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DEGREE EQUATIONS FOR p-BLOCKS OF FINITE GROUPS, II

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1. Introduction. M. Osima's Theorem [6] is famous and available as a sufficient condition for a set of irreducible ordinary characters to be a block. For a prime number p, let B be a p-block of a finite group G with the irreducible ordinary characters $\chi_1, \cdots, \chi_k$. In the previous paper [5], we introduced a residue set $\{\varepsilon_i\}$ associated to B and proved that $\eta_B = \sum_{i=1}^{k} \varepsilon_i\chi_i$ vanishes on G^0, where G^0 denotes the set of p-regular elements of G (see Section 2). Moreover, for a subset J of the index set $\{1, \cdots, k\}$, we put $\{\chi_J\} = \{\chi_j \mid j \in J\}$ and making use of Osima's Theorem, we verified that if $\sum_{j\in J} \varepsilon_j\chi_j$ vanishes on G^0 for every residue set $\{\varepsilon_i\}$ associated to B, then $\{\chi_J\} = B$ or ϕ. Replacing the hypothesis with weaker one, we stated the following;

Conjecture A. Let B be a p-blcok with the irreducible ordinary characters $\chi_1, \cdots, \chi_k$. For $J \subseteq \{1, \cdots, k\}$, assume that $\sum_{j\in J} \varepsilon_j\chi_j(1) = 0$ for every residue set $\{\varepsilon_i\}$ associated to B. Then $\{\chi_J\} = B$ or ϕ.

We also showed that Conjecture A is equivalent to K. Harada's Conjecture. Similarly, replacing the hypothesis with another one, we now state the following new conjecture;

Conjecture B. Let B be a p-block with the irreducible ordinary characters $\chi_1, \cdots, \chi_k$ and $\{\varepsilon_i\}$ be a residue set associated to B. For $J \subseteq \{1, \cdots, k\}$, assume that $\sum_{j\in J} \varepsilon_j\chi_j$ vanishes on G^0. Then $\{\chi_J\} = B$ or ϕ.

In this paper, we prove the next results.

Theorem 1. *If a p-block B has cyclic defect groups, then Conjecture* B *holds.*

Theorem 2. *If $G = \mathrm{PSL}(2, q^f)$ such that q is a prime and f is a positive integer, then Conjecture* B *holds.*

2. Preliminaries. Let the order of G be $p^a g$ such that $(p, q) = 1$. We denote the rational integer ring by $\boldsymbol{Z}$ as usual. Then the following is proved.

Theorem 3. *Let B be a p-block of G with defect d which contains the irreducible ordinary characters $\chi_1, \cdots, \chi_k$ and the principal indecomposable characters $\Phi_1, \cdots, \Phi_t$. Let D denote the decomposition matrix of B. Then the* *following assertions hold.*

(i) *There exist* $m_i \in \boldsymbol{Z}$ $(i = 1, \cdots, k)$ *which satisfy* $(m_1 \cdots m_k)D = (w_1, \cdots, w_\ell)$, *where* $\Phi_s(1) = p^a u w_s$ $(s = 1, \cdots, \ell)$ *with* $\mathrm{GCD}\{\Phi_s(1)\} = p^a u$.

(ii) *If we set* $\chi_i(1) = p^a u m_i + p^{a-d} u \varepsilon_i$ $(i = 1, \cdots, k)$, *then all* ε_i *are integers which satisfy* $(\varepsilon_1 \cdots \varepsilon_k)D = (0 \cdots 0)$ *and* $\eta_B = \sum_{i=1}^k \varepsilon_i \chi_i$ *vanishes on* G^0. *In particular, we have a degree equation* $\eta_B(1) = \sum_{i=1}^k \varepsilon_i \chi_i(1) = 0$.

Proof. See [5] Theorem 1.

We call this $\{\varepsilon_i\}$ a residue set with $\{m_i\}$ associated to B or simply a residue set.

