Math. J. Okayama Univ. 34 (1992), 57—66

# ON THE NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA. X

Dedicated to Professor Kazuo Kishimoto on his 60th birthday

KAORU MOTOSE

We here consider a proper basis $B$ of the radical $J(KU)$ of the modular group algebra $KU$ of an elementary abelian group $U$ over a field $K$ (see [16]). The purpose of this paper is to show some examples such that this basis is useful for computing the nilpotency indices of the radicals of some group algebras. We may assume that a field $K$ contains a finite field $F$ of order $p^r = |U|$ and an elementary abelian group $U$ is the permutation group on $F$ defined by

$$U = \{u_a : x \to x+a \mid a \in F\}.$$

We usually use $\{u_a - 1 \mid a \in F\}$ as a basis of the radical $J(KU)$ of $KU$ but it is not so useful for the products of basis elements and actions on $J(KU)$. The basis $B$ is defined as in the following. Let $\lambda$ be an element of the character group $\hat{F}^* = \mathrm{Hom}(F^*, F^*)$ of the multiplicative group $F^*$ of $F$. Then we set

$$R_\lambda = \textstyle\sum_{a \in F} \lambda(a) u_a \quad \text{where} \quad \lambda(0) = \begin{cases} 0 & \text{if } \lambda \neq 1 \\ 1 & \text{if } \lambda = 1. \end{cases}$$

It is easy to see $\{R_\lambda \mid \lambda \in \hat{F}^*\}$ is a basis of the radical $J(KU)$ of the group algebra $KU$. The reason for the usefullness of this basis is the following equations.

1. $R_\lambda R_\mu = J(\lambda, \mu) R_{\lambda\mu}$ where $J(\lambda, \mu) = \sum_{a \in F} \lambda(a)\mu(1-a)$.
2. $R_\lambda^\sigma = R_{\lambda\sigma^{-1}}$ for an automorphism $\sigma$ of $F$.

We can find a much better basis $B$ of $J(KU)$ as in the following. Let $\eta$ be a generator of $F^*$. Then $\phi : \eta \to \eta^{-1}$ is a generator of $\hat{F}^*$. We set $\Phi_k = R_{\phi^{p^k}}$. It is evident $\Phi_k^p = 0$. We obtain from [16, Proposition 3.2] the next assertion

$$\Phi_0^{p-1} \Phi_1^{p-1} \cdots \Phi_{r-1}^{p-1} \neq 0.$$

We can identify an element

$$\Phi_0^{i_0} \Phi_1^{i_1} \cdots \Phi_{r-1}^{i_{r-1}} \quad (0 \leq i_k < p)$$

a natural number

$$i_0+i_1p+\cdots+i_{r-1}p^{r-1}.$$

For two natural numbers $a=i_0+i_1p+\cdots+i_{r-1}p^{r-1}$ and $b=j_0+j_1p+\cdots+j_{r-1}p^{r-1}$ where $0\le i_k, j_k<p$, we define

$$a\# b=\begin{cases} a+b & \text{if } i_k+j_k<p \text{ for all } k \\ 0 & \text{otherwise.}\end{cases}$$

It follows from our observations that the set

$$B=\{1,2,\cdots,p^r-1\}$$

is a basis of $J(KU)$.

Moreover we shall use some notations. We set $a^*=\sum_{k=0}^{r-1} i_k$ for $a=i_0+i_1p+\cdots+i_{r-1}p^{r-1}$ where $0\le i_k<p$. Let $t(G)$ be the nilpotency index of the radical of the group algebra $KG$ of a finite group $G$. Let $F$ be a finite field of order $p^{pt}$ and let $S$ be a subgroup of $F^*$. Then we consider the next permutation group $M_{p,t,S}$ on $F$.

$$M_{p,t,S}=\{x\to ax^{p^{tk}}+b \mid a\in S, b\in F, k=0,1,\cdots,p-1\}.$$

If the order of $S$ is $h_0=(p^{pt}-1)/(p^t-1)$, we set simply $M_{p,t}=M_{p,t,S}$.

