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FINITE POSETS P AND P-GALOIS EXTENSIONS OF RINGS

Dedicated to Professor Takasi Nagahara on his 60th birthday

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0. Introduction. Let B be a ring with an identity 1. A a subring of B with common identity 1 of B and G a finite group of A-automorphisms of B. B/A is called a separable extension if the epimorphism $\mu: B\otimes_A B \Longrightarrow B$; $\mu(b\otimes c) = bc$ splits as a $B-B$-homomorphism (see [5]). B/A is called a G-Galois extension if (1) $B^G = \{b \in B\ ;\ \Lambda(b) = b \text{ for all } \Lambda \in G\} = A$ and (2) B_A is a finitely generated projective module and End(B_A) is ring isomorphic to a trivial crossed product, $D(B, G) = \sum_{\Lambda\in G}\oplus Bu_\Lambda$, of G over B (see [2]).

A separable extension is closely related to a G-Galois extension. Indeed if B/A is a G-Galois extension then B/A is a separable extension, and if B is a commutative separable extension of A such that $B^G = A$ and G is strongly distinct, then B/A is a G-Galois extension. For this reason, for a finite group G of automorphisms, we call a G-Galois extension is a Galois extension of separable type in this paper. On the other hand, there are various kind of works about constant subrings which correspond to (purely) inseparable cases of fields. For a subset P of End(B_A), $B^P = \{b \in B\ ;\ \Omega(b) = 0$ for all $\Omega \in P$ such that Ω is not a ring automorphism$\}\cap\{b \in B\ ;\ \Lambda(b) = b$ for all $\Lambda \in P$ such that Λ is a ring automorphism$\}$ is called a constant subring of B if B^P forms a subring. For example, if $P = \{d_0 = 1, d_1, \cdots, d_m, \cdots\}$ is a higher derivation of B (see [4]), then B^P is a subring which contains 1. We say B/A is a P-Galois extension if

(1) $B^P = A$ is a constant subring,

(2) B_A is a finitely generated projective module and End(B_A) is ring isomorphic to a trivial crossed product, $D(B, P)$, of P over B which is defined in §2.

In this paper, we consider a finite partially ordered set (= poset) P of End (B_A) which is called a relative sequence of homomorphisms. As will be seen in §1, P is able to contain a finite group of automorphisms, a set of derivations and a set of higher derivations etc. In §2, we shall construct a ring $D(B, P)$ which is a free left (as well as right) B-module with a B-basis $\{u_\Omega\ ;\ \Omega \in P\}$. This ring corresponds to the trivial crossed product $D(B, G)$ in the case of a G-Galois extension B/A and plays an important role in the theory of P-Galois extension. In §3, we shall define a P-Galois extension and study some properties of P-Galois extensions. As is remarked above, one can choose a finite group of automorphisms and a set of derivations (resp. higher derivations) as P. Thus the notion of a P-Galois extension is a generalized notion of a Galois extension of separable type and inseparable type. In §4, we shall study P-Galois extensions B/A when a relative sequence of homomorphisms P satisfies some additional conditions. In §5, we shall study a P-Galois extension B/A such that $B_A \oplus > A_A$, that is, A_A is a direct summand of B_A. Finally in §6, we shall treat of the case of P-Galois extensions of algebras over a commutative ring A.

General constructive studies of G-Galois extensions of inseparable types will be seen in forthcoming paper of the author.

1. A finite poset of End(B_A). Let a subset $P = \{\Omega_1, \Omega_2, \cdots, \Omega_n\}$ of End (B_A) be a poset with the order $\leq$. A minimal (resp. maximal) element of P means a minimal (resp. maximal) element of P with respect to the order. By $P(min)$ (resp. $P(max)$), we donote the set of all minimal (resp. maximal) elements of P. $\Lambda \in P(min)$ (resp. $\Lambda \in P(max)$) is said to be a minimal (resp. maximal) element of Ω_j if $\Lambda < \Omega_j$ (resp. $\Lambda > \Omega_j$). Ω_j is said to be a cover of Ω_i if $\Omega_j > \Omega_i$ and there is no Ω_k such that $\Omega_j > \Omega_k > \Omega_i$. If Ω_j is a cover of Ω_i, we denote it by $\Omega_j \gg \Omega_i$. For Ω_j, a chain of Ω_j means a descending chain

$$\Omega_j = \Omega_{j_0} \gg \Omega_{j_1} \gg \cdots \gg \Omega_{j_m}$$

where Ω_{j_m} is a minimal element of Ω_j, and in this case, $m+1$ is said to be the length of this chain. The reader can find relevent notations of the poset in [1].

For a finite poset P of End(B_A), we shall give the notion of a relative sequence of homomorphisms (abbreviate a r.s.h).

We state following conditions (A.1)-(A.6) and (B.1)-(B.4).

(A.1) $\Omega \neq 0$ for all $\Omega \in P$ and $P(min)$ coincides with all $\Lambda \in P$ such that Λ is a ring automorphism.

(A.2) Any two chain of Ω have the same length.

By $ht(\Omega)$ we denote the length of the chain of Ω.

(A.3) For $\Omega, \Gamma \in P$ if $\Omega\Gamma \neq 0$ then $\Omega\Gamma \in P$ and if $\Omega\Gamma = 0$ then $\Gamma\Omega = 0$.

(A.4) For $\Omega, \Gamma_1, \Gamma_2 \in P$, assume $\Omega\Gamma_1 \in P$ and $\Omega\Gamma_2 \in P$.

(i) $\Omega\Gamma_1 \geq \Omega\Gamma_2$ (resp. $\Gamma_1\Omega \geq \Gamma_2\Omega$) if and only if $\Gamma_1 \geq \Gamma_2$.

(ii) $\Omega\Gamma \geq \Lambda$ if and only if $\Lambda = \Omega_0\Gamma_0$ for some $\Omega_0 \leq \Omega$ and $\Gamma_0 \leq \Gamma$ where $\Omega_0, \Gamma_0 \in P$.

(A.5) $|P(min)| = |P(max)|$, where $|*|$ means the cardinality of the set $*$.

If $\Omega = \Lambda\Gamma$, Λ (resp. Γ) is said to be a left (resp. right) factor of Ω and Γ (resp. Λ) is denoted by $(\Omega/\Lambda)_l$ (resp. $(\Omega/\Gamma)_r$). $(\Omega/\Lambda)_l$ (resp. $(\Omega/\Gamma)_r$) is determined uniquely by (A.4), (i).

(A.6) For any $\Delta \in P(max)$, if $\Omega \leq \Delta$ then Ω is a left (as well as right) factor of Δ.

Remark. If P satisfies conditions (A.1)-(A.4), then $P(min)$ forms a group since it is a finite semigroup which is contained in the group of automorphisms of B.

(B.1) $\Omega(1) = 0$ for all $\Omega \in P - P(min)$.

(B.2) For Ω, there exist $g(\Omega, \Gamma) \in \mathrm{End}(B_A)$ for all Γ such that $g(\Omega, \Gamma) = 0$ if $\Gamma \nleq \Omega$ and

$$\Omega(xy) = \textstyle\sum_{\Gamma \in P} g(\Omega, \Gamma)(x)\Gamma(y) \quad \text{for } x, y \text{ in } B$$

Since $g(\Omega, \Gamma) = 0$ for $\Gamma \nleq \Omega$, we have

(B.2′) $\Omega(xy) = \sum_{\Gamma \leq \Omega} g(\Omega, \Gamma)(x)\Gamma(y)$

where $\sum_{\Gamma \leq \Omega}$ means the sum of all Γ such that $\Gamma \leq \Omega$.

The formulation of (B.2′) is more essential than that of (B.2) and we use the formulation (B.2′) in the rest of this paper when this causes no confusion.

(B.3) (i) $g(\Omega, \Lambda)(xy) = \sum_{\Lambda \leq \Gamma \leq \Omega} g(\Omega, \Gamma)(x)g(\Gamma, \Lambda)(y)$

for $x, y \in B$ where $\sum_{\Lambda \leq \Gamma \leq \Omega}$ means the sum of all Γ such that $\Lambda \leq \Gamma \leq \Omega$.

(ii) Let $\Omega, \Lambda, \Gamma \in P$ and $\Omega\Lambda \geq \Gamma$. Then

$$g(\Omega\Lambda, \Gamma)(x) = \textstyle\sum_{\Omega' \leq \Omega, \Lambda' \leq \Lambda, \Omega'\Lambda' = \Gamma} g(\Omega, \Omega')g(\Lambda, \Lambda')(x)$$

for $x \in B$, where $\sum_{\Omega' \leq \Omega, \Lambda' \leq \Lambda, \Omega'\Lambda' = \Gamma}$ means the sum of all $g(\Omega, \Omega')g(\Lambda, \Lambda')$ such that $\Omega' \leq \Omega$, $\Lambda' \leq \Lambda$ and $\Omega'\Lambda' = \Gamma$.

(B.4) (i) $g(\Omega, \Omega)$ is a ring automorphism for each Ω.

(ii) $g(\Omega, \Lambda) = \Omega$ for all minimal element Λ ($\in P(min)$) of Ω.

(iii) $g(\Omega, \Gamma)(1) = 0$ for $\Gamma < \Omega$.

P is said to be a r.s.h if it satisfies (A.1)-(A.4) and (B.1)-(B.4).

For the convenience of readers, we shall state an example of a r.s.h.

Let D be an A-derivation of B such that $D^n = 0$ and $D^i \neq 0$ for $0 \leq i \leq n-1$. Then $\boldsymbol{D} = \{D^0 = 1, D, D^2, \cdots, D^{n-1}\}$ becomes a poset whose order $D^i \geq D^j$ is defined by $i \geq j$.

We can easily see that $\boldsymbol{D}$ satisfies the conditions (A.1)-(A.4).

Since $P(min)=\{1\}$ and $P(max)=\{D^{n-1}\}$ in $\boldsymbol{D}$, $\boldsymbol{D}$ satisfies (A.5)-(A.6).
For $D^i \in \boldsymbol{D}$ and $x, y \in B$

$$D^i(xy)=\sum_{j=0}^{i}\binom{i}{j}D^{i-j}(x)D^j(y).$$

Hence, if we put $g(D^i, D^j):=\binom{i}{j}D^{i-j}$ where we put $\binom{i}{j}=0$ for $j>i$, then $g(D^i, D^j) \in \operatorname{End}(B_A)$ and

$$D^i(xy)=\sum_{D^j \leq D^i} g(D^i, D^j)(x)D^j(y).$$

Thus $\boldsymbol{D}$ satisfies (B.2).

$$\begin{aligned} g(D^i, D^j)(xy) &= \binom{i}{j}D^{i-j}(xy)=\binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}D^{i-j-k}(x)D^k(y) \\ &= \sum_{k=0}^{i-j}\binom{i}{j+k}D^{i-j-k}(x)\binom{j+k}{k}D^k(y)=\sum_{k=0}^{i-j} g(D^i, D^{j+k})(x)g(D^{j+k}, D^j)(y). \end{aligned}$$

For $D^iD^j \geq D^k$

$$\begin{aligned} g(D^iD^j, D^k)(x) &= \binom{i+j}{k}D^{i+j-k}(x)=\sum_{s+t=k, 0\leq s\leq i, 0\leq t\leq j}\binom{i}{s}D^{i-s}\binom{j}{t}D^{j-t}(x) \\ &= \sum_{s+t=k, 0\leq s\leq i, 0\leq t\leq j} g(D^i, D^s)g(D^j, D^t)(x) \end{aligned}$$

shows that $\boldsymbol{D}$ satisfies (B.3).