Now we suppose that the condition of Conjecture B holds, i.e. $\sum_{j \in J} \varepsilon_j \chi_j$ vanishes on G^0 for $J \subseteq \{1, \cdots, k\}$. Then by the linear independence of the irreducible Brauer characters, we have $\sum_{j \in J} \varepsilon_j d_{js} = 0$ for all $s = 1, \cdots, \ell$, where $D = (d_{is})$. Since $\chi_j(1) = p^a u m_j + p^{a-d} u \varepsilon_j$, we obtain

$$\textstyle\sum_{j \in J} \chi_j(1) d_{js} \equiv 0 \pmod{p^a} \quad \text{for all} \quad s = 1, \cdots, \ell. \tag{1}$$

Since $\eta_B = \sum_{i=1}^k \varepsilon_i \chi_i$ vanishes on G^0, for $J' = \{1, \cdots, k\} - J$, similarly we have

$$\textstyle\sum_{j \in J'} \chi_j(1) d_{js} \equiv 0 \pmod{p^a} \quad \text{for all } s = 1, \cdots, \ell. \tag{2}$$

Therefore the next is proved.

Lemma 4. *If* (1) *or* (2) *does not occur for a non-empty proper subset* J, *then Conjectre* B *holds.*

Lemma 5. *If* $k = \ell + 1$, *then Conjecture* B *holds.*

Proof. If a non-empty proper subset J of $\{1, \cdots, k\}$ satisfies $\sum_{j \in J} \varepsilon_j d_{js} = 0$ for all s, then $J' = \{1, \cdots, k\} - J$ also satisfies $\sum_{j \in J'} \varepsilon_j d_{js} = 0$ for all s. Hence the rank of D is less than $k - 1 = \ell$. This is a contradiction and the proof is complete by Lemma 4.

3. Proof of Theorem 1. Without loss, we may assume $d > 0$. We use Dade's Theorem on block structure with cyclic defect groups (see [2] or [3] for detail). The principal indecomposable character Φ_s of B has the form $\Phi_s = \chi_i + \sum_{\lambda \in \Lambda} \chi_\lambda$ $(1 \leq i \leq \ell)$ or $\Phi_s = \chi_i + \chi_j$ $(1 \leq i \neq j \leq \ell)$. Since $\Phi_s(1) \equiv 0 \pmod{p^a}$ and $\chi_i(1) \not\equiv 0 \pmod{p^a}$, we have $\nu(\chi_i(1)) = \nu(\sum_{\lambda \in \Lambda} \chi_\lambda(1))$ for all $i = 1, \cdots, \ell$, where ν denotes the p-adic exponential valuation with $\nu(p) = 1$. Since

$\sum_{\lambda\in\Lambda}\chi_\lambda(1) = |\Lambda|\chi_\lambda(1)$ and $|\Lambda| = (p^d-1)/\ell$, we have $\nu(\chi_i(1)) = \nu(\chi_\lambda(1)) = a-d$ for all $i = 1, \cdots, \ell$ and $\lambda\in\Lambda$.

If $\Phi_s = \chi_i+\chi_j$, then either $i, j\in J$ or $i, j\in J'$. Next assume $\Phi_s = \chi_i + \sum_{\lambda\in\Lambda}\chi_\lambda$. We claim that $\{i\}\cup\Lambda\subseteq J$ or $\{i\}\cup\Lambda\subseteq J'$. If $i\notin J$, then

$$\begin{aligned}\nu(\textstyle\sum_{j\in J}\chi_j(1)d_{js}) &= \nu(|J\cap\Lambda|\chi_\lambda(1)) \leq \nu(|J\cap\Lambda|)+\nu(\chi_\lambda(1))\\ &< d+a-d = a.\end{aligned}$$

Hence $J\cap\Lambda = \phi$ by Lemma 4 and so $\{i\}\cup\Lambda\subseteq J'$. Consequently we obtain either $\{i\}\cup\Lambda\subseteq J$ or $\{i\}\cup\Lambda\subseteq J'$ as desired. By the definition of blocks, it is impossible to arrange rows and columns of D so that

$$\begin{pmatrix} D_1 & 0\\ 0 & D_2\end{pmatrix}.$$

Therefore we have $\{\chi_J\} = B$ or ϕ. This completes the proof of Theorem 1.

4. Proof of Theorem 2. In the following cases the defect groups of each block are all cyclic.

$$G = \mathrm{PSL}(2, q^f) \quad 2\neq p|q^f-1 \quad\text{or}\quad 2\neq p|q^f+1$$

Then Conjecture B holds for these cases by Theorem 1. Thus it suffices to show the following four cases.