Using the basis $B$, we already obtained the next assertions. The set $\{b\in B \mid b^*=k\}$ is a basis of $J(KU)^k/J(KU)^{k+1}$ (see [16]). If the order of $S$ is multiple of $h_0$, then we have $t(M_{p,t,S})=t(M_{p,t})=(pt+1)(p-1)+1$ (see [9]) since $M_{p,t}$ is a normal subgroup of $M_{p,t,S}$ whose index is not divisible by $p$ and we have the formula to compute Loewy series of the group algebra of $M_{p,t,S}$ over a field $K$ (see [15,16]).

In this paper, using $B$ we shall give an altenative proof of the essential part of H. Fukushima's result (see [1]), the minimal order group with some conditions about the nilpotency index, and some examples of computations of $t(M_{p,t,S})$ in case the order $h$ of $S$ is a proper divisor of $h_0$.

**1.** Let $WV$ be a $p$-nilpotent group with a $p$-group $W$ and a normal abelian $p'$-component $V$, and let $G$ be a semidirect product of a subgroup $WV$ and a normal elementary abelian $p$-subgroup $U$. But we should remark that the assumption $U$ is elementary is superfluous in our discussion. Recently, in case $WV$ is a Frobenius, H. Fukushima characterized the groups with $t(G)=s(p-1)+1$ where $p^s$ is the order of a $p$-Sylow subgroup $WU$ of $G$ (see [1]).

The most difficult part in his proof is to prove that $C_G(x)$ contains a $p$-Sylow subgroup of $G$ for every $x\in U$. However, if we use our basis $B$, then we can easily prove this part. It is easy to see that the radical of $KG$ contains $J(KW)\cdot\hat{V}+J(KU)$ where $\hat{V}=\sum_{v\in V}v$.

The next lemma was proved in [10]. However we shall here restate for the completeness.

**Lemma 1.1.** *If* $t(G)=s(p-1)+1$ *and let* $U_0$ *be a minimal normal subgroup of* $UV$ *contained in* $U$. *Then* $U_0$ *is normal in* $G$.

*Proof.* Assume that $U_0^\tau\neq U_0$ for some $\tau\in W$ which implies $U_0\cap U_0^\tau=1$. There exists a normal subgroup $U_1$ of $UV$ containing $U_0^\tau$ such that $U=U_0\times U_1$ since $V$ is a $p'$-group. The next equation follows from $t(G)=s(p-1)+1$.

$$\begin{aligned}0&=\hat{W}\hat{V}\hat{U}_0(1-\tau)\hat{V}\hat{U}_1=\hat{W}\hat{V}(\hat{U}_0-\hat{U}_0^\tau)\hat{V}\hat{U}_1\\&=|V|\hat{W}\hat{V}(\hat{U}_0-\hat{U}_0^\tau)\hat{U}_1=|V|\hat{W}\hat{V}\hat{U}_0\hat{U}_1=|V|\hat{G}.\end{aligned}$$

This is a contradiction and we have the assertion.

**Proposition 1.2.** *If* $t(G)=s(p-1)+1$ *then* $C_G(u)$ *contains a* $p$*-Sylow subgroup of* $G$ *for all* $u\in U$. *Conversely, if* $|W|=p$ *and* $C_G(u)$ *contains a* $p$*-Sylow subgroup of* $G$ *for all* $u\in U$, *then* $t(G)=s(p-1)+1$.