We can easily see that $\boldsymbol{D}$ satisfies (B.4).
Thus $\boldsymbol{D}$ is an example of a r.s.h P.

Let B be of prime characteristic p and $\partial=\{d_0=1, d_1, \cdots, d_{p^e-1}\}$ ($\subseteq$ End (B_A)) a higher derivation of rank p^e of B (see [4]). Then $P=\{(d_1)^{i_0}(d_p)^{i_1}\cdots(d_{p^{e-1}})^{i_{e-1}};\ 0\leq i_j\leq p-1\}$ becomes a post whose order

$$(d_1)^{i_0}(d_p)^{i_1}\cdots(d_{p^{e-1}})^{i_{e-1}} \geq (d_1)^{j_0}(d_p)^{j_1}\cdots(d_{p^{e-1}})^{j_{e-1}}$$

is defined by

$$\sum_{s=k}^{e-1} p^s\cdot i_s \geq \sum_{s=k}^{e-1} p^s\cdot j_s$$

for each $k=0, 1, \cdots, e-1$ (see [6]). Further we can see that P satisfies (A.1) -(A.6) and (B.1)-(B.4).

We will study P-Galois extensions with these examples of posets P in mind. In the rest of this paper, we assume that P is a r.s.h.

Let $P(s):=\{\Gamma \in P;\ ht(\Gamma)\leq s\}$. Then $P(1)=P(min)$. Further we have the following

Lemma 1.1. (1) *$\Lambda P(s) = P(s)\Lambda = P(s)$ for any $\Lambda \in P(1)$, where $\Lambda P(s)$ (resp. $P(s)\Lambda$) means $\{\Lambda\Gamma ; \Gamma \in P(s)\}$ (resp. $\{\Gamma\Lambda ; \Gamma \in P(s)\}$).*

(2) *If $\Omega \notin P(1)$ then $\Omega\Delta_j = 0$ for any $\Delta_j \in P(max)$.*

Proof. (1) If $\Lambda \in P(1)$ then $\Lambda\Gamma, \Gamma\Lambda \neq 0$ are clear for all $\Gamma \in P(s)$ since Λ is an isomorphism. Further it is easy to see that $ht(\Lambda\Gamma) = ht(\Gamma\Lambda) = ht(\Gamma)$. This yields that $P(s) = \{\Lambda\Gamma ; \Gamma \in P(s)\} = \{\Gamma\Lambda ; \Gamma \in P(s)\}$.

(2) Suppose $\Omega\Delta_j \neq 0$. For a minimal element Λ of Ω, $\Omega\Delta_j > \Lambda\Delta_j$ by (A.4).(i). But this contradicts to the maximality of Δ_j since $\Lambda^{-1}\Omega\Delta_j > \Lambda^{-1}\Lambda\Delta_j = \Delta_j$ again by (A.4).(i).

Let m_Ω be the number of minimal elements of Ω. Then we have the following

Lemma 1.2. *$m_\Omega\Omega = \Omega$.*

Proof. For $x \in B$,

$$\Omega(x) = \Omega(x1) = \textstyle\sum_{\Lambda\leq\Omega} g(\Omega, \Lambda)(x)\Lambda(1)$$
$$= \textstyle\sum_{\Lambda\in P(1),\Lambda\leq\Omega} g(\Omega, \Lambda)(x)\Lambda(1) = m_\Omega\Omega(x)$$

since $g(\Omega, \Lambda) = \Omega$ for any minimal element Λ of Ω by (B.4).(ii).

Corollary 1.3. *Let $\Omega \in P$.*

(1) *If A is an algebra over a field of characteristic 0 then $m_\Omega = 1$.*

(2) *If A is an algebra over a field of prime characteristic $p > 0$ then $m_\Omega = 1$ (mod p).*

Proof. (1) is clear by Lemma 1.2.

(2) Since $(m_\Omega - 1)\Omega = 0$, we have $m_\Omega - 1 \equiv 0$ (mod p).

In the rest, we put $P(max) = \{\Delta_1, \Delta_2, \cdots, \Delta_k\}$, $P_i := \{\Omega \in P ; \Omega \leq \Delta_i\}$ and $H_i := P_i \cap P(1)$. For $\Lambda \in P(1)$, $\Lambda\Delta_1 = \Delta_j$ for some j, and in this case $\Lambda P_1 = \{\Lambda\Omega ; \Omega \in P_1\} = P_j$ and $\Lambda H_1 = \{\Lambda\Omega ; \Omega \in H_1\} = H_j$. If P satisfies (A.5) and $P(1) = \{\Lambda_1, \cdots, \Lambda_k\}$, then $P(max) = \{\Lambda_i\Delta_1 ; i = 1, 2, \cdots, k\} = \{\Delta_1, \Delta_2, \cdots, \Delta_k\}$.

A finite poset S is said to be a pure poset if each maximal element has the same length.

Lemma 1.4. *Assume P satisfies* (A.5). *Then*

(1) *P is a pure poset.*

(2) *$P=\bigcup_{i=1}^{k}P_i$ and P_i is isomorphic to P_j as a poset for $i, j=1, 2, \cdots, k$.*

(3) *Assume $H_1 \ni 1$, Then H_1 is subgroup of $P(1)$ if and only if H_i is a subgroup of $P(1)$ for all H_i such that $H_i \ni 1$. Moreover, if this is the case $H_i=H_1$.*

(4) *If $H_1=\{1\}$, then $m_{\Delta_1}=1$, P_i is a sublattice of P and $P_i \cap P_j=\phi$ for $i \neq j$ and $i, j=1, 2, \cdots, k$.*

Proof. (1) Since $P(max)=\{\Lambda_i\Delta_1\, ;\ i=1, 2, \cdots, k\}$, $ht(\Lambda_i\Delta_1)=ht(\Delta_1)$.

(2) $P=\bigcup_{i=1}^{k} P_i$ is clear. The relation between P_i and P_j is given by $\Lambda P_i = P_j$ for some $\Lambda \in P(1)$. Hence $f_\Lambda : P_i \Rightarrow P_j\, ;\ \Omega \mapsto \Lambda\Omega$ gives an isomorphism.

(3) H_i is obtained by $\Lambda_i H_1$ for some $\Lambda_i \in P(1)$. Assume H_1 forms a subgroup of $P(1)$ and $1 \in H_i=\Lambda_i H_1$. Then Λ_i must be in H_1, and hence, $H_i = \Lambda_i H_1=H_1$. The converse is clear.

(4) Assume $H_1=\{1\}$. Since each P_i is obtained by $\Lambda_i P_1$ for some $\Lambda_i \in P(1)$, $H_i=\Lambda_i H_1=\{\Lambda_i\}$ shows that $m_{\Delta_i}=1$ and thus P_i is a lattice with the join Δ_i and the meet Λ_i. If $i \neq j$ and $H_i \cap H_j=\phi$ and hence $P_i \cap P_j=\phi$.

For a poset P, rank P is the maximal length of maximal elements of P. Then we can see that if P is a pure poset and

(1) rank $P=1$, then P is a finite group of automorphisms,

(2) rank $P \geq 2$, then for each $\Omega \in P$ with $ht(\Omega)=2$,

$$\begin{aligned}\Omega(xy) &= \textstyle\sum_{\Lambda \leq \Omega, \Lambda \in P(1)} g(\Omega, \Lambda)(x)\Lambda(y)+g(\Omega, \Omega)(x)\Omega(y)\\ &= \Omega(x)(\textstyle\sum_\Lambda \Lambda(y))+g(\Omega, \Omega)(x)\Omega(y)\end{aligned}$$

shows that Ω is a $(g(\Omega, \Omega), \sum_\Lambda \Lambda)$-derivation of B. In particular, if $\sum_\Lambda \Lambda=1$ and $g(\Omega, \Omega)=1$ then Ω is a derivation of B.

2. The trivial crossed product of P over B. In this section we shall define a ring $D(B, P)$ which is generated by elements $\{u_\Omega\, ;\ \Omega \in P\}$ over B and shall study the relationship between $D(B, P)$ and $\mathrm{End}(B_A)$.

Let $D(B, P)=\sum_{\Omega \in P} \oplus B u_\Omega$ be a free left B-module with a B-basis $\{u_\Omega\, ;\ \Omega \in P\}$. Then $D(B, P)$ becomes a right B-module via

$$u_\Omega \cdot b=\sum_{\Lambda \leq \Omega} g(\Omega, \Lambda)(b) u_\Lambda.$$

For,

$$u_\Omega \cdot (bc)=\textstyle\sum_{\Lambda \leq \Omega} g(\Omega, \Lambda)(bc) u_\Lambda$$

$$= \Sigma_{\Lambda}(\Sigma_{\Gamma\leq\Omega}\, g(\Omega, \Gamma)(b)g(\Gamma, \Lambda)(c))u_{\Lambda} \quad \text{by (i) of (B.3), and}$$

$$\begin{aligned}(u_{\Omega}\cdot b)\cdot c &= (\Sigma_{\Gamma\leq\Omega}\, g(\Omega, \Gamma)(b)u_{\Gamma})c \\ &= \Sigma_{\Gamma\leq\Omega}\, g(\Omega, \Gamma)(b)(\Sigma_{\Lambda\leq\Gamma}\, g(\Gamma, \Lambda)(c))u_{\Lambda} \\ &= \Sigma_{\Lambda}(\Sigma_{\Lambda\leq\Gamma}\, g(\Omega, \Gamma)(b)g(\Gamma, \Lambda)(c))u_{\Lambda}.\end{aligned}$$

Since $u_{\Omega}\cdot(b+c) = u_{\Omega}\cdot b + u_{\Omega}\cdot c$ is clear, the above shows that $D(B, P)$ is a right B-module.

Let $D' = \Sigma_{\Omega\in P}\, u_{\Omega}\cdot B$ be a right B-submodule of $D(B, P)$. Then we can obtain the following

Theorem 2.1. *D' coincides with $D(B, P)$ and $\{u_{\Omega}\,;\ \Omega\in P\}$ is a right B-basis of D'.*

Proof. Let $\Lambda\in P(1)$ and let b an element of B. Then, $D'\ni u_{\Lambda}b = g(\Lambda, \Lambda)(b)u_{\Lambda}$ yields $bu_{\Lambda}\in D'$ since $g(\Lambda, \Lambda) = \Lambda$ is an isomorphism. Assume now $bu_{\Gamma}\in D'$ for any $\Gamma\in P(s)$. If Ω is a cover of $\Gamma\in P(s)$, we have

$$u_{\Omega}b = g(\Omega, \Omega)(b)u_{\Omega} + \Sigma_{\Omega'<\Omega}\, g(\Omega, \Omega')(b)u_{\Omega'}$$

where the sum $\Sigma_{\Omega'<\Omega}$ runs over all $\Omega'\in P(s)$ with $\Omega' < \Omega$ since $\Omega\in P(s+1)$. Hence, each $g(\Omega, \Omega')(b)u_{\Omega'}\in D'$ by induction hypothesis. Consequently we have $bu_{\Omega}\in D'$. Thus $D' = D(B, P)$.