(I) $G = \mathrm{PSL}(2, q^f)$ $p=2$ $q\neq 2$ $q^f\equiv 1 \pmod 4$,

(II) $G = \mathrm{PSL}(2, q^f)$ $p=2$ $q\neq 2$ $q^f\equiv -1 \pmod 4$,

(III) $G = \mathrm{PSL}(2, 2^f)$ $p=2$,

(IV) $G = \mathrm{PSL}(2, p^f)$ $p\neq 2$.

The decomposition matrices which we use in this proof are determined by R. Burkhardt in [1].

(I) $G = \mathrm{PSL}(2, q^f)$ $p=2$ $q\neq 2$ $q^f\equiv 1 \pmod 4$. In this case the irreducible characters of G are $1, \alpha, \gamma_1, \gamma_2, \zeta_i$ $(1\leq i\leq (q^f-5)/4)$, δ_i $(1\leq i\leq (q^f-1)/4)$. Each degree is $\alpha(1) = q^f$, $\gamma_1(1) = \gamma_2(1) = (q^f+1)/2$, $\zeta_i(1) = q^f+1$, $\delta_i(1) = q^f-1$. Let $P\times C$ be a cyclic subgroup of order $(q^f-1)/2$, where P is a subgroup of order 2^{a-1} and C is a subgroup of order c, $2\nmid c$. Thus there are only one block of maximal defect and $(c-1)/2$ blocks of defect $a-1$ and the other blocks are all of defect 0.

The decomposition matrix of the principal block is

		1	φ_1	φ_2	degree
	1	1	0	0	1
	α	1	1	1	$2^a c+1$
	γ_1	1	1	0	$2^{a-1}c+1$
	γ_2	1	0	1	$2^{a-1}c+1$
$2^{a-2}-1$ {	ζ	2	1	1	$2^a c+2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	ζ	2	1	1	$2^a c+2$

Hence by Lemma 4, we have $\{\chi_J\} = B$ or ϕ.

The decomposition matrix of the block of defect $a-1$ is

		1	degree
2^{a-1} {	ζ	1	$2^a c+2$
	$\vdots$	$\vdots$	$\vdots$
	ζ	1	$2^a c+2$

Thus clearly $\{\chi_J\} = B$ or ϕ.

(II) $G = \mathrm{PSL}(2, q^f)$ $p = 2$ $q \neq 2$ $q^f \equiv -1 \pmod 4$. In this case there are irreducible characters $1, \alpha, \gamma_1, \gamma_2, \zeta_i$ $(1 \leq i \leq (q^f-3)/4)$, δ_i $(1 \leq i \leq (q^f-3)/4)$ of G. Each degree is $\alpha(1) = q^f$, $\gamma_1(1) = \gamma_2(1) = (q^f-1)/2$, $\zeta_i(1) = q^f+1$, $\delta_i(1) = q^f-1$. Let $P\times C$ be a cyclic subgroup of order $(q^f+1)/2$, where P is a subgroup of order 2^{a-1} and C is a subgroup of order c, $2 \nmid c$. There are only one block of maximal defect and $(c-1)/2$ blocks of defect $a-1$. The other blocks are all of defect 0.

The decomposition matrix of the principal block is

		1	φ_1	φ_2	degree
	1	1	0	0	1
	α	1	1	1	$2^a c-1$
	γ_1	0	1	0	$2^{a-1}c-1$
	γ_2	0	0	1	$2^{a-1}c-1$
$2^{a-2}-1$ {	δ	0	1	1	$2^a c-2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	δ	0	1	1	$2^a c-2$

The same argument as in (I) yields the result.

The decomposition matrix of the block of defect $a-1$ is

$$
\begin{array}{c|cc}
 & 1 & \text{degree} \\
\hline
2^{a-1}\left\{\begin{array}{c}\delta \\ \vdots \\ \delta\end{array}\right. & \begin{array}{c}1 \\ \vdots \\ 1\end{array} & \begin{array}{c}2^a c-2 \\ \vdots \\ 2^a c-2\end{array}
\end{array}
$$

Thus the result is clear.