*Proof.* Since $t(G)=s(p-1)+1$, $W$ is elementary. Let $U_0$ be a minimal normal subgroup of $UV$ contained in $U$. Then there exists a normal subgroup $U_1$ of $UV$ such that $U=U_0\times U_1$ since $V$ is a $p'$-group, and it follows from lemma that $U_0$ and $U_1$ are normal subgroup of $G$. Thus $G_0=WVU_0=G/U_1$ and so by virtue of the inequality $t(G)\geq t(G_0)+t(U_1)-1$, $G_0$ satisfies the same condition as $G$. Hence, we may assume that $U$ is a minimal normal subgroup of $UV$ (of $G$). We set $H=C_G(U)$, $G_1=G/H$, $W_1=WH/H$ and $V_1=VH/H$. Then we may assume that $W_1$ and $V_1$ are nontrivial. $W_1$ and $V_1$ have the same actions on $U$ as $W$ and $V$, respectively. Since $W_1V_1$ and $V_1$ act faithfully and irreducibly on $U$, respectively and $V_1$ is a normal abelian subgroup of $W_1V_1$, by [17, p.244 Proposition 19.8], we can see that $F=GF(q^p)$ where $q=p^t$ and $W_1$ and $V_1$ are regarded as a permutation group on $F$ such that

$$W_1=\langle\sigma_1: x\to x^q\rangle\quad\text{and}\quad V_1=\{x\to ax\,;\ a\in S\}$$

where $\sigma_1=\sigma \bmod H$ for some $\sigma\in W$ and $S$ is a subgroup of the multiplicative group of $GF(q^p)$.

We may regard $B=\{1,2,\cdots,q^p-1\}$ as a basis of the radical of $KU$. Let $a$ be an arbitrary element of $\{1,2,\cdots,q^p-2\}$ and $a+b=q^p-1$. Then it follows from the condition $t(G) = s(p-1)+1$ that for an arbitrary element $\tau$ of $W$,

$$\hat{W}\hat{V}(a-a^{\tau})\hat{V}b = \hat{W}\hat{V}a(1-\tau)\hat{V}b = 0.$$

Thus we have

$$\hat{V}a\hat{V}b = \hat{V}a^{\tau}\hat{V}b.$$

We can set $a^{v} = \theta(v)a$ for $v \in V$ where $\theta(v) \in K$ (see the definition of $B$). It is easy to see that $\theta$ is a linear character of $V$ and

$$\hat{V}a\hat{V} = \Big(\sum_{v\in V}\theta(v)\Big)\hat{V}a.$$

Assume first that $\theta$ is trivial. Then

$$|V|\,\hat{V}a \# b = \hat{V}a\hat{V}b = \hat{V}a^{\tau}\hat{V}b = |V|\,\hat{V}a^{\tau} \# b$$

and so

$$a \# b = a^{\tau} \# b.$$

Since $a^{\tau}$ is an element of $B$ (see the definition of $B$) and $a \# b$ is not zero, we can see that $a^{\tau} = a$. Next we assume that $\theta$ is nontrivial. Then $\sum_{v\in V}\theta(v) = 0$ and so $\hat{V}a\hat{V} = 0 = \hat{V}a^{\tau}\hat{V}$.

Thus we have $(\hat{V}a\hat{V}) = \hat{V}a^{\tau}\hat{V}$ for all $a \in B$ and $\tau \in W$. Since $B$ together with the identity of $U$ forms a basis of $KU$, we obtain that $\hat{V}u\hat{V} = \hat{V}u^{\tau}\hat{V}$ and so $\sum_{v\in V} u^{v} = \sum_{v\in V} u^{\tau v}$ for all $u \in U$ and $\tau \in W$. $L = \{v \in V \mid u^{v} = u\}$ is a subgroup of $V$ and $\{v \in V \mid u^{\tau v} = u^{\tau}\} = L^{\tau}$. Thus we have

$$\sum_{k} u^{v_k} = \sum_{k} u^{\tau v_k^{\tau}}$$

where $\{v_k\}$ is representatives of right cosets of $L$ in $V$. Hence $u = u^{v\tau}$ for some $v$ in $V$ which implies that $G = VG_G(u)$ and so the index $(G : C_G(u))$ is not divisible by $p$.

