & & \nearrow & \\ & & G & & \end{array}$$

Then, as ${}_BG \in \mathrm{kT}(T_B)$, the assumption for ${}_BY$ implies that there is a unique B-homomorphism β_1 : $\mathrm{Ext}_A^1(T, W_1) \to Y$ such that $\alpha\beta_1 = id_Y$ (, and so $\mathrm{Ker}\ \beta_1 \simeq G$). By assumption for Y, $\mathrm{Im}(\mathrm{Ext}_A^1(T, V_2) \to \mathrm{Ext}_A^1(T, W_1))$ is contained in $\mathrm{Ker}\ \beta_1$, and hence

$$\mathrm{Ext}_A^1(T_B, W_2) \simeq \mathrm{Ker}(\mathrm{Ext}_A^1(T, V_2) \to \mathrm{Ext}_A^1(T, W_1)) \in \mathrm{kT}(T_B),$$

where $e \geqq 3$. By using Lemma 1.9, there is a unique B-homomorphism β_2 which render the diagram

$$\begin{array}{ccccc} {}_B\mathrm{Ext}_A^1(T, W_1) & & \longrightarrow & & {}_B\mathrm{Ext}_A^2(T, W_2) \\ & \beta_1 \searrow & & \nearrow \beta_2 & \\ & & Y & & \end{array}$$

commutative, so that ${}_B\mathrm{Ker}\ \beta_2 \in \mathrm{kT}(T_B)$. Then

$$\mathrm{Im}(\mathrm{Ext}_A^2(T, V_3) \to \mathrm{Ext}_A^2(T, W_2)) \subseteqq \mathrm{Ker}\ \beta_2,$$

and hence

$$\mathrm{Ker}(\mathrm{Ext}_A^2(T, V_3) \to \mathrm{Ext}_A^2(T, W_2)) \in \mathrm{kT}(T_B),$$

and so on. Repeating this argument, we see that, for each $i = 2, \cdots, e-1$,

$$_B\mathrm{Ext}_A^j(T, W_i) \in \mathrm{kT}(T_B) \quad (1 \leqq j \leqq i-1).$$

Furthermore, by using Lemma 1.3, $\mathrm{Ext}_A^j(T, X) = 0$ ($j = 1, \cdots, e-1$), and $\mathrm{Ext}_B^j(N', \mathrm{Ext}_A^e(T, X)) = 0$ ($j = 0, 1$) for all $_BN' \in \mathrm{KT}(T_B)$. Then the sequence

$$_B\mathrm{Ext}_A^{e-1}(T, V_e) \to {}_B\mathrm{Ext}_A^{e-1}(T, W_{e-1}) \xrightarrow{\beta_{e-1}} {}_BY \to 0$$

is exact, and we have a short exact sequence

$$0 \to {}_BY \to {}_B\mathrm{Ext}_A^e(T, X) \to {}_BG' \to 0,$$

where $_BG' \in \mathrm{KT}(T_B)$. Then the above sequence splits, and so $G' = 0$. Thus

$$_BY \xrightarrow{\sim} {}_B\mathrm{Ext}_A^e(T, X),$$

as desired.

Remark. Take a commutative diagram with exact rows:

$$\begin{array}{ccccccccccccccc}
0 \to & _AX & \to & V_e & \to & V_{e-1} & \to \cdots \to & V_2 & \to & V_1 & \to & V_0 & \to 0 \\
 & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & X & \to & I_0 & \to & I_1 & \to \cdots \to & I_{e-2} & \to & I_{e-1} & \to & J^eX & \to 0
\end{array}$$

where each $_AI_i$ is injective. Then, as is seen from the proof, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}(_AT, {}_AV_1) & \to & \mathrm{Hom}(_AT, {}_AV_0) & \to & Y & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \wr & & \\
\mathrm{Hom}(_AT, I_{e-1}) & \to & \mathrm{Hom}(_AT, J^eX) & \to & \mathrm{Ext}_A^e(T, X) & \to & 0
\end{array}$$

Dualizing Theorem 1.12, we obtain another theorem which holds under Condition $_AI$.

Theorem 1.14. *Assume that* $_AT_B$ *satisfies Condition* $_AI$. *Let* $e \geqq 0$ *be an integer, and let* $_AX$ *be an A-module such that* $\mathrm{Ext}_A^j(T, X) = 0$ $(j < e)$, $_B\mathrm{Ext}_A^j(T_B, X) \in \mathrm{KT}(T_B)$ $(j > e)$ *and such that* $\mathrm{Ext}_A^j(X, N) = 0$ $(j = 0, 1)$ *for any* $_AN \in \mathrm{KE}(_AT)$. *Put* $Y = {}_B\mathrm{Ext}^e(_AT_B, {}_AX)$. *Then* $\mathrm{Tor}_j(T_B, {}_BY) = 0$ $(j < e)$, $_A\mathrm{Tor}_j(_AT_B, {}_BY) \in \mathrm{KE}(_AT)$ $(j > e)$, $_A\mathrm{Tor}_e(_AT_B, {}_BY) \xrightarrow{\sim} {}_AX$, *and* $\mathrm{Ext}^i(_BN', {}_BY) = 0$ $(i = 0, 1)$ *for any* $N' \in \mathrm{KT}(T_B)$.

To prove the above, we use the following lemma which corresponds to Lemma 1.13.