Assume now $\alpha = \Sigma_{\Gamma\in P}\, u_{\Gamma}b_{\Gamma} = 0$ $(b_{\Gamma}\in B)$. Since $\alpha\in\Sigma_{\Omega\in P}\oplus Bu_{\Omega}$, we can write $\alpha = \Sigma c_{\Omega}u_{\Omega}$ for some $c_{\Omega}\in B$ and $c_{\Omega} = 0$ for all Ω. Let $\Delta\in P(max)$. Then $b_{\Delta} = g(\Delta, \Delta)^{-1}(c_{\Delta}) = 0$. Next let $Q_1 = P - P(max)$ and $Q_{i+1} = Q_i - Q_i(max)$ for $i = 1, 2, \cdots, k$. Assume now $b_{\Omega} = 0$ for all $\Omega\in Q_s(max)$ for $s = 1, 2, \cdots, t$. Then, $b_{\Gamma} = g(\Gamma, \Gamma)^{-1}(c_{\Gamma}) = 0$ for an arbitrary $\Gamma\in Q_{t+1}(max)$. Thus $\{u_{\Omega}\,;\ \Omega\in P\}$ is right linearly independent over B.

Theorem 2.2. *$D(B, P)$ becomes a ring under the multiplication defined by*

$$(au_{\Lambda})(bu_{\Gamma}) = \Sigma_{\Lambda'\leq\Lambda}\, ag(\Lambda, \Lambda')(b)u_{\Lambda'\Gamma}$$

where $u_{\Lambda'\Gamma} = 0$ if $\Lambda'\Gamma = 0$.

Proof. It suffices to show that $(u_{\Omega}au_{\Lambda})b = u_{\Omega}(au_{\Lambda}b)$. Let $(u_{\Omega}au_{\Lambda})b = \Sigma_{\Gamma}\, c_{\Gamma}u_{\Gamma}$ and $u_{\Omega}(au_{\Lambda}b) = \Sigma_{\Gamma}\, d_{\Gamma}u_{\Gamma}$ for $c_{\Gamma}, d_{\Gamma}\in B$. Then

$$\begin{aligned}(u_{\Omega}au_{\Lambda})b &= \Sigma_{\Omega''\leq\Omega,\Omega''\Lambda\neq 0}\, g(\Omega, \Omega'')(a)u_{\Omega''\Lambda}b \\ &= \Sigma_{\Omega''\leq\Omega,\Omega''\Lambda\neq 0}\, g(\Omega, \Omega'')(a)(\Sigma_{\Gamma'\leq\Omega''\Lambda}\, g(\Omega'', \Gamma')(b)u_{\Gamma'})\end{aligned}$$

Hence, for a fixed Ω'' such that $\Omega''\Lambda\geq\Gamma$, the coefficient of u_{Γ} is

$$g(\Omega, \Omega'')(a)g(\Omega''\Lambda, \Gamma)(b)$$

and hence,

$$c_\Gamma = \sum_{\Omega''\leq\Omega, \Omega''\Lambda\geq\Gamma} g(\Omega, \Omega'')(a)g(\Omega''\Lambda, \Gamma)(b).$$

On the other hand,

$$\begin{aligned} u_\Omega(au_\Lambda b) &= u_\Omega(a\sum_{\Lambda'\leq\Lambda} g(\Lambda, \Lambda')(b)u_{\Lambda'}) \\ &= \sum_{\Lambda'\leq\Lambda}(\sum_{\Omega'\leq\Omega, \Omega'\Lambda'\neq 0} g(\Omega, \Omega')(ag(\Lambda, \Lambda')(b))u_{\Omega'\Lambda'}) \\ &= \sum_{\Lambda'\leq\Lambda, \Omega'\leq\Omega, \Omega'\Lambda'\neq 0}(\sum_{\Omega'\leq\Omega''\leq\Omega} g(\Omega, \Omega'')(a)g(\Omega'', \Omega')g(\Lambda, \Lambda')(b))u_{\Omega'\Lambda'}. \end{aligned}$$

Thus, for a fixed Ω'' such that $\Omega''\Lambda \geq \Gamma$, the coefficient of u_Γ is

$$\sum_{\Omega'\leq\Omega'', \Lambda'\leq\Lambda, \Omega'\Lambda'=\Gamma} g(\Omega, \Omega'')(a)(g(\Omega'', \Omega')(b)) = g(\Omega, \Omega'')(a)g(\Omega''\Lambda, \Gamma)(b)$$

by (B.3).(ii). Therefore d_Γ is also $\sum_{\Omega''\leq\Omega, \Omega''\Lambda\geq\Gamma} g(\Omega, \Omega'')(a)g(\Omega''\Lambda, \Gamma)(b)$.

Let j be the map of $D(B, P)$ to $\mathrm{End}(B_A)$ defined by

$$j(bu_\Omega): x \Rightarrow b\Omega(x).$$

Then j is a ring homomorphism. Indeed, $j(bu_\Lambda cu_\Gamma(x)) = b\sum_{\Lambda'\leq\Lambda} g(\Lambda, \Lambda')(c)\Lambda'\Gamma(x)$. While, $j(bu_\Lambda)j(cu_\Gamma)(x) = j(bu_\Lambda)(c\Gamma(x)) = b(\sum_{\Lambda'\leq\Lambda} g(\Lambda, \Lambda')(c)\Lambda'\Gamma(x))$. Since j is a ring homomorphism, $\mathrm{End}(B_A)$ can be regarded as a left $D(B, P)$-module via j.

3. A P-Galois extension and a P-Galois system. In this section we shall study P-Galois extensions for a r.s.h P. We put $P(max) = \{\Delta_1, \Delta_2, \cdots, \Delta_k\}$ and 1 is a minimal element of Δ_1.

We use following notations:

(i) $T = \sum_{\Lambda\in P(1)} \Lambda$.

(ii) $T\Delta_i = \sum_{\Lambda\in P(1)} \Lambda\Delta_i$.

For $P(max) \ni \Delta_i, \Delta_j$, if $\Delta_i = \Lambda\Delta_j$ for some $\Lambda \in P(1)$, we call Δ_i and Δ_j are similar. Then we may choose a set $N = \{\Delta_1, \Delta_2, \cdots, \Delta_h\}$ which consists of all non-similar elements of $P(max)$ for some $h \leq k$.

Lemma 3.1. *Assume Δ_m and Δ_n are elements of N.*

(1) *$\Lambda\Delta_m = \Lambda'\Delta_n$ for some $\Lambda, \Lambda' \in P(1)$ if and only if $\Lambda = \Lambda'$ and $m = n$.*

(2) *$P(max) = \{\Lambda\Delta_1, \Lambda\Delta_2, \cdots, \Lambda\Delta_h$; Λ runs over all elements of $P(1)\}$.*

(3) *If j is an isomorphism and $m \neq n$, then $T\Delta_m \neq T\Delta_n$.*

Proof. (1) If $\Lambda\Delta_m = \Lambda'\Delta_n$, then Δ_m and Δ_n are similar, and hence, $m = n$

since $\Delta_m, \Delta_n \in N$. Then $\Lambda = \Lambda'$ by (A.4).(i). The converse is clear.

(2) For distinct Δ_m and Δ_n of N, $\Lambda\Delta_m \neq \Lambda\Delta_n$ for any $\Lambda, \Lambda' \in P(1)$. Next, for any $\Delta_s \in P(max)$, Δ_s is similar to some $\Delta_i \in N$. Thus $P(max) = \{\Lambda\Delta_1, \Lambda\Delta_2, \cdots, \Lambda\Delta_h; \Lambda$ runs over all elements of $P(1)\}$.

(3) First we note that $\Lambda\Delta_m \neq \Lambda'\Delta_n$ for any $\Lambda, \Lambda' \in P(1)$ by (1). Hence $\{u_{\Lambda\Delta_m}, u_{\Lambda\Delta_n}; \Lambda$ runs over all elements of $P(1)\}$ is linearly independent over B, and hence $\sum_{\Lambda\in P(1)} u_{\Lambda\Delta_m} \neq \sum_{\Lambda\in P(1)} u_{\Lambda\Delta_n}$. This shows that $T\Delta_m = j(\sum_{\Lambda\in P(1)} u_{\Lambda\Delta_m}) \neq j(\sum_{\Lambda\in P(1)} u_{\Lambda\Delta_n}) = T\Delta_n$.

Remark. Since $|P(1)|h = (|P(max)|$, $|P(1)|$ is a divisor of $|P(max)|$.

Further we put as follows:

(iii) $\Delta = \sum_{i=1}^{h} T\Delta_i$ $(= \sum_{i=1}^{k} \Delta_i)$.

(iv) For $\Gamma \in P$, $g(T\Delta_i, \Gamma) = \sum_{\Lambda\in P(1)} g(\Lambda\Delta_i, \Gamma)$, where $g(\Lambda\Delta_i, \Gamma) = 0$ if $\Lambda\Delta_i$ is not a maximal element of Γ (Cf. (B.2')). Further, $g(\Delta, \Gamma) = \sum_{i=1}^{h} g(T\Delta_i, \Gamma)$.

(v) $B_1 = B^{P(1)} = \{b \in B; \Lambda(b) = b$ for all $\Lambda \in P(1)\}$.

(vi) $B_0 = \{b \in B; \Omega(b) = 0$ for all $\Omega \in P - P(1)\}$.

(vii) $B^P = B_1 \cap B_0$.

Since $\Lambda T = T$ for all $\Lambda \in P(1)$, we have

(1') $T(B) \subseteq B_1$.

By Lemma 1.1.(2), we have $\Omega\Delta_j = 0$ for any $\Omega \in P - P(1)$ and a maximal element Δ_j of P. Hence

(2') $\Delta_j(B) \subseteq B_0$.

In virtue of (1') and (2'), we have

(3') $\Delta(B) \subseteq B_0 \cap B_1$.

A subset S of P is called an ideal if $\Omega \in S$ and $\Gamma \leq \Omega$ then $\Gamma \in S$.

Lemma 3.2. *If S is an ideal of P then $B^S = \{b \in B; \Lambda(b) = b$ for all $\Lambda \in S \cap P(1)\} \cap \{b \in B; \Omega(b) = 0$ for all $\Omega \in S - P(1)\}$ is a subring of B which contains A.*

Proof. For $x, y \in B^S$, $x - y \in B^S$ is clear. For $\Omega \in S$, $\Omega(xy) = \sum_{\Gamma\leq\Omega} g(\Omega, \Gamma)(x)\Gamma(y) = \sum_{\Lambda\leq\Omega, \Lambda\in S(min)} g(\Omega, \Lambda)(x)\Lambda(y)$ and each $g(\Omega, \Lambda)(x) = \Omega(x) = 0$ by (B.4).(ii) if $\Omega \notin S(min)$. Thus B^S is a subring of B. $A \subseteq B^S$ is clear.

Definition 3.3. B/A is called a P-Galois extension if

(a) $B^P = A$

(b) B_A is a finitely generated projective module
(c) j is an isomorphism.

In the rest, we shall assume following additional conditions:
(i) P satisfies (A.6)
(ii) P is a pure poset.
Further, in the rest we denote u_Ω by Ω and $\sum_{\Lambda \in P(1)} u_{\Lambda\Delta_s}$, by $T\Delta_s$, when this causes no confusion.