*Proof of the converse*: For arbitrary elements $u \in U$ and $\tau \in W$, it follows from the assumption that $\tau^{h} \in C_G(u)$ for some $h \in V$. Hence $u^{\tau^{h}} = u$. Let $e$ and $f$ be primitive idempotents of $KV$ corresponing to character $\nu$ and $\mu$ of $V$ such that $e^{\rho} = e$ and $f^{\rho} = f$ for all $\rho \in W$, respectively. Then we have

$$euf = eu^{\tau^{h}}f = \nu(h^{-1})\mu(h)\tau^{-1}ehuh^{-1}f\tau = (euf)^{\tau}.$$

It follows from this equation that

$$\Big(\sum_{e} J(KW)eKG\Big)^{p} = 0,$$

and on the other hand, Morita's theorem [7] shows that

$$J(KG)=J(KU)KG+\sum_{e}J(KW)eKG$$

where $e$ runs over primitive idempotents of $KV$ such that $e^{\rho}=e$ for all $\rho\in W$. Thus we have $t(G)=t(U)+p-1$ which implies our assertion.

The next corollary follows from our theorem.

**Corollary 1.3** (H. Fukushima [1]). *Assume that $WV$ is a Frobenius group with the kernel $V$ and a complement $W$. Then $t(G)=s(p-1)+1$ if and only if $|W|=p$ and $C_G(u)$ contains a $p$-Sylow subgroup of $G$ for all $u\in U$.*

The following corollary follows from the proof of our theorem and [10, Theorem 12].

**Corollary 1.4.** *If $G$ is a group of the minimal order satisfying the next conditions, then $G$ is isomorphic to $M_{p,1}$*

1. *$G$ is a $p$-solvable group with a $p$-Sylow subgroup $P$ of order $p^s$.*
2. *$P$ is not elementary abelian.*
3. $t(G)=s(p-1)+1$.
4. *$O_{p'}(G/O_p(G))$ is abelian.*

**2.** The purpose of this section is to compute the nilpotency index of $M_{p,t,s}$ in case the order $h$ of $S$ is a proper divisor of $h_0$. We set

$$W=\langle w: x\rightarrow x^q\rangle \quad \text{and} \quad V=\{x\rightarrow ax\ ;\ a\in S\}.$$

Moreover we set $G=M_{p,t,s}$ and $f=\hat{V}$. If the order $h$ of $S$ is a proper divisor of $h_0$, it is not so easy to compute $(M_{p,t,s})$ (see [14]). However, using a proper basis of $J(KU)$ which play an important role in [16], we can reduce this problem to the computation of a certain number. This number can be computed by a computer. Some of these are cited in the last of this section. We should note that $w$ shifts $t$ times components of elements in $B$. We can see easily $J(KG)=J(KW)f+J(KU)KG$ since $WV$ is a Frobenius group. We use the notation $\alpha^{1-w}=\alpha-\alpha^{w}$ for $\alpha\in KG$.

The next is a key lemma in this section.

**Lemma 2.1.** *The set $fJ(KU)^{s(1-w)^k}fJ(KW)$ is contained in $fJ(KU)^{s(1-w)^{k+1}}f+fJ(KW)fJ(KU)^{s(1-w)^k}f$ for $s\geq 1$, $k\geq 0$, and the set*

*$fJ(KG)^{s(1-w)^k}f$ is also contained in $J(KG)^{s+k}$.*

*Proof.* The next equation follows from $fw = wf$ and $b(1-w) = b^{(1-w)} + (1-w)b^w$ for $b \in KG$. For $a \in J(KU)^s$, we have

$$fa^{(1-w)^k}f(1-w) = fa^{(1-w)^{k+1}}f + f(1-w)fa^{w(1-w)^k}.$$

This equation shows the first assertion. Together wih induction on $k$, the second assertion follows.

The next lemma follows from Lemma 2.1 and the equation $J(KG) = J(KW)f + J(KU)KG$.