Lemma 1.15. *Assume that* $_AT_B$ *satisfies Condition* $_AI$, *and let* $e \geqq 1$ *be an integer. Let the sequence of left B-modules*

$$\begin{array}{ccccccccccc} 0 \to {}_BV_0 \to & V_1 & \longrightarrow & V_2 \to \cdots \to & V_{e-1} & \longrightarrow & V_e \to {}_BY \to 0 \\ & & \searrow \ \nearrow & & & \searrow \ \nearrow & \\ & & W_1 & \cdots & & W_{e-1} & \end{array}$$

be an exact sequence such that ${}_A\mathrm{Tor}_j({}_AT_B, {}_BV_i) \in \mathrm{kE}({}_AT)$ $(j \geqq 1)$ $(i = 0, \cdots, e-1)$, ${}_A\mathrm{Tor}_j({}_AT_B, {}_BV_e) \in \mathrm{KE}({}_AT)$ $(j \geqq 1)$, *and* $\mathrm{Ext}^j({}_BN', {}_BV_i) = 0$ $(j = 0, 1)$ $(i = 0, \cdots, e)$ *for any* ${}_BN' \in \mathrm{KT}(T_B)$. *Furthermore, assume that the sequence*

$$0 \to {}_AX \to T\otimes_B V_0 \to T\otimes_B V_1 \to \cdots \to T\otimes_B V_{e-1} \to {}_AT\otimes_B V_e \to 0$$

is exact, and $\mathrm{Ext}^j({}_AX, {}_AN) = 0$ $(j = 0, 1)$ *for any* ${}_AN \in \mathrm{KE}({}_AT)$. *Then* $\mathrm{Tor}_j(T_B, {}_BY) = 0$ $(j < e)$, ${}_A\mathrm{Tor}_j({}_AT_B, {}_BY) \in \mathrm{KE}({}_AT)$ $(j > e)$, *and*

$${}_A\mathrm{Tor}_e({}_AT_B, {}_BY) \simeq {}_AX.$$

In the sequel we put $\mathrm{End}({}_AT) = B^*$ and $\mathrm{End}(T_B) = A^*$.

Assume that ${}_AT_B$ satisfies Condition ${}_BP$. Then, for any projective ${}_BP \in B$-Mod, Ker h_P and Cok h_P lie in $\mathrm{KT}(T_B)$. In particular, if we put ${}_BP = {}_BB$ then we know that $\mathrm{Tor}_j(T_B, {}_BB^*) = 0$ $(j \geqq 1)$, and $T\otimes_B B^* \simeq T$, $t\otimes b^* \mapsto tb^*$. Take a projective resolution of T_B:

$$\cdots \to Q_3 \to Q_2 \to Q_1 \to Q_0 \to T_B \to 0.$$

Then this induces a projective resolution of T_{B^*}:

$$\cdots \to Q_3\otimes_B B^* \to Q_2\otimes_B B^* \to Q_1\otimes_B B^* \to Q_0\otimes_B B^* \to T \to 0.$$

By using these we obtain the following

Proposition 1.16. (1) *Assume that* ${}_AT_B$ *satisfies Condition* ${}_BP$. *Then* ${}_AT_{B^*}$ *satisfies Condition* ${}_{B^*}P$. *For any* ${}_{B^*}Y \in B^*$-Mod, *we have* $\mathrm{Tor}_j(T_B, {}_BY) \simeq \mathrm{Tor}_j(T_{B^*}, {}_{B^*}Y)$ *for all* $j \geqq 0$. *For any* $W_{B^*} \in$ Mod-B^*, *we have* $\mathrm{Ext}^j(T_B, W_B) \simeq \mathrm{Ext}^j(T_{B^*}, W_{B^*})$ *for all* $j \geqq 0$. *Therefore,* $\mathrm{KT}(T_{B^*}) = \{{}_{B^*}N' \mid {}_BN' \in \mathrm{KT}(T_B)\}$, *and* $\mathrm{KE}(T_{B^*}) = \{N_{B^*} \mid N_B \in \mathrm{KE}(T_B)\}$.

(2) *Assume that* ${}_AT_B$ *satisfies Conditions* ${}_BP$ *and* P_A. *Then* ${}_AT_{B^*}$ *satisfies Conditions* ${}_{B^*}P$ *and* P_A. *Therefore* ${}_{A^*}T_{B^*}$ *satisfies Conditions* ${}_{B^*}P$ *and* P_{A^*}.

(3) *If* ${}_AT_B$ *satisfies Conditions* ${}_BP$ *and* ${}_AI$ *then* ${}_AT_{B^*}$ *satisfies Conditions* ${}_{B^*}P$ *and* ${}_AI$.

Proof. (1) We have to prove that ${}_AT_{B^*}$ satisfies Condition ${}_{B^*}P$. We take any free B^*-module ${}_{B^*}F^* = {}_{B^*}B^*\otimes_B F$, where ${}_BF$ is a free B-module. Then $T\otimes_B F \simeq T\otimes_{B^*}F^*$ canonically, and we have an exact sequence

$$0 \to {}_{B^*}F^* \xrightarrow{h} {}_{B^*}\mathrm{Hom}({}_AT_{B^*}, {}_AT\otimes_{B^*}F^*) \to \mathrm{Cok}\, h \to 0.$$

Then, by making use of the long exact sequence associated with this, we can see that ${}_B\mathrm{Cok}\ h \in \mathrm{KT}(T_B)$, or equivalently, ${}_{B^*}\mathrm{Cok}\ h \in \mathrm{KT}(T_{B^*})$. Furthermore, as is easily seen, ${}_{B^*}\mathrm{Ext}^i({}_AT_{B^*}, {}_AT\otimes_{B^*}F^*) \in \mathrm{kT}(T_{B^*})$ for all $i \geqq 1$. Thus we obtain (1) of Condition ${}_{B^*}P$. It is evident that (2) of Condition ${}_{B^*}P$ holds.