Theorem 3.4. *Assume $B^P = A$ and j is an isomorphism. Then $j(\sum_{i=1}^h (T\Delta_i \cdot B)) = Hom(B_A, A_A) = B^*$, and A_A is a direct summand of B_A if and only if there exist $x_1, x_2, \cdots, x_h \in B$ such that $\sum_{i=1}^h T\Delta_i(x_i) = 1$.*

Proof. First we note that $B_0 \cap B_1 = A$ since $B^P = A$. If $P = P(1)$ (and hence $P = P(max)$) then P is a finite group of automorphisms of B and $\Delta = T$. Let $f \in B^*$. Then $f = j(\sum_{\Lambda \in P = P(1)} \Lambda b_\Lambda)$ for $b_\Lambda \in B$. Since $j(\Gamma)f = f$ for any $\Gamma \in P$, $\sum_{\Lambda \in P} \Lambda b_\Lambda = \sum_{\Lambda \in P} \Gamma\Lambda b_\Lambda$ yields $b_\Lambda = b_1$ for all $\Lambda \in P(1)$. (cf. [2]).

Assume now $P \neq P(1)$ and $\Omega \in P - P(1)$. Then we can easily see that $\Omega \cdot T\Delta_i = 0$ by Lemma 1.1.(2) and $\Lambda \cdot T\Delta_i = T\Delta_i$ for $\Lambda \in P(1)$. Then $j(\sum_{i=1}^h (T\Delta_i \cdot B)) \subseteq B^*$ by (3′). For $f \in B^*$, f is obtained by $j(V)$ for $V = \sum_{\Omega \in P} \Omega b_\Omega$ $(\in D(B, P))$ since j is an isomorphism. Then

$$j(\Gamma)f = j(\Gamma V) = \textstyle\sum_{\Lambda \in P(1)} j(\Gamma\Lambda) b_\Lambda + \sum_{\Omega \notin P(1)} j(\Gamma\Omega) b_\Omega$$
$$= \begin{cases} 0 & \text{if } \Gamma \notin P(1) \\ f & \text{if } \Gamma \in P(1). \end{cases} \qquad \cdots\cdots\cdots (*)$$

First we assert that $V = \sum_{i=1}^k \Delta_i \cdot b_{\Delta_i}$. For choosing Γ from $P - P(1)$, we can see $b_\Lambda = 0$ for all $\Lambda \in P(1)$ by (*) and the fact that $\Gamma\Lambda \neq \Gamma\Omega$ for $\Lambda \in P(1)$ and $\Omega \in P - P(1)$ (by (A.4).(i)). Hence we assume that $b_\Omega = 0$ for all Ω such that $ht(\Omega) \leq m < ht(\Delta_1)$. Let Γ be an arbitrary element of $ht(\Gamma) = m+1$. Then Γ is a cover of some Ω with the height m. Assume $\Gamma \notin P(max)$ and Δ_i is an maximal element of Γ. Then there exists $\Gamma_i \in P$ such that $\Delta_i = \Gamma_i\Gamma$ by (A.6). If $\Gamma_i \in P(1)$, then $ht(\Delta_i) = ht(\Gamma_i\Gamma) = ht(\Gamma)$ implies a contradiction $\Gamma \in P(max)$, since P is a pure poset. Thus $\Gamma_i \notin P(1)$, and hence.

$$0 = j(\Gamma_i)f = j(\Delta_i) b_\Gamma + \textstyle\sum_{\Omega \neq \Gamma, \Omega \notin P(1)} j(\Gamma_i\Omega) b_\Omega.$$

Noting that $\Gamma_i\Omega \neq \Delta_i$ for any $\Omega \neq \Gamma$, we have $b_\Gamma = 0$. Consequently we have

$$V = \textstyle\sum_{i=1}^k \Delta_i \cdot b_{\Delta_i} = \sum_{i=1}^h (\sum_{\Lambda \in P(1)} \Lambda\Delta_i \cdot b_{\Lambda\Delta_i}).$$

Since $\Lambda_0 V = V$, $\Lambda_0(\sum_{\Lambda \in P(1)} \Lambda\Delta_i \cdot b_{\Lambda\Delta_i}) = \sum_{\Lambda \in P(1)} \Lambda_0\Lambda\Delta_i \cdot b_{\Lambda\Delta_i} = \sum_{\Lambda \in P(1)}$

$\Lambda\Delta_i\cdot b_{\Lambda\Delta_i}$. Hence, for a fixed $\Lambda\in P(1)$, take $\Lambda_0=\Lambda^{-1}$. Then we have $b_{\Lambda\Delta_i}=b_{\Delta_i}$. Therefore $b_{\Lambda\Delta_i}=b_{\Delta_i}$ for all $\Lambda\in P(1)$. Thus

$$V=\textstyle\sum_{i=1}^{h}T\Delta_i\cdot b_{\Delta_i}\in\sum_{i=1}^{h}(T\Delta_i\cdot B).$$

Let $B_A\oplus>A_A$. Then the projection $\pi: B_A\Rightarrow A_A$ is obtained by $\sum_{i=1}^{h}T\Delta_i\cdot x_i$ for some $x_i\in B$ and so $1=(\sum_{i=1}^{h}T\Delta_i\cdot x_i)(1)=\sum_{i=1}^{h}T\Delta_i(x_i)$. Conversely, if there exist $x_1, x_2,\cdots,x_h\in B$ such that $\sum_{i=1}^{h}T\Delta_i(x_i)=1$, then $\varphi: b\Rightarrow\varphi(b)=\sum_{i=1}^{h}T\Delta_i(x_ib)$ is an epimorphism with $\varphi(a)=a$ for all $a\in A$. Thus $B=A\oplus\mathrm{Ker}\,\varphi$.

Theorem 3.5. *If B/A is a P-Galois extension, then there exists a system $\{x_i, y_{it};\ i=1,2,\cdots,s$ and $t=1,2,\cdots,h\}\subseteq B$ such that*

$$\sum_{i=1}^{s}x_i(\sum_{t=1}^{h}g(T\Delta_t,\Gamma)(y_{it}))=\delta_{1,\Gamma}$$

for all $\Gamma\in P$.
Moreover, if this is the case,

$$\sum_{i=1}^{s}\Omega(x_i)(\sum_{t=1}^{h}g(T\Delta_t,\Gamma)(y_{it}))=\delta_{\Omega,\Gamma}$$

for all $\Gamma\in P$.

Proof. Since B_A is finitely generated projective, there exists a projective coordinate system $\{x_i, f_i;\ i=1,2,\cdots,s, x_i\in B, f_i\in B^*\}$, and each f_i is obtained by $\sum_{t=1}^{h}T\Delta_t\cdot y_{it}$, $y_{it}\in B$, by Theorem 3.4. Namely,

$$\begin{aligned}D(B,P)\ni 1&=\textstyle\sum_{i=1}^{s}x_i(\sum_{t=1}^{h}T\Delta_t\cdot y_{it})\\&=\textstyle\sum_{i=1}^{s}x_i(\sum_{t=1}^{h}g(T\Delta_t,1)(y_{it}))\cdot 1\\&\quad+\textstyle\sum_{i=1}^{s}x_i\sum_{\Gamma\neq 1}(\sum_{t=1}^{h}g(T\Delta_t,\Gamma)(y_{it}))\Gamma.\end{aligned}$$

Therefore $\sum_{i=1}^{s}\sum_{t=1}^{h}x_ig(T\Delta_t,\Gamma)(y_{it})=\delta_{1,\Gamma}$.

For $\Omega\in P$,

$$\begin{aligned}\Omega=\Omega\cdot 1&=\Omega(\textstyle\sum_{i=1}^{s}x_i(\sum_{t=1}^{h}T\Delta_t\cdot y_{it}))\\&=\textstyle\sum_{\Gamma\leq\Omega}(\sum_{i=1}^{s}g(\Omega,\Gamma)(x_i)\sum_{t=1}^{h}\Gamma T\Delta_t\cdot y_{it})\\&=\textstyle\sum_{\Lambda\in P(1),\Lambda\leq\Omega}(\sum_{i=1}^{s}g(\Omega,\Lambda)(x_i)\sum_{t=1}^{h}\Lambda T\Delta_t\cdot y_{it})\\&=\textstyle\sum_{i=1}^{s}(m_\Omega\Omega(x_i)\sum_{t=1}^{h}T\Delta_t\cdot y_{it})\\&=\textstyle\sum_{i=1}^{s}(\Omega(x_i)\sum_{t=1}^{h}T\Delta_t\cdot y_{it})\quad\text{(by Lemma 1.2)}\\&=\textstyle\sum_{i=1}^{s}\Omega(x_i)\sum_{t=1}^{h}g(T\Delta_t,\Omega)(y_{it})\Omega\\&\quad+\textstyle\sum_{i=1}^{s}\Omega(x_i)\sum_{\Gamma\neq\Omega}\sum_{t=1}^{h}g(T\Delta_t,\Gamma)(y_{it})\Gamma.\end{aligned}$$

This implies

$$\sum_{i=1}^{s} \Omega(x_i)\sum_{t=1}^{h} g(T\Delta_t, \Gamma)(y_{it}) = \delta_{\Omega,\Gamma}.$$

Definition 3.6. Let $\Omega \in P$. For this fixed Ω, a system $\{x_i, y_{it}\,;\ i = 1, 2, \cdots, s \text{ and } t = 1, 2, \cdots, h\} \subseteq B$ is called a (P, Ω)-Galois system for B/A if it satisfies

$$\sum_{i=1}^{s} x_i(\sum_{t=1}^{h} g(T\Delta_t, \Gamma)(y_{it})) = \delta_{\Omega,\Gamma}$$

for any $\Gamma \in P$. In particular, a $(P, 1)$-Galois system for B/A is called a P-Galois system for B/A.

Let $\{x_i, y_{it}\,;\ i = 1, 2, \cdots, s$ and $t = 1, 2, \cdots, h\}$ be a P-Galois system for B/A. Then

$$\sum_{i=1}^{s} x_i(\sum_{t=1}^{h} g(T\Delta_t, 1)(y_{it})) = 1 \text{ and } \sum_{i=1}^{s} x_i(\sum_{t=1}^{h} g(T\Delta_t, \Omega)(y_{it})) = 0$$

for $\Omega \neq 1$.
Further, for $\Lambda_0 \in P(1)$,

$$g(T\Delta_t, \Lambda_0) = \sum_{\Lambda\Delta_t \geq \Lambda_0, \Lambda \in P(1)} g(\Lambda\Delta_t, \Lambda_0) = \sum_{\Lambda\Delta_t \geq \Lambda_0, \Lambda \in P(1)} \Lambda\Delta_t.$$

Hence we have

$$\begin{aligned} \sum_{i=1}^{s}(x_i(\sum_{t=1}^{h}\sum_{\Lambda\Delta_t \geq 1, \Lambda \in P(1)} \Lambda\Delta_t(y_{it})) &= 1 \\ \sum_{i=1}^{s}(x_i(\sum_{t=1}^{h}\sum_{\Lambda\Delta_t \geq \Lambda_0, \Lambda \in P(1)} \Lambda\Delta_t(y_{it})) &= 0 \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(**)$$

for all $\Lambda_0(\neq 1) \in P(1)$.
Thus we have the following

Corollary 3.7. *If B/A is a P-Galois extension, then $P(1) = \{1\}$ if and only if P_i contains $P(1)$, where $P_i = \{\Omega \in P\,;\ \Omega \leq \Delta_i\}$.*

Proof. Assume each P_i contains $P(1)$. If $P(1)$ contains $\Lambda(\neq 1)$, then $P(max) = \{\text{maximal elements of } 1\} = \{\text{maximal elements of } \Lambda\}$, and this contradicts to $(**)$. The converse is clear.