**Lemma 2.2.** *We have*

$$J(KG)^n = J(KU)^n KG + \sum_{j+\ell+j'=n} J(KU)^j J(KW)^\ell f J(KU)^{j'}$$
$$+ \sum J(KU)^j J(KW)^\ell f \cdot \prod_i (fJ(KU)^{s_i(1-w)^{k_i}}f) \cdot J(KU)^{j'}$$

*where the last summation is extended over $\sum_i s_i + \sum_i k_i + \ell + j + j' = n$, $s_i \geq 1$, $k_i \geq 1$ and $\ell \geq 1$.*

We can now prove our theorem in this section.

**Theorem 2.3.** *We set*

$$d = \mathrm{Max}\{\sum k_i \mid b_1^{(1-w)^{k_1}} \# b_2^{(1-w)^{k_2}} \# \cdots \# b_m^{(1-w)^{k_m}} \neq 0\}$$

*where $b_1, b_2, \cdots, b_m$ runs the set of multiples of $h$ and nonmultiples of $h_0$ in $B$ and $k_n < p$ for all $n$. Then we have $t(G) = d + (tp+1)(p-1)+1$.*

*Proof.* It follows from [16, Lemma 3.1(4)] that $d$ is the maximum integer in $\sum k_i$ with $\prod_i (fJ(KU)^{s_i(1-w)^{k_i}}f) \neq 0$ where $s_i \geq 1$ and $k_i \geq 1$. We put $n = d + (tp+1)(p-1)+1$. Assume that $J(KG)^n \neq 0$. Then we can see that $\sum s_i + j + j' \leq tp(p-1)$, $\sum k_i \leq d$ and $\ell \leq p-1$ by using the same notations in lemma 2.2 where $n = \sum s_i + \sum k_i + \ell + j + j'$. This yields a contradiction $n = \sum s_i + \sum k_i + \ell + j + j' \leq d + (tp+1)(p-1) < n$ which implies $J(KG)^n = 0$. On the other hand there exists a nonzero element

$$z' = \prod_{i=1}^m (fa_i^{(1-w)^{k_i}}f)$$

such that $d = \sum_{i=1}^m k_i$. Since the nonzero element $z'$ is equal to $zf$ where $z = \prod_i^m a_i^{(1-w)^{k_i}}$ (see [16, Lemma 3.1(4)]), there exists a nonzero term $b$ of $z$ with a nonzero coefficient $\gamma$. We can find an element $c$ such that $c \# b = q^p - 1 = \hat{U}$

and $c \# b' = 0$ for every nonzero term $b' \neq b$ of $z$ by noting $b'^* = \sum_{i=1}^{m} a_i^* = b^*$ since $b'$ and $b$ have the same form $a^{w^{j_1}} \# \cdots \# a^{w^{j_m}}$ with only differences of $j_1$, $j_2, \cdots, j_m$. Then we have $cz = \gamma\hat{U}$. Since $J(KG)^{d+tp(p-1)}$ contains $cz'$ by using Lemma 2.1 and $c^* + \sum_{i=1}^{m} a_i^* = tp(p-1)$, $J(KG)^{n-1}$ contains a nonzero element $cz'(1-w)^{p-1}f = cz\hat{W}\hat{V} = \gamma\hat{G}$. This completes the proof.

It is not so easy to calculate $d$ and so using a computer, we can present some examples for proper divisors of $h_0$.

| $p$ | $t$ | $q^p$ | $h_0$ | $h$ | $t(G)$ |
|---|---|---|---|---|---|
| 2 | 3 | 64 | 9 | 3 | 10 |
| | 5 | 1024 | 33 | 3 | 16 |
| | | | | 11 | 14 |
| | 6 | 4096 | 65 | 5 | 18 |
| | | | | 13 | 17 |
| | 7 | 16384 | 129 | 43 | 18 |
| 3 | 2 | 729 | 91 | 7 | 22 |
| | | | | 13 | 20 |
| 5 | 1 | 3125 | 781 | 71 | 33 |

**3.** The purpose of this section is to prove that every $p$-solvable group with a regular $p$-Sylow subgroup has almost $p$-length 1 (Theorem 3.1(b)) and some results relating to this (Theorem 3.1(a) and (c)). It is well known from [4, p.456, 4.8 Theorem, b)] but we here give a much simpler proof and the useful form for us.