(2) For any $W_{B^*} \in \text{Mod-}B^*$, $\mathrm{Ext}^i(T_{B^*}, W_{B^*}) \xrightarrow{\sim} \mathrm{Ext}^i(T_B, W_B)$ holds for all integer $i \geqq 0$, by (1) above. Therefore ${}_AT_{B^*}$ satisfies Condition P_A. Then, by (1), ${}_{A^*}T_{B^*}$ satisfies Condition ${}_{B^*}P$ and P_{A^*}.

(3) This is evident from (1).

Remark. Assume that ${}_AT_B$ satisfies Condition ${}_BP$. Then there is an exact sequence

$$0 \to \mathrm{Ker}\ h_B \to B \xrightarrow{h_B} B^* \to \mathrm{Cok}\ h_B \to 0,$$

where ${}_B\mathrm{Ker}\ h_B$, ${}_B\mathrm{Cok}\ h_B \in \mathrm{KT}(T_B)$. Then, by Lemma 1.3, $\mathrm{Ext}^j({}_BN', {}_BB^*) = 0$ $(j = 0, 1)$ for all ${}_BN' \in \mathrm{KT}(T_B)$. These characterize the ring homomorphism $h_B : B \to B^*$ up to B-ring isomorphism.

We now assume that ${}_AT_B$ satisfies Conditions P_A and ${}_AI$, and seek some cases in which Condition ${}_{A^*}I$ holds. In the following we put $h_A = h$, and so Ker h, Cok $h \in \mathrm{KT}({}_AT)$. Then it is easily seen that ${}_A\mathrm{Hom}({}_A\mathrm{Ker}\ h_A, {}_AI)$, ${}_A\mathrm{Hom}({}_A\mathrm{Cok}\ h_A, {}_AI) \in \mathrm{KE}({}_AT)$ for any injective ${}_AI \in A\text{-Mod}$. For any injective A^*-module ${}_{A^*}J$ we take a monomorphism from ${}_AJ$ to an injective A-module ${}_AI$. Then we obtain a splitting monomorphism ${}_{A^*}J \to {}_{A^*}\mathrm{Hom}({}_AA^*_{A^*}, {}_AI)$. Therefore it is sufficient to consider an injective A^*-module ${}_{A^*}\mathrm{Hom}({}_AA^*_{A^*}, {}_AI)$ in place of ${}_{A^*}J$, by Lemmas 1.4 and 1.5. Then, from the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to & 0 & \to & T\otimes_B\mathrm{Hom}({}_{A^*}T, {}_{A^*}\mathrm{Hom}({}_AA^*, {}_AI)) \\
 & \downarrow & & \downarrow k^* \\
0 \to & \mathrm{Hom}({}_A\mathrm{Cok}\ h, {}_AI) & \to & \mathrm{Hom}({}_AA^*, {}_AI) \\
 \to & T\otimes_B\mathrm{Hom}({}_AT, {}_AI) & \to & 0 \\
 & \downarrow k & & \downarrow \\
 \to & I & \to & \mathrm{Hom}({}_A\mathrm{Ker}\ h, {}_AI) \to 0
\end{array}$$

it follows an exact sequence

$$0 \to \mathrm{Ker}\ k^* \to \mathrm{Ker}\ k \to {}_A\mathrm{Hom}({}_A\mathrm{Cok}\ h_A, {}_AI) \to \mathrm{Cok}\ k^* \to \mathrm{Cok}\ k \\ \to {}_A\mathrm{Hom}({}_A\mathrm{Ker}\ h_A, {}_AI) \to 0.$$

By assumption, ${}_A\mathrm{Ker}\ k$ and ${}_A\mathrm{Cok}\ k$ lie in $\mathrm{KE}({}_AT)$. Then we have the following

Lemma 1.17. *Assume that ${}_AT_B$ satisfies Conditions ${}_AI$ and P_A. If one of the* *following conditions holds then* ${}_{A^*}T_B$ *satisfies Condition* ${}_{A^*}I$.

(i) T_B *is finitely generated.*

(ii) $\mathrm{kE}({}_AT) = \mathrm{KE}({}_AT)$.

(iii) *The right A-module* A^*_A *is flat.*

Proof. By Proposition 1.16, it is evident that (2) of Condition ${}_{A^*}I$ holds. For any injective A^*-module ${}_{A^*}J$ we have to prove that (1) of Condition ${}_{A^*}I$ holds. We may assume that ${}_{A^*}J = {}_{A^*}\mathrm{Hom}({}_AA^*_{A^*}, {}_AI)$ for some injective A-module ${}_AI$. Then it is evident that the latter half of (1) of Condition ${}_{A^*}I$ holds. To prove that the first half of (1) of Condition ${}_{A^*}I$ holds, it is sufficient to show that ${}_A\mathrm{Ker}\ k^*$ or ${}_A\mathrm{Cok}\ k^*$ lies in $\mathrm{KE}({}_AT)$. If (i) holds then it is easily seen that $\mathrm{Cok}\ k^* = 0$ (cf. [7; Lemma 1.7]). If (ii) holds then it is evident. If (iii) holds then ${}_A\mathrm{Hom}({}_AA^*_A, {}_AI)$ is injective, and so ${}_A\mathrm{Ker}\ k^* \in \mathrm{KE}({}_AT)$.

Concerning the condition (ii) above we note the following

Lemma 1.18. (1) *If* ${}_B\mathrm{Ext}^2({}_AT_B, {}_AX) \in \mathrm{KT}(T_B)$ *for any* ${}_AX \in A$-Mod *then* $\mathrm{kE}({}_AT) = \mathrm{KE}({}_AT)$.

(2) *If* ${}_A\mathrm{Tor}_2({}_AT_B, {}_BY) \in \mathrm{KE}({}_AT)$ *for any* ${}_BY \in B$-Mod *then* $\mathrm{kT}(T_B) = \mathrm{KT}(T_B)$.