Lemma 3.8. *Let $\Lambda \in P(1)$. If B has a (P, Λ)-Galois system $\{x_i, y_{it}\,;\ i = 1, 2, \cdots, s$ and $t = 1, 2, \cdots, h\}$ for B/A, then*

(1) *$m_\Omega \cdot 1$ is a unit element of B.*

(2) *$\sum_{i=1}^{s} \Omega(x_i)(\sum_{t=1}^{h} g(T\Delta_t, \Gamma)(y_{it})) = \delta_{\Omega\Lambda,\Gamma}$ for any $\Omega \in P$.*

Proof. Since $\sum_{i=1}^{s} x_i(\sum_{t=1}^{h} T\Delta_t \cdot y_{it}) = \sum_{i=1}^{s}(x_i(\sum_{t=1}^{h} g(T\Delta_t, \Lambda)(y_{it}))\Lambda) = \Lambda$,

$$\begin{aligned}\Omega\Lambda &= \Omega\cdot(\textstyle\sum_{i=1}^{s} x_i(\sum_{t=1}^{h} T\Delta_t\cdot y_{it})\\ &= \textstyle\sum_{i=1}^{s}(\sum_{\Omega'\leq\Omega} g(\Omega,\Omega')(x_i)\sum_{t=1}^{h}\Omega' T\Delta_t\cdot y_{it})\\ &= \textstyle\sum_{i=1}^{s} m_\Omega \Omega(x_i)\sum_{t=1}^{h} T\Delta_t\cdot y_{it}\\ &= \textstyle\sum_{i=1}^{s}\Omega(x_i)\sum_{t=1}^{h} T\Delta_t y_{it} \quad \text{(by Lemma 1.2)}\\ &= \textstyle\sum_{i=1}^{s}\Omega(x_i)(\sum_\Gamma\sum_{t=1}^{h} g(T\Delta_t,\Gamma)(y_{it})\Gamma).\end{aligned}$$

Thus,

$$\begin{aligned}&1 = \textstyle\sum_{i=1}^{s}\Omega(x_i)(\sum_{t=1}^{h} g(T\Delta_t,\Omega\Lambda)(y_{it})) \quad \text{and}\\ &\textstyle\sum_{i=1}^{s}\Omega(x_i)\sum_{t=1}^{h} g(T\Delta_t,\Gamma)(y_{it}) = 0 \quad \text{for } \Gamma\neq\Omega\Lambda.\end{aligned}$$

The following theorem gives a characterization for B/A to be a P-Galois extension.

Theorem 3.9. *Let $B^P = A$. Then B/A is a P-Galois extension if and only if B has a P-Galois system $\{x_i, y_{it}\ ;\ i = 1, 2, \cdots, s$ and $t = 1, 2, \cdots, h\}$ for B/A.*

Proof. Assume B has a P-Galois system $\{x_i, y_{it}\ ;\ i = 1, 2, \cdots, s$ and $t = 1, 2, \cdots, h\}$. First we shall show that j is an isomorphism. For $f\in \mathrm{End}(B_A)$, we put $V = \sum_{i=1}^{s} f(x_i)\sum_{t=1}^{h} T\Delta_t\cdot y_{it}$ $(\in D(B, P))$.
Then, for $b\in B$,

$$\begin{aligned}j(V)(b) &= \textstyle\sum_{i=1}^{s} f(x_i)\sum_{t=1}^{h} T\Delta_t(y_{it}b) = f(\sum_{i=1}^{s} x_i\sum_{t=1}^{h} T\Delta_t(y_{it}b))\\ &= f(\textstyle\sum_{\Omega\in P}(\sum_{i=1}^{s} x_i\sum_{t=1}^{h} g(T\Delta_t,\Omega)(y_{it}))\Omega(b))\\ &\qquad\qquad\qquad\qquad (\text{since } \textstyle\sum_{t=1}^{h} T\Delta_t(y_{it})\in A)\\ &= f(\textstyle\sum_{t=1}^{h}\sum_{i=1}^{s} x_i g(T\Delta_i, 1)(y_{it})b) = f(b)\end{aligned}$$

shows that j is an epimorphism. Next we shall show that j is a monomorphism.

$$\begin{aligned}&b(\textstyle\sum_{i=1}^{s} j(\Omega)(x_i)\sum_{t=1}^{h} T\Delta_t\cdot y_{it}) = b(\sum_{i=1}^{s}\Omega(x_i)\sum_\Gamma\sum_{t=1}^{h} g(T\Delta_t,\Gamma)(y_{it})\Gamma)\\ &\quad = b(\textstyle\sum_{i=1}^{s}\Omega(x_i)\sum_{t=1}^{h} g(T\Delta_t,\Omega)(y_{it})\Omega) = b\Omega\end{aligned}$$

by Lemma 3.8. Let $W = \sum_{\Omega\in P} b_\Omega\Omega$ be an arbitrary element of $D(B, P)$. Then

$$W = \textstyle\sum_{\Omega\in P}(\sum_{i=1}^{s} b_\Omega j(\Omega)(x_i)\sum_{t=1}^{h} T\Delta_t\cdot y_{it}) = \sum_{i=1}^{s}(jW)(x_i)\sum_{t=1}^{h} T\Delta_t\cdot y_{it})$$

yields that $W = 0$ if $j(W) = 0$. Since $\{x_i, \sum_{t=1}^{h} T\Delta_t\cdot y_{it}\ ;\ i = 1, 2, \cdots, s$ and $t = 1, 2, \cdots, h\}$ is a projective coordinate system for B/A, B_A is finitely generated projective. The converse is proved in Theorem 3.5.

Let P satisfy also (A.5). Thus P is a r.s.h with (A.5) and (A.6), $P(max) = \{\Lambda\Delta_1\ ;\ \Lambda\in P(1)\}$ and $\Delta = T\Delta_1 = \Delta_1 T$. Applying theorems 3.4-3.9, we have the following simpler formulation in this case.

Corollary 3.10. *Let $B^P = A$. Then B/A is a P-Galois extension if and only if there exists a P-Galois system $\{x_i, y_i\,;\ i = 1, 2, \cdots, s\} \subseteq B$ for B/A (i.e., $\sum_{i=1}^{s} g(\Delta, \Gamma)(y_i) = \delta_{1,\Gamma}$). Moreover if this is the case, A_A is a direct summand of B_A if and only if there exists an element $x \in B$ such that $\Delta(x) = 1$.*

Let P be a r.s.h with $\Delta = \sum_{t=1}^{h} T\Delta_t$ again, and let Φ_t be the map from $B\otimes_A B$ to $D(B, P)$ defined by $\Phi_t(b\otimes c) = bT\Delta_t\cdot c$ for each $t = 1, 2, \cdots, h$. Then Φ_t is a $D(B, A)-B$-homomorphism, where the $D(B, P)$-module structure of $B\otimes_A B$ is defined by $d\Omega(b\otimes c) = d\Omega(b)\otimes c$. For, $\Phi_t(d\Omega(b)\otimes c) = d\Omega(b)T\Delta_t\cdot c$ and $d\Omega\Phi_t(b\otimes c) = d\Omega\cdot bT\Delta_t\cdot c = d(\sum_{\Gamma\leq\Omega} g(\Omega, \Gamma)(b)\Gamma T\Delta_t\cdot c)$ $= dm_\Omega\Omega(b)T\Delta_t\cdot c = d\Omega(b)T\Delta_t\cdot c$ since $\Gamma T\Delta_t = 0$ if $\Gamma \notin P(1)$ by Lemma 1.1.(2)

Theorem 3.11. *If B/A is a P-Galois extension then $\Phi(B\otimes_A B) = D(B, P)$, where $\Phi = \sum_{t=1}^{h}\Phi_t$. In particular, if $h = 1$, that is, $\Delta = T\Delta_1$, then $\Phi = \Phi_1$ is an isomorphism.*

Proof. Let $\{x_i, y_{it}\,;\ i = 1, 2, \cdots, s$ and $t = 1, 2, \cdots, h\}$ be a P-Galois system for B/A. For $\Omega\cdot b \in D(B, P)$, we shall show that there exist $\alpha_1, \alpha_2, \cdots, \alpha_h \in B\otimes_A B$ such that $\sum_{t=1}^{h}\Phi_t(\alpha_t) = \Omega\cdot b$. Now,

$$\begin{aligned}
\textstyle\sum_{t=1}^{h}\Phi_t(\sum_{i=1}^{s} \Omega(x_i)\otimes y_{it})b &= \textstyle\sum_{t=1}^{h}(\sum_{i=1}^{s} \Omega(x_i)T\Delta_t\cdot y_{it})b \\
&= \textstyle\sum_{i=1}^{s} \Omega(x_i)(\sum_{t=1}^{h} T\Delta_t\cdot y_{it})b \\
&= \textstyle\sum_{i=1}^{s} \Omega(x_i)\sum_{t=1}^{h}(\sum_{\Gamma\in P} g(T\Delta_t, \Gamma)(y_{it})\Gamma)b = \Omega\cdot b
\end{aligned}$$

since $\sum_{i=1}^{s} \Omega(x_i)\sum_{t=1}^{h} g(T\Delta_t, \Gamma)(y_{it}) = \delta_{\Omega,\Gamma}$ by Lemma 3.8. This means that Φ is an epimorphism.

Assume now $\Delta = T\Delta_1$. Then we already know that Φ_1 is an epimorphism. If $0 = \Phi_1(b\otimes c) = b\Delta c = b(\sum_{\Omega\in P} g(T\Delta_1, \Omega)(c)\Omega)$, then $bg(T\Delta_1, \Omega)(c) = 0$ for all $\Omega \in P$. Consequently we have

$$\begin{aligned}
&b\textstyle\sum_{i=1}^{s}(T\Delta_1, \Omega)(cx_i)\otimes g(T\Delta_1, \Delta_1)(y_i)) \\
&\quad = b\textstyle\sum_{i=1}^{s}((\sum_{\Omega\in P} g(T\Delta_1, \Omega)(c)\Omega(x_i))\otimes g(T\Delta_1, \Delta_1)(y_i)) = 0.
\end{aligned}$$

While

$$\begin{aligned}
0 &= b(\textstyle\sum_{i=1}^{s} T\Delta_1(cx_i)\otimes g(T\Delta_1, \Delta_1)(y_i)) \\
&= b\otimes(\textstyle\sum_{i=1}^{s} T\Delta_1(cx_i)g(T\Delta_1, \Delta_1)(y_i)) \\
&= b\otimes\textstyle\sum_{i=1}^{s}(\sum_{\Omega\in P} g(T\Delta_1, \Omega)(c)\Omega(x_i)g(T\Delta_1, \Delta_i)(y_i))
\end{aligned}$$

$$= b\otimes\sum_{i=1}^{s}(\sum_{\Omega\in P} g(T\Delta_1, \Omega)(c)\Omega(x_i)g(\Delta_1, \Delta_1)(y_i))$$
$$= b\otimes\sum_{i=1}^{s}(g(T\Delta_1, \Delta_1)(c)\Delta_1(x_i)g(\Delta_1, \Delta_1)(y_i)) \quad \text{(by Theorem 3.5)}$$
$$= b\otimes g(\Delta_1, \Delta_1)(c) = (1\otimes g(\Delta_1, \Delta_1))(b\otimes c)$$

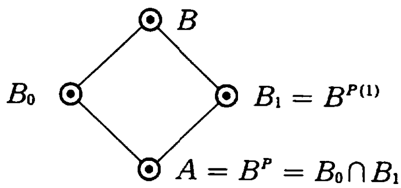
since $g(T\Delta_1, \Delta_1) = g(\Delta_1, \Delta_1)$. Noting that $1\otimes g(\Delta_1, \Delta_1)$ is an isomorphism, we can obtain that $b\otimes c = 0$. Then it is easy to see that $\sum_j b_j\otimes c_j = 0$ if $\Phi_1(\sum_j b_j\otimes c_j) = 0$.