(III) $G = \mathrm{PSL}(2, 2^f)$ $p = 2$. The irreducible characters are 1, α, ζ_i $(1 \le i \le 2^{f-1}-1)$, γ_i $(1 \le i \le 2^{f-1})$ and their degrees are $\alpha(1) = 2^f$, $\zeta_i(1) = 2^f+1$, $\gamma_i(1) = 2^f-1$. Thus $\{\alpha\}$ is a block of defect 0 and only principal block is a block of positive defect. So we consider the principal block B. The number k of the irreducible ordinary characters of B is $1+(2^{f-1}-1)+2^{f-1} = 2^f$. It is known that there is a one-to-one correspondence between the irreducible Brauer characters of B and the proper subsets of $\{1, \cdots, f\}$. Thus the number ℓ of the irreducible Brauer characters of B is 2^f-1. Hence $k = \ell+1$ and by Lemma 5 the result follows.

(IV) $G = \mathrm{PSL}(2, p^f)$ $p \neq 2$. If $p^f \equiv 1 \pmod 4$, then the irreducible characters are 1, ζ_i $(1 \le i \le (p^f-5)/4)$, δ_i $(1 \le i \le (p^f-1)/4)$, γ_1, γ_2 and their degrees are $\zeta_i(1) = p^f+1$, $\delta_i(1) = p^f-1$, $\gamma_1(1) = \gamma_2(1) = (p^f+1)/2$. There is only one block B. Set $F = \{I = (i_1 \cdots i_f) \mid 1 \le i_r \le p-1, \sum_{r=1}^f i_r \equiv 0 \pmod 2\} - \{(p-1 \cdots p-1)\}$. Then it is known that there is a one-to-one correspondence between the irreducible Brauer characters φ_I of B and the elements I of F. In particular, the principal Brauer character corresponds to $(0 \cdots 0)$. Thus the number ℓ is equal to $|F| = (p^f-1)/2$. Since $k = (p^f+3)/2$, we have $k = \ell+2$. A residue set $\{\varepsilon_i\}$ with $\{m_i\}$ associated to B is arranged as follows.

$$
\begin{aligned}
1(1) &= p^f u m_0 + u\varepsilon_0 \\
\zeta_i(1) &= p^f u m_i + u\varepsilon_i \quad (1 \le i \le h) \\
\delta_i(1) &= p^f u m'_i + u\varepsilon'_i \quad (1 \le i \le h+1) \\
\gamma_i(1) &= p^f u m''_i + u\varepsilon''_i \quad (1 \le i \le 2)
\end{aligned} \tag{3}
$$

where $h = (p^f-5)/4$. The rows and the columns of the decomposition matrix D of B are arranged as follows. The first row is 1, the second is $\zeta_1, \cdots$, the $(h+1)$-st is ζ_h, the $(h+2)$-nd is $\delta_1, \cdots$, the $(2h+2)$-nd is δ_{h+1}, the $(2h+3)$-rd is γ_1 and the last is γ_2. The first column is $(0 \cdots 0)$, from the second $I = (i_1 \cdots i_f)$

$\neq (0 \cdots 0)$ such that no $i_r = p-1$, the next is $I = \{i_1 \cdots i_f\}$ such that exactly one $i_r = p-1, \cdots$ and the last is $I = (i_1 \cdots i_f)$ such that exactly $(f-1)$ $i_r = p-1$. Now we define for $I = (i_1 \cdots i_f)$,

$$\Sigma(I) = \left\{\frac{1}{2}\sum_{r=1}^{f} \tau_r(p-1-i_r)p^{r-1} \mid \tau_r \in \{1, -1\}\right\}.$$

If $\zeta_i = \sum_{I \in F} d_{iI}\varphi_I$ is the decomposition of ζ_i into the irreducible Brauer characters, then

$$\begin{aligned} d_{iI} &= 1 \quad \text{for } \{i, \tfrac{1}{2}(p^f-1)-i\} \cap \Sigma(I) \neq \phi \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

If $\delta_i = \sum_{I \in F} d_{iI}\varphi_I$, then

$$\begin{aligned} d_{iI} &= 1 \quad \text{for } \{i, \tfrac{1}{2}(p^f+1)-i\} \cap \Sigma(I) \neq \phi \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