Proof. (1) Let ${}_AN, {}_AN' \in \mathrm{KE}({}_AT)$, and let

$$0 \to \mathrm{Ker}\ f \to {}_AN \xrightarrow{f} {}_AN' \to \mathrm{Cok}\ f \to 0$$
$$\searrow \quad \nearrow$$
$$W$$

be an exact sequence. Then $\mathrm{Hom}({}_AT, {}_A\mathrm{Ker}\ f) = \mathrm{Hom}({}_AT, {}_AW) = 0$, ${}_B\mathrm{Ext}^i({}_AT_B, {}_AW) \xrightarrow{\sim} {}_B\mathrm{Ext}^{i+1}({}_AT_B, {}_A\mathrm{Ker}\ f)$ $(i \geqq 0)$ and ${}_B\mathrm{Ext}^i({}_AT_B, {}_A\mathrm{Cok}\ f) \xrightarrow{\sim} {}_B\mathrm{Ext}^{i+1}_A(T_B, W)$ $(i \geqq 0)$. Then $\mathrm{Ext}^1({}_AT, {}_A\mathrm{Ker}\ f) = 0$, and so $\mathrm{Hom}({}_BN'', {}_B\mathrm{Ext}^2({}_AT_B, {}_A\mathrm{Ker}\ f)) = 0$ for any ${}_BN'' \in \mathrm{KT}(T_B)$, by Lemma 1.3. By assumption, $\mathrm{Ext}^2({}_AT, {}_A\mathrm{Ker}\ f) = 0$. Then $\mathrm{Hom}({}_BN'', {}_B\mathrm{Ext}^3({}_AT_B, {}_A\mathrm{Ker}\ f)) = 0$ for any $N'' \in \mathrm{KT}(T_B)$, by Lemma 1.3. Then $\mathrm{Ext}^3_A(T, \mathrm{Ker}\ f) = 0$. Repeating this argument, we see that $\mathrm{Ker}\ f \in \mathrm{KE}({}_AT)$. Thus ${}_AN \in \mathrm{kE}({}_AT)$. Dualizing (1) above, we obtain (2).

2. Modules over finite dimensional algebras over a field. In this section, both A and B are finite dimensional algebras over a field K, and all modules are finite dimensional over K. Under this restriction we use notations $\mathrm{Kt}(-)$, $\mathrm{Ke}(-)$, $\mathrm{kt}(-)$, and $\mathrm{ke}(-)$ instead of $\mathrm{KT}(-)$, $\mathrm{KE}(-)$, $\mathrm{kT}(-)$, and $\mathrm{kE}(-)$, respectively.

Condition ${}_Bp$. (1) For a left B-module ${}_BB$, there hold $\operatorname{Ker} h_B$, $\operatorname{Cok} h_B \in \operatorname{Kt}(T_B)$, and ${}_B\operatorname{Ext}^i({}_AT_B, {}_AT) \in \operatorname{kt}(T_B)$ for all $i \geqq 1$, where $h_B : {}_BB \to {}_B\operatorname{Hom}({}_AT_B, {}_AT)$, $b \mapsto (x \to xb)$ $(b \in B, x \in T)$.

(2) There is an integer $r \geqq 0$ such that ${}_B\operatorname{Ext}^i({}_AT_B, {}_AX) \in \operatorname{Kt}(T_B)$ for any $i > r$ and any ${}_AX \in A$-mod.

Condition ${}_Ai$. (1) For a left A-module ${}_A\mathrm{D}(A)$, there hold $\operatorname{Ker} k_{\mathrm{D}(A)}$, $\operatorname{Cok} k_{\mathrm{D}(A)} \in \operatorname{Ke}({}_AT)$, and ${}_A\operatorname{Tor}_i({}_AT_B, {}_B\operatorname{Hom}({}_AT_B, {}_A\mathrm{D}(A))) \in \operatorname{ke}({}_AT)$ for any $i \geqq 1$, where D is the duality functor, and $k_{\mathrm{D}(A)} : {}_AT \otimes_B \operatorname{Hom}({}_AT_B, {}_A\mathrm{D}(A)) \to {}_A\mathrm{D}(A)$, $x \otimes f \mapsto (x)f$.

(2) There is an integer $r \geqq 0$ such that ${}_A\operatorname{Tor}_i({}_AT_B, {}_BY) \in \operatorname{Ke}({}_AT)$ for any $i > r$ and any ${}_BY \in B$-mod.

However, it is easily seen that Condition ${}_Ai$ for ${}_AT_B$ is equivalent to Condition p_A for ${}_AT_B$. Symmetrically, Condition i_B is equivalent to Condition ${}_Bp$. Therefore we have the following

Theorem 2.1. *Assume that* ${}_AT_B$ *satisfies Condition* ${}_Bp$. *Let* $e \geqq 0$ *be an integer.*

(1) *Let* ${}_BY \in B$-mod *be such that* $\operatorname{Tor}_i(T_B, {}_BY) = 0$ $(i < e)$, ${}_A\operatorname{Tor}_i({}_AT_B, {}_BY) \in \operatorname{Ke}({}_AT)$ $(i > e)$ *and such that* $\operatorname{Ext}^j({}_BN', {}_BY) = 0$ $(j = 0, 1)$ *for any* ${}_BN' \in \operatorname{Kt}(T_B)$. *Put* ${}_AX = {}_A\operatorname{Tor}_e({}_AT_B, {}_BY)$. *Then* $\operatorname{Ext}^i({}_AT, {}_AX) = 0$ $(i < e)$, ${}_B\operatorname{Ext}^i({}_AT_B, {}_AX) \in \operatorname{Kt}(T_B)$ $(i > e)$, ${}_BY \xrightarrow{\sim} {}_B\operatorname{Ext}^e({}_AT_B, {}_AX)$, *and* $\operatorname{Ext}^j({}_AX, {}_AN) = 0$ $(j = 0, 1)$ *for any* ${}_AN \in \operatorname{Ke}({}_AT)$.