B/A is said to be a projective Frobenius extension if B_A is finitely generated projective and ${}_AB_B \cong {}_AB_B^*$. Then we have the following as a corollary of Theorem 3.11.

Corollary 3.12. *Assume P satisfies* (A.5). *If B/A is a P-Galois extension, then B/A is a projective Frobenius extension.*

Proof. Since P satisfies (A.5), P is pure by Lemma 1.4 and $\Delta = T\Delta_1$. Then ${}_AB_B \cong {}_A\Delta\cdot B_B \cong {}_AB_B^*$ by $b \mapsto \Delta\cdot b \mapsto j(\Delta\cdot b)$.

4. The case P satisfies (A.5) and (A.6). In this section, we assume that P is a r.s.h with (A.5) and (A.6). If $B^P = A$ then we have the diagram



Let B/A be a P-Galois extension.

(i) If $P = P(1)$ then $B_1 = A$ and B/A is a P-Galois extension of separable type.

(ii) If $P(1) = \{1\}$ then $B_0 = A$ and B/A is a P-Galois extension of inseparable type which will study in the following paper.

Since P satisfies (A.5), if $P(max) = \{\Delta_1, \Delta_2, \cdots, \Delta_k\}$, then $P(1) = \{\Lambda_1 = 1, \Lambda_2, \cdots, \Lambda_k\}$, $P(max) = \{\Lambda_i\Delta_1 ; i = 1, 2, \cdots, k\}$ and $\Delta = T\Delta_1$. Further any Δ_i and Δ_j of $P(max)$ are similar. Hence we put $\Delta_i = \Lambda_i\Delta_1$ in the rest. Moreover B_0 coincides with $B^{P_1} = \{b \in B ; \Omega(b) = 0 \text{ for all } \Omega \in P_1 - P(1)\}$, and hence $B^P = B_0\cap B_1 = B^{P_1}\cap B^{P(1)}$.

Lemma 4.1. (1) *P_1 is a r.s.h if and only if $H_1P_1 \subseteq P_1$.*

(2) *$m_\Delta = 1$ if and only if $m_{\Delta_i} = 1$ for all $i \geq 2$. In this case P_1 is a r.s.h and*

if $\Omega\Gamma \in P_1$ *(resp.* $\Gamma\Omega \in P_1$*) for* $\Omega \in P_1$ *and* $\Gamma \in P$, *then* $\Gamma \in P_1$. *Moreover* Λ_i *is a unique minimal element of* Δ_i *and* $\Delta_i = \Delta_1\Lambda_i$ *for all* i.

Assume $m_\Delta = 1$.

(3) *Let* $\Omega_i \in P_i$. *Then* $\Omega_i = \Lambda_i\Omega_1$ (*resp.* $\Omega_i = \Omega_1'\Lambda_i$) *for some* $\Omega_1 \in P_1$ (*resp.* $\Omega_1' \in P_1$) *and* $g(\Delta_i, \Omega_i) = \Lambda_i g(\Delta_1, \Omega_1) = g(\Delta_1, \Omega_1')\Lambda_i$.

(4) $g(\Delta_1, \Delta_1) = 1$ $(= \Lambda_1)$ *if and only if* $g(\Delta_i, \Delta_i) = \Lambda_i$ *for all* $i \geq 2$.

Proof. (1) Assume $H_1P_1 \subseteq P_1$. P_1 becomes a r.s.h if we show that $\Omega\Gamma \in P_1$ for Ω, $\Gamma \in P_1$ such that $\Omega\Gamma \neq 0$. Let Ω_0 and Γ_0 be respective minimal elements of Ω and Γ. Then $\Omega\Gamma \geq \Omega_0\Gamma_0$ implies $\Gamma_0^{-1}\Omega_0^{-1}\Omega\Gamma \geq 1$. Thus $\Gamma_0^{-1}\Omega_0^{-1}\Omega\Gamma \in P_1$, and hence $\Omega\Gamma \in H_1(H_1P_1) \subseteq H_1P_1 \subseteq P_1$. The converse is clear.

(2) Since $\Delta_i = \Lambda_i\Delta_1$, $m_{\Delta_1} = 1$ if and only if $m_{\Delta_i} = 1$. If $m_{\Delta_1} = 1$ then $H_1 = \{1\}$ and hence P_1 is a r.s.h by (1). Since $m_{\Delta_i} = 1$, Λ_i is a unique minimal element of Δ_i and so $\Delta_i = \Delta_1\Lambda_i$ for all i. Let $\Omega\Gamma \in P_1$ for $\Omega \in P_1$ and $\Gamma \in P$. If $\Gamma \in P_i \neq P_1$ then $\Delta_1 \geq \Omega\Gamma \geq \Lambda_1\Lambda_i = \Lambda_i \neq 1$ and this contradicts to that $\Lambda_1 = 1$ is a unique minimal element of Δ_1.

(3) Let $\Omega_i \leq \Delta_i$. Then $\Lambda_i \leq \Omega_i \leq \Delta_i$ implies that $1 \leq \Lambda_i^{-1}\Omega_i \leq \Lambda_i^{-1}\Delta_i = \Delta_1$. Hence $\Lambda_i^{-1}\Omega_i = \Omega_1 \in P_1$ and $\Omega_i = \Lambda_i\Omega_1$. By the similar way we can see that $\Omega_i = \Omega_1'\Lambda_i$ for some $\Omega_1' \in P_1$. For $b \in B$,

$$\begin{aligned}\Delta_i\cdot b &= \textstyle\sum_{\Gamma_i\Delta_i} g(\Delta_i, \Gamma_i)(b)\Gamma_i = \Lambda_i(\sum_{\Gamma_1\leq\Delta_1} g(\Delta_1, \Gamma_1)(b)\Gamma_1)\\ &= \textstyle\sum_{\Gamma_1\Delta_1} \Lambda_i g(\Delta_1, \Gamma_1)(b)\Lambda_i\Gamma_1\end{aligned}$$

show that $g(\Delta_i, \Omega_i) = \Lambda_i g(\Delta_1, \Omega_1)$. By the similar way we can see that $g(\Delta_i, \Omega_i) = g(\Delta_1, \Omega_1')\Lambda_i$.

(4) This is a direct consequence of the latter half of (3).

Theorem 4.2. *Let* B/A *be a* P*-Galois extension.*

(1) *Assume* P_1 *is a r.s.h. Then* B/B_0 *is a* P_1*-Galois extension if and only if* $m_{\Delta_1} = 1$.

Assume $g(\Delta_1, \Delta_1) = 1$. *Then*

(2) B/B_1 *is a* $P(1)$*-Galois extension.*

(3) B *coincides with* $B_0[B_1]$, *the subring generated by* B_0 *and* B_1. *More precisely,* $B = \sum_{i=1}^s B_0v_i = \sum_{i=1}^s w_iB_0$ *for* $v_i, w_i \in B_1$ *and* $B = \sum_{i=1}^s B_1v_i' = \sum_{i=1}^s w_i'B_1$ *for* $v_i', w_i' \in B_0$.

Proof. Let $\{x_i, y_i\,;\ i = 1, 2, \cdots, s\}$ be a P-Galois system for B/A.

(1) Let B/B_0 be a P_1-Galois extension. Then there exists a P_1-Galois system $\{u_i, v_i\,;\ i = 1, 2, \cdots, t\}$ for B/B_0. Namely,

$$\sum_{i=1}^{t} u_i g(\varDelta_1, \Omega)(v_i) = \delta_{1,\Omega} \quad \text{for any } \Omega \in P_1.$$

If $\varDelta_1$ is a minimal element $\Lambda \neq 1$, then we have a contradiction that

$$0 = \sum_{i=1}^{t} u_i g(\varDelta_1, \Lambda)(v_i) = \sum_{i=1}^{t} u_i \varDelta_1(v_i) = \sum_{i=1}^{t} u_i g(\varDelta_1, 1)(v_i) = 1.$$

Conversely, assume $m_{\varDelta_1} = 1$ and $\Omega \in P_1$. Then $\Omega \notin P_i$ for $i \neq 1$, and hence $g(\varDelta, \Omega) = g(\varDelta_1, \Omega)$. Thus

$$\delta_{1,\Omega} = \sum_{i=1}^{s} x_i g(\varDelta, \Omega)(y_i) = \sum_{i=1}^{s} x_i g(\varDelta_1, \Omega)(y_i)$$

for any $\Omega \in P_1$ shows that $\{x_i, y_i\,;\ i = 1, 2, \cdots, s\}$ is a P_1-Galois system for B/B_0.

(2) $\sum_{i=1}^{s} \varDelta_1(x_i) g(\varDelta, \Omega)(y_i) = \delta_{\varDelta_1,\Omega}$ by Lemma 3.8.(2). While, for each $\Lambda_j \in P(1)$, noting that Lemma 4.1.(4) and $g(\varDelta_1, \varDelta_1) = 1$, we have

$$\sum_{i=1}^{s} \varDelta_1(x_i)\Lambda_j(y_i) = \sum_{i=1}^{s} \varDelta_1(x_i) g(\varDelta_j, \varDelta_j)(y_i) = \sum_{i=1}^{s} \varDelta_1(x_i) g(\varDelta, \varDelta_j)(y_i) = \begin{cases} 1 & \text{if } \varDelta_j = \varDelta_1 \\ 0 & \text{if } \varDelta_j \neq \varDelta_1 \end{cases}$$

and this shows that $\sum_{i=1}^{s} \varDelta_1(x_i)\Lambda_j(y_i) = \delta_{1,\Lambda_j}$ and $\{\varDelta_1(x_i), y_i\,;\ i = 1, 2, \cdots, s\}$ is a $P(1)$-Galois system.

(3) Let $\Omega \leq \varDelta_1$. Since $\{\Omega(x_i), y_i\,;\ i = 1, 2, \cdots, s\}$ is a (P, Ω)-Galois system for B/A and $\Lambda_j = g(\varDelta_j, \varDelta_j)$ is the minimal element of $\varDelta_j$ by Lemma 4.1.(4), we have

$$\begin{aligned} \sum_{i=1}^{s} \Omega(x_i) T(y_i) &= \sum_{i=1}^{s} \Omega(x_i)\Big(\sum_{\Lambda_j \in P(1)} \Lambda_j(y_i)\Big) \\ &= \sum_{i=1}^{s} \Omega(x_i)\Big(\sum_{\varDelta_j \in P(max)} g((\varDelta_j, \varDelta_j)(y_i)\Big) \\ &= \begin{cases} 1 & \text{if } \Omega = \varDelta_1 \text{ and } \varDelta_j = \varDelta_1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus, for any $b \in B$,

$$\begin{aligned} B_0[B_1] \ni \sum_{i=1}^{s} \varDelta_1(bx_i) T(y_i) &= \sum_{\Omega \leq \varDelta_1}\Big(\sum_{i=1}^{s} g(\varDelta_1, \Omega)(b)\Omega(x_i) T(y_i)\Big) \\ &= \sum_{i=1}^{s} g(\varDelta_1, \varDelta_1)(b)\varDelta_1(x_i) T(y_i) = \sum_{i=1}^{s} b\varDelta_1(x_i) y_i = b. \end{aligned}$$

Consequently, we have $B = B_0[B_1] = \sum_{i=1}^{s} B_0 \cdot T(y_i)$.