Let $G$ be a $p$-solvable group with a $p$-Sylow subgroup $P$ and let $F/O_{p'}(G)$ be the Frattini subgroup of $O_{p',p}(G)/O_{p'}(G)$. A subgroup $H$ of $GL(2, p)$ acts naturally on the elementary abelian group $E$ of order $p^2$. Let $p^2 \cdot H$ be a semidirect product of $E$ by $H$ with respect to this action.

The following theorem shall give generalizations and improvements of some theorems:

Our assertion (a) in the next theorem contains [6, Lemma 7]. We can see that [18, Theorem 1] (see [12]), [10, Corollary 13] and [13, Corollary] are easy consequences of (b) and a ring theoretical condition in [12, Proposition] and [13, Lemma] is superfluous by (b). Examples in (i) of (c) point out a mistake in the proof of [18, Theorem 1] (see [12]). As a result of (c), we have [6, Lemmas 8 and

9]. It follows from (c) that a ring theoretical condition in [8, Proposition 1], [5, Theorem] and [11, Proposition] is superfluous.

**Theorem 3.1.** *Assume that $G$ has $p$-length at least 2. Then we obtain the following assertions*:

(a) $|P| \geq p^p$.

(b) *If $P$ is regular* (*see* [3, p.321, *Definition* 10.1]), *then $p$ is a Fermat prime and 2-Sylow subgroup of $G/O_{p'}(G)$ is nonabelian.*

(c) *If every proper subgroup of $P$ is metacyclic then one of the next holds.*

i. $p = 3$ *and* $G/F$ *is either* $3^2 \cdot GL(2, 3)$ *or* $3^2 \cdot SL(2, 3)$

ii. $p = 2$ *and* $G/F = 2^2 \cdot GL(2, 3) = S_4$.

*Proof.* In all cases, we may assume $F = 1$ by [2, p.7, Corollary]. Thus $U = O_p(G)$ is elementary abelian and $G/U$ is a subgroup of $GL(U) = GL(n, p)$ where $U$ is regarded as a $n$-dimensional vector space over $GF(p)$ (see [2, Lemma 1.2.3]). Since $G/U$ contains an element $y$ of order $p$ and $y$ is conjugate to a triangular matrix with 1's in the diagonal, we have an inequality $p-1 \leq n$ in view of Hall Higman's theorem B [2, Theorem B]. By the Frattini argument, we can see $G = N_G(V)U$ where $V$ is a $p'$-group such that $O_{p,p'}(G) = UV$.

(a) An inequality $n \geq p-1$ yields our assertion.

(b) Let $x$ be an element of $P$ and let $u$ be an element of $U$. Then it follows that $(xu)^p = x^p$ since $P$ is regular and $\langle x, u\rangle' \subseteq \langle x, U\rangle' \subseteq U$. So if $\tilde{x}$ is a residue class of an element $x$ in $G$, then we have

$$u^{(\tilde{x}-1)^{p-1}} = u^{\tilde{x}^{p-1}+\cdots+\tilde{x}+1} = u^{x^{p-1}} \cdots u^x u = x^{-p}(xu)^p = 1$$

where $u^{\tilde{x}^s} = u^{x^s} = x^{-s}ux^s$ and $(\tilde{x}-1)^{p-1} = \tilde{x}^{p-1} + \cdots + \tilde{x} + 1$ is the sum of endomorphisms $\tilde{x}^{p-1}, \cdots, \tilde{x}, 1$ of $U$. Hence Hall Higman's theorem B yields our results.