(2) *Let* $Y'_B \in \text{mod-}B$ *be such that* $\operatorname{Ext}^i(T_B, Y'_B) = 0$ $(i < e)$, $\operatorname{Ext}^i({}_AT_B, Y'_B)_A \in \operatorname{Kt}({}_AT)$ $(i > e)$ *and such that* $\operatorname{Ext}^j(Y'_B, N_B) = 0$ $(j = 0, 1)$ *for any* $N_B \in \operatorname{Ke}(T_B)$. *Put* $X'_A = \operatorname{Ext}^e({}_AT_B, Y'_B)_A$. *Then* $\operatorname{Tor}_i(X'_A, {}_AT) = 0$ $(i < e)$, $\operatorname{Tor}_i(X'_A, {}_AT_B)_B \in \operatorname{Ke}(T_B)$ $(i > e)$, $\operatorname{Tor}_e(X'_A, {}_AT_B)_B \xrightarrow{\sim} Y'_B$, *and* $\operatorname{Ext}^j(N'_A, X'_A) = 0$ $(j = 0, 1)$ *for any* $N'_A \in \operatorname{Kt}({}_AT)$.

3. Examples. In this section, we shall consider some cases to which Theorem 1.12 or Theorem 1.14 can be applied.

Proposition 3.1. (1) *Assume that a faithful module* T_B *satisfies* $(\mathrm{P})_r$, $(\mathrm{E})_r$ (*cf.* [7]), *and put* $\operatorname{End}(T_B) = A$. *Furthermore, assume that* $\operatorname{pdim}_A T \leqq r$, *and that* $\operatorname{Ext}^i_A(T, \oplus T) = 0$ *for any* $i \geqq 1$ *and any direct sum* ${}_A\oplus T$ *of copies of* ${}_AT$. *Then* ${}_AT_B$ *satisfies Condition* ${}_BP$.

(2) *Let* ${}_AT'$ *be a tilting module of* $\operatorname{pdim}_A T' \leqq r < \infty$, *and put* ${}_A\oplus T' = {}_AT$ (*direct sum of copies of* ${}_AT'$) *and* $\operatorname{End}({}_AT) = B$. *Then* ${}_AT_B$ *satisfies*

Condition ${}_BP$.

Proof. (1) As T_B is faithful, Ker $h_P = 0$ for any projective module ${}_BP \in$ B-Mod. By assumption, $\mathrm{Ext}_A^i(T, T\otimes_B P) = 0$ for all $i \geqq 1$. Thererfore if we put ${}_B\mathrm{Hom}_A(T_B, T\otimes_B P) = {}_BP^*$, we have $\mathrm{Tor}_i(T_B, {}_BP^*) = 0$ for all $i \geqq 1$, and $T\otimes_B P^* \rightleftarrows T\otimes_B P$ canonically, by [7 ; Lemma 1.8]. Then the latter implies that $T\otimes_B P \rightleftarrows T\otimes_B P^*$, $t\otimes p \mapsto t\otimes(t' \to t'\otimes p)$ $(t, t' \in T, p \in P)$. Therefore we have Cok $h_P \in \mathrm{KT}(T_B)$. The remainder is evident.

(2) Evidently $\mathrm{pdim}_A T \leqq r$, and ${}_AT$ satisfies $(\mathrm{G})_r$ (cf. [7]). Furthermore, as ${}_AT'$ has a projective resolution of finitely generated projective modules, we see that $\mathrm{Ext}_A^i(T, \oplus T) = 0$ for any $i \geqq 1$ and any direct sum ${}_A\oplus T$ of copies of ${}_AT$. Then T_B asatisfies $(\mathrm{P})_r$, $(\mathrm{E})_r$ and $\mathrm{End}(T_B) \rightleftarrows A$, by [7 ; Proposition 1.4]. Then by (1) above, ${}_AT_B$ satisfies Condition ${}_BP$.

The following is evident from [7; Lemma 1.8 and 1.9].

Proposition 3.2. *Assume that* ${}_AT$ *satisfies* $(\mathrm{P})_r$, $(\mathrm{E})_r$, *and that* $\mathrm{End}({}_AT) =$ B. *Then* ${}_AT_B$ *satisfies Condition* I_B *and Condition* ${}_BP$.

Therefore Theorems 1.12 and 1.14 yield more precise forms of [7 ; Theorems 1.14 and 1.15]. Since T_B satisfies $(\mathrm{G})_r$ (cf. [7]), both $\mathrm{KE}(T_B)$ and $\mathrm{KT}(T_B)$ are trivial in this case.

Another example comes from a divisible R-module over a commutative integral domain R. Let ∂_R the Fuchs' divisible R-module over a commutative integral domain R. We put $\mathrm{End}(\partial_R) = E$. Then the bimodule ${}_E\partial_R$ has the following properties (cf. [1, 2]).

0. ∂_R is a divisible R-module which generates every divisible R-module.
1. $\mathrm{End}({}_E\partial) \rightleftarrows R$.
2. $1 = \mathrm{pdim}\ \partial_R = \mathrm{pdim}\ {}_E\partial$.
3. There is an exact sequence of R-homomorphisms

$$0 \to R_R \to \partial_R \to \phi(\partial)_R \to 0,$$

where $\phi(\partial)_R$ is an R-direct summand of ∂_R.