Next we consider $\sum_{i=1}^{s} T(x_i)\varDelta_1(y_i b) \in B_1[B_0]$ for $b \in B$.

For $P(1) \ni \Lambda \neq 1$, $\sum_{i=1}^{s} \Lambda(x_i)\varDelta_1(y_i b) = \sum_{i=1}^{s} \Lambda(x_i) g(\varDelta, 1)(y_i b) = 0$ since $\{\Lambda(x_i), y_i\,;\ i = 1, 2, \cdots, s\}$ is a (P, Λ)-Galois system. Hence

$$\begin{aligned} \sum_{i=1}^{s}(T(x_i)\varDelta_1(y_i b)) &= \sum_{i=1}^{s} x_i \varDelta_1(y_i b) \\ &= \sum_{\Omega \leq \varDelta_1}\Big(\sum_{i=1}^{s} x_i g(\varDelta_1, \Omega)(y_i)\Omega(b)\Big) = \sum_{i=1}^{s} x_i g(\varDelta, 1)(y_i) b = b. \end{aligned}$$

Thus $b \in B_1[B_0]$ and hence $B = B_1[B_0] = \sum_{i=1}^{s} T(x_i)B_0$.

Next we shall show that $B = \sum_{i=1}^{s} \Delta_1(x_i)B_1$.

$$\begin{aligned} B &\supseteq \textstyle\sum_{i=1}^{s} \Delta_1(x_i)B_1 \ni \sum_{i=1}^{s} \Delta_1(x_i)T(y_i b) \\ &= \textstyle\sum_{i=1}^{s} \Delta_1(x_i)(\sum_{\Lambda_j \in P(1)} \Lambda_j(y_i)\Lambda_j(b))) \\ &= \textstyle\sum_{i=1}^{s} \Delta_1(x_i)\Lambda_1(y_i)\Lambda_1(b) + \sum_{i=1}^{s} \Delta_1(x_i)(\sum_{j\neq 1} \Lambda_j(y_i)\Lambda_j(b))) \\ &= \textstyle\sum_{i=1}^{s}(\Delta_1(x_i)g(\Delta, \Delta_1)(y_i)\Lambda_1(b)) + \sum_{i=1}^{s}(\Delta_1(x_i)(\sum_{j\neq 1} g(\Delta, \Delta_j)(y_i)\Lambda_j(b)) \\ &= \textstyle\sum_{i=1}^{s} \Delta_1(x_i)g(\Delta, \Delta_1)(y_i)\Lambda_1(b) = b \end{aligned}$$

for $b \in B$ since $\{\Delta_1(x_i), y_i\,;\ i = 1, 2, \cdots, s\}$ is a (P, Δ_1)-Galois system and the minimal element of Δ_1 is 1. Thus $B = \sum_{i=1}^{s} \Delta_1(x_i)B_1$.

Finally

$$\begin{aligned} B &\supseteq \textstyle\sum_{i=1}^{s} B_1\Delta_1(y_i) \ni \sum_{i=1}^{s} T(bx_i)\Delta_1(y_i) \\ &= \textstyle\sum_{i=1}^{s}((\sum_{\Lambda_i \in P(1)} \Lambda_j(b)\Lambda_j(x_i))\Delta_1(y_i)) \\ &= \textstyle\sum_{i=1}^{s}(\sum_{\Lambda_j \in P(1)} \Lambda_j(b)\Lambda_j(x_i)g(\Delta_1, 1)(y_i) = \sum_{i=1}^{s} bx_i g(\Delta_1, 1)(x_i) = b \end{aligned}$$

since $\{x_i, y_i\,;\ i = 1, 2, \cdots, s\}$ is a P-Galois system. Thus $\sum_{i=1}^{s} B_1\Delta_1(y_i) = B$.

Let $m_{\Delta_1} = 1$, $g(\Delta_1, \Delta_1) = 1$ and B/A a P-Galois extension. Then B/B_0 is a P_1-Galois extension and B/B_1 is a $P(1)$-Galois extension by Theorem 4.2. Further B_0 is a $P(1)$-admissible, $B_0^{P(1)} = A$, and if B_1 is P_1-admissible then $B_1^{P_1} = A$.

Then it is natural to ask that whether B_0/A (resp. B_1/A) is a $P(1)$-Galois extension (resp. P_1-Galois extension). As will be seen in the next section, these are true if $B_A \oplus > A_A$. But, first we shall prove the converse of this problem.

Theorem 4.3. *Let $m_{\Delta_1} = 1$ and $B^P = A$. If B_0/A is a $P(1)$-Galois extension and B_1/A is a P_1-Galois extension then B/A is a P-Galois extension.*

Proof. Let $\{u_i, v_i\,;\ i = 1, 2, \cdots, t\}$ be a $P(1)$-Galois system for B_0/A and let $\{x_i, y_i\,;\ i = 1, 2, \cdots, s\}$ be a P_1-Galois system for B_1/A. Since $\Gamma_k \in P_k$ is obtained by $\Gamma_1\Lambda_k$ for some $\Gamma_1 \in P_1$ by Lemma 4.1.(3), $g(\Delta_k, \Gamma_k) = g(\Delta_1, \Gamma_1)\Lambda_k$ by Lemma 4.1.(3). Therefore

$$\begin{aligned} &\textstyle\sum_{i=1}^{s} x_i g(\Delta_k, \Gamma_k)(y_i) = \sum_{i=1}^{s} x_i g(\Delta_1, \Gamma_1)(\Lambda_k(y_i)) = \sum_{i=1}^{s} x_i g(\Delta_1, \Gamma_1)(y_i) \\ &\quad = \delta_{1,\Gamma_1} = \delta_{\Lambda_k, \Gamma_k}. \end{aligned}$$

We now consider

$$\begin{aligned} &\textstyle\sum_{j=1}^{t} u_j(\sum_{i=1}^{s} x_i g(\Delta, \Gamma)(y_i v_j)) \\ &\quad = \textstyle\sum_{j=1}^{t} u_j(\sum_{i=1}^{s} x_i(\sum_{\Omega \in P} g(\Delta, \Omega)(y_i)g(\Omega, \Gamma)(v_j))). \end{aligned}$$

Then, for $\Omega \in P_k$, $\Omega = \Omega_1\Lambda_k$ for some $\Omega_1 \in P_1$, and hence

$$\sum_{i=1}^{s} x_i g(\Delta, \Omega)(y_i) = \sum_{i=1}^{s} x_i g(\Delta_k, \Omega)(y_i) = \sum_{i=1}^{s} x_i g(\Delta_1, \Omega_1)\Lambda_k(y_i) = \sum_{i=1}^{s} x_i g(\Delta_1, \Omega_1)(y_i) = 0$$

if $\Omega \notin P(1)$ since $\Lambda_k(y_i) = y_i$.
Next

$$\sum_{i=1}^{s} x_i g(\Delta, \Omega)(y_i) = 0 \quad \text{if } \Omega \notin P(1) \quad \text{and}$$
$$\sum_{i=1}^{s} x_i g(\Delta, \Lambda_k)(y_i) = \sum_{i=1}^{s} x_i g(\Delta_k, \Lambda_k)(y_i) = 1 \quad \text{for } \Lambda_k \in P(1).$$

Thus we have

$$\sum_{j=1}^{t} u_j\left(\sum_{i=1}^{s} x_i g(\Delta, \Gamma)(y_i v_j)\right) = \begin{cases} 0 & \text{if } \Gamma \notin P(1) \\ \delta_{1,\Gamma} & \text{if } \Gamma = \Lambda_k \in P(1). \end{cases}$$

Consequently, we have

$$\sum_{j=1}^{t} u_j\left(\sum_{i=1}^{s} x_i g(\Delta, \Gamma)(y_i v_j)\right) = \delta_{1,\Gamma} \quad \text{for } \Gamma \in P,$$

and this means that B has a P-Galois system for B/A.

5. P-Galois extensions B/A with $B_A \oplus > A_A$. In this section we assume the following conditions:

(i) P satisfies (A.5) and (A.6), and so we may put $P(max) = \{\Delta_1, \Delta_2, \cdots, \Delta_k\} = \{\Lambda_i\Delta_1 ;\ i = 1, 2, \cdots, k\}$ where $P(1) = \{\Lambda_1 = 1, \Lambda_2, \cdots, \Lambda_k\}$ and $\Delta_i = \Lambda_i\Delta_1 = \Delta_1\Lambda_i$. Moreover P_1 forms a r.s.h by Lemma 4.1.(1).

(ii) $m_{\Delta_1} = 1$ and $g(\Delta_1, \Delta_1) = 1$.

(iii) $B_A \oplus > A_A$.

Lemma 5.1. *Let $\Omega \in P_1$.*

(1) *If $\Gamma \in P_i$ and $\Omega\Gamma = \Delta_i$ then $\Gamma\Omega = \Delta_i$.*

(2) *$\Omega\Lambda = \Lambda\Omega$ for all $\Lambda \in P(1)$.*

(3) *B_1 is P_1-admissible.*

Proof. Assume $ht(\Delta_1) = n+1$. Then $ht(\Delta_i) = n+1$ for all $\Delta_i \in P(max)$ since P is a pure poset.

(1) Let $\Omega\Gamma = \Delta_i$. Then we have a chain

$$\Delta_i = \Omega\Gamma = \Omega_0\Gamma_0 \gg \Omega_{i_1}\Gamma_{i_1} \gg \Omega_{i_2}\Gamma_{i_2} \gg \cdots \gg \Omega_{i_{n-1}}\Gamma_{i_{n-1}} \gg \Lambda_i$$

for some $\Omega > \Omega_{i_j}$ and $\Gamma > \Gamma_{i_j}$ for $j = 1, 2, \cdots, n-1$ by (A.4).(ii). Further $\Omega_{i_j} \in P_1$ by Lemma 4.1.(2) and hence $\Gamma_{i_j} \in P_i$. By (A.3), $\Gamma_{i_j}\Omega_{i_j} \neq 0$ and it is contained in P_i. Thus

$$\Delta_i \geq \Gamma\Omega = \Gamma_0\Omega_0 \gg \Gamma_{i_1}\Omega_{i_1} \gg \cdots \gg \Gamma_{i_{n-1}}\Omega_{i_{n-1}} \gg 1$$

shows that $ht(\Gamma\Omega) = n+1 = ht(\Delta_i)$. Thus $\Gamma\Omega = \Delta_i$.

(2) For $\Lambda_i \in P(1)$, assume that $\Omega\Lambda_i = \Lambda_i\Omega'$. Then $\Omega' \in P_1$ since $\Lambda_i\Omega'$ has a unique minimal element Λ_i. Let $\Gamma\Omega = \Delta_1$ for $\Gamma \in P_1$. Then $\Delta_1\Lambda_i = \Delta_i = \Gamma\Omega\Lambda_i = \Gamma\Lambda_i\Omega'$. Since $\Omega\Gamma = \Delta_1$ by (1), we have $\Gamma\Omega\Lambda_i = \Omega\Gamma\Lambda_i$. Noting that $\Gamma\Lambda_i \in P_i$ and $\Omega' \in P_1$, $\Delta_i = \Gamma\Lambda_i\Omega' = \Omega'\Gamma\Lambda_i$ by (1) again. Hence $\Omega\Gamma = \Omega'\Gamma$ and so $\Omega = \Omega'$by (A.4).(i).