(c) We can easily see that $n = 2$ since $U$ is metacyclic, and so $p \leq 3$ follows from the inequality $p-1 \leq n = 2$. Assume $G/U$ acts reducibly on $U$, namely, $G/U$ is a group consisting of triangular matrices of degree 2. Then $P/U$ is normal in $G/U$ contrary to the assumption. Thus $G$ acts irreducibly on $U$. Since $N_U(V)$ is normal in $N_G(V)U = G$ and $U$ is a minimal normal subgroup, we have $N_U(V) = 1$ by $F = 1$ and so $G$ is a semidirect product of $U$ by $N_G(V)$. Thus we obtain $G = 2^2 \cdot GL(2, 2)$ if $p = 2$. We may consider only the case $p = 3$. Since $N_G(V)$ is a subgroup of $GL(2, 3)$ and its order has a divisor 24 by (b), we can see $N_G(V) = SL(2, 3)$ or $GL(2, 3)$. This proves our result.

**Remark.** For every Fermat prime $p$, there exists an example of a $p$-solvable group $G$ of $p$-length 2 with a regular $p$-Sylow subgroup of order $p^p$ such that a 2-Sylow subgroup of $G/O_{p'}(G)$ is nonabelian (see [2, Theorem 3.5.3]). It seems to be the ultimate example of for our theorem.

## REFERENCES


[1] H. FUKUSHIMA: On groups $G$ of $p$-length 2 whose nilpotency indices of $J(KG)$ are $a(p-1)+1$, Hokkaido Math. J. **20** (1991), 523—530.

[2] P. HALL and G. HIGMAN: On the $p$-length of $p$-soluble groups and reduction theorems for Burnside's problem, Proc. London Math. Soc. **6** (1956), 1—42.

[3] B. HUPPERT: Endliche Gruppen I, Springer Verlag, 1967.

[4] B. HUPPERT and N. BLACKBURN: Finite Ggroups II, Springer Verlag, 1982.

[5] S. KOSHITANI: A remark on the nilpotency index of the radical of a group algebra of a $p$-solvable group, Proc. Edinburgh Math. Soc. **25** (1982), 31—34.

[6] S. KOSHITANI: On the Jacobson radical of a block ideal in a finite $p$-solvable group for $p \geq 5$, J. Algebra **80** (1983), 134—144.

[7] K. MORITA: On group rings over a modular field possess radicals expressible as principal ideals, Sci. Rep. Tokyo Bunrika Daigaku **4** (1951), 177—194.

[8] K. MOTOSE: On the nilpotency index of the radical of a group algebra. II, Math. J. Okayama Univ. **22** (1980), 141—143.

[9] K. MOTOSE: On the nilpotency index of the radical of a group algebra. III, J. London Math. Soc. **25** (1982), 39—42.

[10] K. MOTOSE: On the nilpotency index of the radical of a group algebra. IV, Math. J. Okayama Univ. **25** (1983), 35—42.

[11] K. MOTOSE: On a result of S. Koshitani, Edinburgh Math. Soc. **27** (1984), 57.

[12] K. MOTOSE: On a theorem of Y. Tsushima, Math. J. Okayama Univ. **26** (1984), 11—12.

[13] K. MOTOSE: On the nilpotency index of the radical of a group algebra. VIII, Proc. Amer. Math. Soc. **92** (1984), 327—328.

[14] K. MOTOSE: On the nilpotency index of the radical of a group algebra. IX, Mathematics Studies 126 (Group and Semigroup Rings), North-Holland, (1986), 193—196.

[15] K. MOTOSE: On Loewy series of group algebras of some solvable groups II, Sci. Rep. Hirosaki Univ. **36** (1989), 1—8.

[16] K. MOTOSE: On Loewy series of group algebras of some solvable groups, J. Algebra **130** (1990), 261—272.

[17] D. S. PASSMAN: Permutation Groups, Benjaminn, 1968.

[18] Y. TSUSHIMA: Some notes on the radical of a finite group ring II, Osaka J. Math. **16** (1979), 35—38.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HIROSAKI UNIVERSITY
HIROSAKI 036, JAPAN

(*Received December 20, 1991*)