4. $\mathrm{Ext}_R^1(\partial, \oplus\partial) = 0$ for any direct sum $\oplus\partial$ of copies of ∂_R.
5. ${}_E\partial$ is finitely presented, and $\mathrm{Ext}_E^1(\partial, \partial) = 0$.

Therefore, by Proposition 3.2, ${}_E\partial_R$ satisfies Condition I_R and Condition ${}_RP$. On the other hand, by Proposition 3.1(1), ${}_R\partial_{E^{op}}$ satisfies Condition ${}_{E^{op}}P$, or equivalently, ${}_E\partial_R$ satisfies Condition P_E. Furthermore we have the follwing

Proposition 3.3. *${}_E\partial_R$ satisfies Codition ${}_EI$.*

Proof. There is an exact sequence

$$0 \to R_R \to Q_R \to Q/R_R \to 0,$$

where Q_R is the quotient field of R. Let ${}_EI$ be any injective module of E-Mod. Then, as ∂_R is divisible, ${}_R\mathrm{Hom}({}_E\partial_R, {}_EI)$ is R-torsion free, and hence we have an exact sequence

$$0 \to \mathrm{Hom}_E(\partial, I) \to Q \otimes_R \mathrm{Hom}_E(\partial, I) \to (Q/R) \otimes_R \mathrm{Hom}_E(\partial, I) \to 0.$$

Then, since the middle term is R-flat, we see that $\mathrm{Tor}_1^R(\partial, \mathrm{Hom}_E(\partial, I)) = 0$. Therefore, by [7; Theorem 1.15], $\mathrm{Ext}_E^i(\partial, \partial \otimes_R \mathrm{Hom}_E(\partial, I)) = 0$ for all $i \geqq 1$, and $\mathrm{Hom}_E(\partial, \partial \otimes_R \mathrm{Hom}_E(\partial, I)) \xrightarrow{\sim} \mathrm{Hom}_E(\partial, I)$ as R-modules. This implies that ${}_E\mathrm{Ker}\, k_I \in \mathrm{KE}({}_E\partial)$, where $k_I : {}_E\partial \otimes_R \mathrm{Hom}({}_E\partial_R, {}_EI) \to {}_EI$ is the canonical map. Furthermore, as is easily seen, ${}_E\mathrm{Cok}\, k_I \in \mathrm{KE}({}_E\partial)$. As pdim $\partial_R = 1$, this completes the proof.

Thus ${}_E\partial_R$ satisfies Conditions I_R, ${}_RP$, P_E, and ${}_EI$. Thus we get a complete set of tilting type correspondences between R-modules and E-modules, by virtue of Theorems 1.12 and 1.14. Note that $\mathrm{KE}(\partial_R)$ and $\mathrm{KT}(\partial_R)$ are trivial in this case. Facchini's Theorem ([1, 2]) corresponds to the fact that ${}_E\partial_R$ satisfies Conditions I_R and P_E.

Finally we state an example which comes from a certain ring extension. Let $R \subseteq Q$ be rings with common identity, and suppose that $\mathrm{Tor}_i(Q_R, {}_RQ) = 0$ $(i \geqq 1)$, and $Q \otimes_R Q \xrightarrow{\sim} Q$, $q_1 \otimes q_2 \mapsto q_1q_2$.

Then, for any ${}_QX_1$, ${}_QX_2 \in Q$-Mod and $Y_Q \in$ Mod-Q,

$$\mathrm{Ext}^i({}_RX_1, {}_RX_2) \xrightarrow{\sim} \mathrm{Ext}^i({}_QX_1, {}_QX_2) \qquad (i \geqq 0),$$

and

$$\mathrm{Tor}_i(Y_Q, {}_QX_1) \xrightarrow{\sim} \mathrm{Tor}_i(Y_R, {}_RX_1) \qquad (i \geqq 0).$$

To see these, we take a projective resolution of Q_R :

$$\cdots \to P_2 \to P_1 \to P_0 \to Q_R \to 0.$$

Then, by assumption, we get an exact sequence

$$\cdots \to P_2 \otimes_R Q_Q \to P_1 \otimes_R Q_Q \to P_0 \otimes_R Q_Q \to Q \otimes_R Q_Q \to 0.$$

However, as $Q \otimes_R Q_Q \xrightarrow{\sim} Q_Q$, the above sequence splits. Therefore, applying $\otimes_Q X_1$ to the above sequence, we get an exact sequence. Hence, $\mathrm{Tor}_i(Q_R, {}_RX_1) = 0$ $(i \geqq 1)$ and ${}_QQ \otimes_R X_1 \xrightarrow{\sim} {}_QX_1$, $q \otimes x_1 \mapsto qx_1$. Therefore, from a projective resolution of ${}_RX_1$, we obtain a projective resolution of ${}_QX_1$, by applying $Q \otimes_R$ on the left side. From this fact we can easily see the preceding isomorphisms. We put $K = Q/R$, which is an R-R-bimodule:

$(**)\quad 0 \to R \to Q \to K \to 0.$

Then we have the following

Proposition 3.4. *Let $K = Q/R$ be as above. Then ${}_RK_R$ satisfies Conditions ${}_RP$ and ${}_RI$, and* Q-Mod $= \mathrm{KE}({}_RK) = \mathrm{kE}({}_RK) = \mathrm{KT}(K_R) = \mathrm{kT}(K_R)$.