If $\gamma_{1/2} = \sum_{I \in F} d_I\varphi_I$, then

$$\begin{aligned} d_I &= 1 \quad \text{for } \tfrac{1}{4}(p^f-1) \in \Sigma(I) \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

If $I = (0 \cdots 0)$, then

$$\left|\{\zeta_i \mid d_{iI} = 1\}\right| + 1 = \left|\{\delta_i \mid d_{iI} = 1\}\right| = 2^{f-1}.$$

If $I = (i_1 \cdots i_f)$ such that exactly t $i_r = p-1$ and $(p^f-1)/4 \notin \Sigma(I)$, then

$$\left|\{\zeta_i \mid d_{iI} = 1\}\right| = \left|\{\delta_i \mid d_{iI} = 1\}\right| = 2^{f-t-1}.$$

If $I = (i_1 \cdots i_f)$ such that exactly t $i_r = p-1$ and $(p^f-1)/4 \in \Sigma(I)$, then

$$\left|\{\zeta_i \mid d_{iI} = 1\}\right| + 1 = \left|\{\delta_i \mid d_{iI} = 1\}\right| = 2^{f-t-1}.$$

Then the degrees of the principal indecomposable characters Φ_I are $p^f(2^f-1)$, $p^f 2^f, \cdots, p^f 2^{f-1}, \cdots, p^f 2, \cdots$. Hence $(\cdots w_I \cdots) = (2^f-1\ 2^f \cdots 2^{f-1} \cdots 2 \cdots)$ and $u = 1$ where $\Phi_I(1) = p^f u w_I$.

Next we consider the linear homogeneous equation

$$(x_0 x_1 \cdots x_h x_1' \cdots x_{h+1}' x_1'' x_2'')D = (0 \cdots 0). \tag{4}$$

Now

$$(1\ \underbrace{1 \cdots 1}_{h}\ \underbrace{-1 \cdots -1}_{h+1}\ 0\ 1) \quad \text{and} \quad (\underbrace{0 \cdots 0}_{2h+2}\ 1\ -1)$$

are linearly independent solutions of (4). Furthermore

$$(-1\ \underbrace{0 \cdots 0}_{h}\ \underbrace{2 \cdots 2}_{h+1}\ 0\ 0)$$

is a solution of the equation

$$(m_0 m_1 \cdots m_h m'_1 \cdots m'_{h+1} m''_1 m''_2)D = (\cdots w_l \cdots) \tag{5}$$

Since the rank of D is ℓ and $k = \ell+2$, $(-1\ 0 \cdots 0\ 2 \cdots 2\ 0\ 0)+z_1(1\ 1 \cdots 1\ -1 \cdots -1\ 0\ 1)+z_2(0 \cdots 0\ 1\ -1)$ $(z_1, z_2 \in Z)$ are all of the solutions of (5). Hence $m_i = z_1$ $(1 \leq i \leq h)$ and $m'_i = 2-z_1$ $(1 \leq i \leq h+1)$. Therefore by (3) we have $\varepsilon_i = p^f(1-z_1)+1$ $(1 \leq i \leq h)$ and $\varepsilon'_i = p^f(z_1-1)-1$ $(1 \leq i \leq h+1)$. By Lemma 4, we may assume that $1, \gamma_1, \gamma_2 \notin \{\chi_J\}$. We claim that $\{\chi_J\} = \phi$. Suppose that $n_1 \zeta_i$'s and $n_2 \delta_i$'s are contained in $\{\chi_J\}$. Then by $\sum_{j \in J} \varepsilon_j \chi_j(1) = 0$,

$$n_1\{p^f(1-z_1)+1\}(p^f+1)+n_2\{p^f(z_1-1)-1\}(p^f-1) = 0,$$
$$0 \leq n_1 \leq h, \quad 0 \leq n_2 \leq h+1.$$

Hence $n_1+n_2 \equiv 0 \pmod{p^f}$, $0 \leq n_1+n_2 \leq 2h+1 = (p^f-3)/2$ and so $n_1 = n_2 = 0$. We obtain $\{\chi_J\} = \phi$ as required.

If $p^f \equiv -1 \pmod 4$, then the same argument implies the result. This completes the proof of Theorem 2.

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