Proof. First we prove the last assertion. Let ${}_RN' \in R$-Mod be such that $\mathrm{Tor}_i(K_R, {}_RN') = 0$ $(i = 0, 1)$. Then ${}_RN' \xrightarrow{\sim} ({}_RR \otimes_R N' \xrightarrow{\sim})\, Q \otimes_R N'$, $x \mapsto 1 \otimes x$, and so $N' \in Q$-Mod. Conversely, if $N' \in Q$-Mod then $\mathrm{Tor}_i(Q_R, {}_RN') \xrightarrow{\sim} \mathrm{Tor}_i(Q_Q, {}_QN') = 0$ $(i \geqq 1)$, and $Q \otimes_R N' \xrightarrow{\sim} N'$, $q \otimes x \mapsto qx$. Then $N' \xrightarrow{\sim} Q \otimes_R N'$, $x \mapsto 1 \otimes x$. Applying the functor $\otimes_R N'$ to the short exact sequence $(**)$, we see that ${}_RN' \in \mathrm{KT}(K_R)$. Hence Q-Mod $= \mathrm{KT}(K_R)$, and so $\mathrm{KT}(K_R) = \mathrm{kT}(K_R)$. Next we let $\mathrm{Ext}^i({}_RK, {}_RN) = 0$ $(i = 0, 1)$. Then ${}_R\mathrm{Hom}({}_RQ, {}_RN) \xrightarrow{\sim} {}_R\mathrm{Hom}({}_RR, {}_RN) \xrightarrow{\sim} {}_RN$, and so $N \in Q$-Mod. Conversely, if $N \in Q$-Mod then $\mathrm{Hom}({}_RQ, {}_RN) \xrightarrow{\sim} \mathrm{Hom}({}_QQ, {}_QN) \xrightarrow{\sim} N$, and $\mathrm{Ext}^i({}_RQ, {}_RN) \xrightarrow{\sim} \mathrm{Ext}^i({}_QQ, {}_QN) = 0$ $(i \geqq 1)$. Applying the functor $\mathrm{Hom}_R(-, N)$ to $(**)$, we can see that $N \in \mathrm{KE}({}_RK)$. Hence Q-Mod $= \mathrm{KE}({}_RK) = \mathrm{kE}({}_RK)$. Now, for any ${}_RX \in R$-Mod, we have an exact sequence

$$\mathrm{Ext}^1({}_RR, {}_RX) \to \mathrm{Ext}^2({}_RK, {}_RX) \to \mathrm{Ext}^2({}_RQ, {}_RX) \to \mathrm{Ext}^2({}_RR, {}_RX).$$

Therefore $\mathrm{Ext}^2({}_RK, {}_RX) \xrightarrow{\sim} \mathrm{Ext}^2({}_RQ, {}_RX) \in Q$-Mod. Let ${}_RP \in R$-Mod be projective. Then the exact sequence

$$0 \to P \to Q \otimes_R P \to K \otimes_R P \to 0$$

induces isomorphisms

$$\mathrm{Hom}({}_RK, {}_RK \otimes_R P) \xrightarrow{\sim} \mathrm{Ext}^1({}_RK, {}_RP),\ \mathrm{Ext}^1({}_RK, {}_RK \otimes_R P) \xrightarrow{\sim} \mathrm{Ext}^2({}_RK, {}_RP).$$

Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to \mathrm{Hom}({}_RQ, {}_RP) \to & \mathrm{Hom}({}_RR, {}_RP) & \to & \mathrm{Ext}^1({}_RK, {}_RP) & \to \mathrm{Ext}^1({}_RQ, {}_RP) \to 0 \\ & \wr\uparrow & & \wr\uparrow & \\ & P & \xrightarrow{h_P} & \mathrm{Hom}({}_RK, {}_RK \otimes_R P) & \end{array}$$

and hence $\mathrm{Ker}\, h$, $\mathrm{Cok}\, h \in Q$-Mod. Thus ${}_RK_R$ satisfies Condition ${}_RP$. Let ${}_RI$ be an injective R-module. Then

$$0 \to \mathrm{Hom}({}_RK, {}_RI) \to \mathrm{Hom}({}_RQ, {}_RI) \to I \to 0$$

is exact. Since the middle term is a left Q-module, we have $K\otimes_R I = 0$, and

$$\mathrm{Tor}_i(K_R, {}_RI) \simeq \mathrm{Tor}_{i-1}(K_R, {}_R\mathrm{Hom}({}_RK_R, {}_RI))$$

for all $i \geqq 1$. On the other hand we have an exact sequence

$$0 \to \mathrm{Tor}_1(Q_R, {}_RI) \to \mathrm{Tor}_1(K_R, {}_RI) \longrightarrow I \to Q\otimes_R I \to 0$$
$$\searrow \quad \nearrow$$
$$\mathrm{t}(I)$$

where $\mathrm{t}(I) = \mathrm{Ker}(I \to Q\otimes_R I)$. Therefore, for any $i \geqq 1$,

$$\mathrm{Ext}^i({}_RK, {}_R\mathrm{Tor}_1({}_RK_R, {}_RI)) \simeq \mathrm{Ext}^i({}_RK, {}_RI) = 0.$$

As $Q\otimes_R I \in \mathrm{KE}({}_RK)$, we have an exact sequence

$$0 \to L \to K\otimes_R\mathrm{Hom}({}_RK_R, {}_RI) \to \mathrm{t}(I) \to 0.$$

If we put ${}_RM = {}_R\mathrm{Hom}({}_RK_R, {}_RI)$ then, since ${}_RK_R$ satisfies Condition ${}_RP$, we have a canonical isomorphism

$${}_RM \simeq {}_R\mathrm{Hom}({}_RK_R, {}_RK\otimes_R M),$$

and we can see that $\mathrm{Ext}^i({}_RK, {}_RL) = 0$ $(i = 0, 1)$. Hence ${}_RL \in \mathrm{KE}({}_RK)$. Thus ${}_RK$ satisfies Condition ${}_RI$.

Proposition 3.5. *Under the same assumption as in Proposition 3.4, if we put* $\mathrm{End}({}_RK) = H$, $\mathrm{End}(K_R) = H'$ *then we obtain a bimodule* ${}_{H'}K_H$ *which satisfies Conditions* ${}_HP$, $P_{H'}$, ${}_{H'}I$, *and* I_H.

Proof. We have an isomorphism $K\otimes_R H \simeq K, x\otimes h \mapsto (x)h$, and so $\mathrm{End}(K_H) = \mathrm{End}(K_R)$. Similarly we have an isomorphism $H'\otimes_R K \simeq K$, and so $\mathrm{End}({}_{H'}K) = \mathrm{End}({}_RK)$. Since $\mathrm{KE}(K_R) = \text{Mod-}Q = \mathrm{kE}(K_R)$ and $\mathrm{KE}({}_RK) = Q\text{-Mod} = \mathrm{kE}({}_RK)$, the remainder follows from Lemmas 1.16 and 1.17.

For this example we refer to [6] wholly, and further, to [5]. In particular, Theorems 1.12 and 1.14 contain [6; Theorem 3.4 and 3.8], by Proposition 3.4. The proof of Proposition 3.4 is related to the proof of [6; Proposition 5.2].

4. Supplement. (1) Assume that ${}_AT_B$ satisfies Condition ${}_BP$. Let ${}_BY \in B\text{-Mod}$ be such that $\mathrm{Ext}^j_B(N', Y) = 0$ $(j = 0, 1)$ for any $N' \in \mathrm{KT}(T_B)$. Put $h = h_B$ and $\mathrm{End}({}_AT) = B^*$. Then, as ${}_B\mathrm{Ker}\, h$, ${}_B\mathrm{Cok}\, h \in \mathrm{KT}(T_B)$, we see that $\mathrm{Hom}({}_BB^*, {}_BY) \simeq (\mathrm{Hom}({}_BB, {}_BY) \simeq) Y, f \mapsto (1)f$. Therefore ${}_BY$ is uniquely extended to a left B^*-module ${}_{B^*}Y$. In particular, if we put $Y = {}_BB^*$ then, by Lemma 1.3, $\mathrm{Hom}({}_BB^*, {}_B\mathrm{B}^*) \simeq \mathrm{Hom}({}_BB, {}_BB^*)$, that is, $B^* \simeq \mathrm{End}({}_BB^*)$ by right multiplication. On the other hand, let $X_B \in \text{Mod-}B$ be such that $\text{Ext}^j_B(X, N) = 0$ $(j = 0, 1)$ for any $N_B \in \text{KE}(T_B)$. Then, for any injective B-module I_B, $\text{Hom}({}_B\text{Ker}\, h_B, I_B)_B$ and $\text{Hom}({}_B\text{Cok}\, h_B, I_B)_B$ belong to $\text{KE}(T_B)$. Hence $\text{Hom}(X_B, \text{Hom}({}_BB^*_B, I_B)) \xrightarrow{\sim} \text{Hom}(X_B, \text{Hom}({}_BB_B, I_B)_B)$. Therefore $X \xrightarrow{\sim} X \otimes_B B^*$, $x \mapsto x \otimes 1$. Hence X_B is uniquely extended to a right B^*-module X_{B^*}. (cf. [6 ; Proposition 5.1, 5.5, and 5.6])

(2) The following is also true, and its proof is similar to the one of Lemma 1.2.

Let e, s be non-negative integers, and assume that $\text{Tor}_i(T_B, {}_BY') = 0$ $(0 \leqq i \leqq e+s$, and $i \neq e)$, and $\text{Tor}_j(N'_A, {}_AT) = 0$ $(0 \leqq j \leqq e+s+1)$. Then $\text{Tor}_t(N'_A, {}_A\text{Tor}_e(T_B, {}_BY')) = 0$ $(0 \leqq t \leqq s+1)$.

(3) Let A be a commutative ring, and assume that a left A-module $T(\neq 0)$ satisfies $(\text{P})_r$, $(\text{E})_r$ for some $r \geqq 0$. Then ${}_AT$ is projective. To see this, by localization, we may assume that A is a local ring. Then ${}_AT$ has a minimal projective resolution of finitely generated projective modules:

$$0 \to P_r \to P_{r-1} \to \cdots \to P_0 \to {}_AT \to 0.$$

Assume that $P_r \neq 0$ and $r \geqq 1$. Then, by assumption, $P_r \to P_{r-1}$ yields an epimorphism $\text{Hom}_A(P_{r-1}, T) \twoheadrightarrow \text{Hom}_A(P_r, T)$. Since the image of P_r in P_{r-1} is contained in $\text{rad}(A) \cdot P_{r-1}$, $P_r \cdot \text{Hom}_A(P_r, T)$ is contained in $\text{rad}(A) \cdot T$. But, as $P_r \neq 0$, ${}_AP_r$ is free, and so $T = \text{rad}(A) \cdot T$, a contradiction.

(4) Assume the situation of Theorem 1.12. Additionally, assume that; $\text{Ext}^i({}_AT, {}_AT \otimes_B P) = 0$ $(i \geqq 1)$ for any projective ${}_BP$, and that $\text{pdim}_A T \leqq r < \infty$. Furthermore, we additionally assume that ${}_BY$ (in Theorem 1.12) satisfies $\text{Tor}_i(T_B, {}_BY) = 0$ $(i \neq e)$. Then $\text{Ext}^i({}_AT, {}_A\text{Tor}_e({}_AT_B, {}_BY)) = 0$ $(i \neq e)$ holds.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA
TSUKUBA 305, JAPAN

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