(3) Let b and Ω be arbitrary elements of B_1 and P_1. Then, for $\Lambda \in P(1)$, $\Lambda\Omega(b) = \Omega\Lambda(b) = \Omega(b)$ show that $\Omega(b) \in B_1$.

Theorem 5.2. *Let B/A be a P-Galois extension.*

(1) $\mathrm{Hom}(B_{0_A}, A_A)$ *is a homomorphic image of the submodule* $u_T\Delta_1(B) = (\sum_{\Lambda\in P(1)} u_\Lambda)\Delta_1(B)$ *of* $D(B, P)$ *and* B_0/A *is a* $P(1)$*-Galois extension.*

(2) $\mathrm{Hom}(B_{1_A}, A_A)$ *is a homomorphic image of the submodule* $u_{\Delta_1}T(B)$ *of* $D(B, P)$ *and* B_1/A *is a* P_1*-Galois extension.*

Proof. Since B/A is a P-Galois extension with $B_A\oplus > A_A$, there exists $x \in B$ such that $1 = \Delta(x) = \Delta_1(T(x)) = T(\Delta_1(x))$. Hence $\Delta_1(B) = B_0$, $B_{B_0}\oplus > B_{0_{B_0}}$, $T(B) = B_1$ and $B_{B_1}\oplus > B_{1_{B_1}}$. Thus, for any $f \in B^*$, $f|B_0 \in \mathrm{Hom}(B_{0_A}, A_A) = B_0^*$ gives an epimorphism of B^* to B_0^* and $f|B_1 \in \mathrm{Hom}(B_{1_A}, A_A) = B_1^*$ gives an epimorphism of B^* to B_1^*.

Thus we have $j(u_\Delta\cdot B)|B_0 = B_0^*$ and $j(u_\Delta\cdot B)|B_1 = B_1^*$.

(1) Since $u_T\cdot\Delta_1(B) \longrightarrow j(u_\Delta\cdot B)|B_0$: $u_T\cdot\Delta_1(b) \mapsto j(\Delta b)|B_0$ gives an epimorphism, B_0^* is a homomorphic image of $u_T\cdot\Delta_1(B)$. B_{0_A} is also projective since B_A is projective and $B_{B_0}\oplus > B_{0_{B_0}}\cdot B = z_1A+z_2A+ \cdots u_tA$ $(z_i \in B)$ implies $B_0 = \Delta_1(B) = \Delta_1(z_1)A+\Delta_1(z_2)+ \cdots +\Delta_1(z_t)A$. Therefore B_{0_A} is finitely generated projective.

The map J_0 of a B_0-submodule $\sum_{\Lambda\in P(1)}\oplus B_0u_\Lambda$ of $D(B, P)$ into $\mathrm{End}(B_{0_A})$ defined by $J_0(bu_\Lambda)(x_0) = b\Lambda(x_0)$ is a monomorphism. For if $J_0(\sum_{\Lambda\in P(1)}b_\Lambda u_\Lambda) = 0$, then $\sum_{\Lambda\in P(1)}b_\Lambda\Lambda(x_0) = 0$ for all $x_0 \in B_0$. Since $B_0 = \Delta_1(B)$, this means that $\sum_{\Lambda\in P(1)}b_\Lambda\Lambda\Delta_1(y) = 0$ for all $y \in B$, and hence $j^{-1}(\sum_{\Lambda\in P(1)}b_\Lambda\Lambda\Delta_1) = \sum_{\Lambda\in P(1)}b_\Lambda u_\Lambda\Delta_1 = 0$. Thus $b_\Lambda = 0$ for all $\Lambda \in P(1)$.

Let $\{x_i, g_i\,;\ i = 1, 2, \cdots, s,\ x_i \in B_0,\ g_i \in B_0^*\}$ be a projective coordinate system for B_0/A. Since g_i is obtained by $J_0(\sum_{\Lambda\in P(1)} u_\Lambda\cdot\Delta_1(v_i))$,

$$\begin{aligned}\mathrm{End}(B_{0_A}) \ni J_0(u_1) = 1 &= J_0(\textstyle\sum_{i=1}^s x_i(\sum_{\Lambda\in P(1)} u_\Lambda\cdot\Delta_1(v_i)) \\ &= J_0(\textstyle\sum_{i=1}^s x_i(\sum_{\Lambda\in P(1)} \Lambda\Delta_1(v_i)u_\Lambda\end{aligned}$$

and this implies

$$\sum_{i=1}^{s} x_i \Lambda \Delta_1(v_i) = \delta_{1,\Lambda}.$$

This shows that $\{x_i, \Delta_1(v_i)\,;\ i = 1, 2, \cdots, s\}$ is a $P(1)$-Galois system for B_0/A.

(2) $u_{\Delta_1}\cdot T(B) \longrightarrow j(\Delta\cdot B)|B_1 := B_1^* : u_{\Delta_1}\cdot T(b) \mapsto j(u_\Delta \cdot b)|B_1$ gives an epimorphism and B_{1_A} is finitely generated projective since $B = z_1 A + \cdots + z_n A$ ($z_i \in B$) yields $B_1 = T(B) = T(z_1)A + \cdots + T(z_n)A$.

The map J_1 of a B_1-submodule $\sum_{\Omega \in P_1} \oplus B_1 u_\Omega$ of $D(B, P)$ into $\mathrm{End}(B_{1_A})$ defined by $J_1(bu_\Omega)(x_1)$ for $x_1 \in B_1$ is a monomorphism. For, since $P_1(B_1) \subseteq B_1$, $b\Omega(x_1) \in B_1$. If $J_1(\sum_{\Omega \in P_1} b_\Omega u_\Omega) = 0$ then $\sum_{\Omega \in P_1} b_\Omega \Omega(x_1) = 0$ for all $x_1 \in B_1$. Since $B_1 = T(B)$, this means that $\sum_{\Omega \in P_1} b_\Omega \Omega T(y) = 0$ for all $y \in B$, and hence, $j^{-1}(\sum_{\Omega \in P_1} b_\Omega \Omega T) = \sum_{\Omega \in P_1} b_\Omega (\sum_{\Lambda \in P_1} u_{\Omega\Lambda}) = 0$. Thus $b_\Omega = 0$ for all $\Omega \in P_1$.

Let $\{y_i, g_i\,;\ i = 1, 2, \cdots, s,\ y_i \in B_1, g_i \in B_1^*\}$ be a projective coordinate system for B_1/A. Since g_i is obtained by $J_1(u_{\Delta_1}\cdot T(v_i))$,

$$\begin{aligned}\mathrm{End}(B_{1_A}) \ni J_1(u_1) &= J_1(\textstyle\sum_{i=1}^{s} y_i(u_{\Delta_1}\cdot T(v_i)) \\ &= J(\textstyle\sum_{i=1}^{s} y_i(\sum_{\Omega \le \Delta_1} g(\Delta_1, \Omega)\cdot T(v_i) u_\Omega)\end{aligned}$$

and this implies $\sum_{i=1}^{s} y_i g(\Delta_1, \Omega)(T(v_i)) = \delta_{1,\Omega}$. Thus B_1/A is a P_1-Galois extension.

Combining Theorem 4.3 with Theorem 5.2, we have the following

Corollary 5.3. *Let $B^P = A$. Then B/A is a P-Galois extension (with $B_A \oplus > A_A$) if and only if B_0/A is a $P(1)$-Galois extension with $B_{0_A} \oplus > A_A$ and B_1/A is a P_1-Galois extension with $B_{1_A} \oplus > A_A$.*

In the rest we shall study generating elements of B over A when B/A is a P-Galois extension.

Theorem 5.4. *Let B/A be a P-Galois extension and let $\{x_i, y_i\,;\ i = 1, 2, \cdots, s\}$ be a P-Galois system for B/A. Then B coincides with $A[\{y_i\,;\ i = 1, 2, \cdots, s\}]$, the subring generated by $\{y_i\,;\ i = 1, 2, \cdots, s\}$ over A. More precisely, $B = \sum_{i=1}^{s} A y_i$.*

Proof. Let $T = A[\{y_i\,;\ i = 1, 2, \cdots, s\}]$ and let $\{\sum b_i \otimes t_i\,;\ b_i \in B,\ t_i \in T\}$ be a submodule of $B \otimes_A B$. We denote it by $B \otimes T$. For $\alpha = \sum_{i=1}^{s} b\Omega(x_i) \otimes y_i \in B \otimes T$, $\Phi(\alpha) = \sum_{i=1}^{s} b\Omega(x_i)\Delta\cdot y_i = \sum_{i=1}^{s} b\Omega(x_i)(\sum_{\Gamma \in P} g(\Delta, \Gamma)(y_i)\Gamma = b\Omega$ since $\{\Omega(x_i), y_i\,;\ i = 1, 2, \cdots, t\}$ is a (P, Ω)-Galois system. But this means that $\Phi(B \otimes T) = D(B, P) = \Phi(B \otimes_A B)$ and we obtain $B \otimes T =$

$B\otimes_A B$ since Φ is an isomorphism by Theorem 3.11. Let $x\in B$ be an element such that $\Delta(x)=1$. Then $x\otimes b=\sum b_i\otimes t_i$, $b_i\in B$ and $t_i\in T$, and so

$$\begin{aligned}(\Delta\otimes 1)(x\otimes b)&=\Delta(x)\otimes b=1\otimes b \quad\text{and}\\ (\Delta\otimes 1)(x\otimes b)&=(\Delta\otimes 1)(\textstyle\sum b_i\otimes t_i)=\sum\Delta(b_i)\otimes t_i\\ &=\textstyle\sum(1\otimes\Delta(b_i)t_i)\in A\otimes_A T=T\end{aligned}$$

shows that $B=T$.

Let $S=\sum_{i=1}^{s}Ay_i$ ($\subseteq T$). Since $b\Omega$ is obtained by $\Phi(\sum_{i=1}^{s}b\Omega(x_i)\otimes y_i)$, we have $B\otimes S=B\otimes_A B$ again. Thus we can see $S=B$ by the same way.

In the rest, we assume that B/A is a P-Galois extension. Then there exists $T(x)\in B_1$ such that $\Delta_1(T(x))=1$. We put $T(x)=x_{\Delta_1}$ and for this x_{Δ_1} we put $x_\Omega=(\Delta_1/\Omega)_\ell(x_{\Delta_1})$ for $\Omega\in P_1$ (and so $(\Delta_1/\Omega)_\ell\in P_1$ by Lemma 4.1.(2)). Then $\Omega(x_\Omega)=\Omega(\Delta_1/\Omega)_\ell(x_{\Delta_1})=\Delta_1(x_{\Delta_1})=1$ and $x_1=1$ since $(\Delta_1/1)_\ell=\Delta_1$ and $x_1=(\Delta_1/1)_\ell(x_{\Delta_1})$.

Lemma 5.5. *Let $\Gamma\in P_1$.*

(1) *If $\Gamma(x_{\Delta_1})=1$. Then $\Gamma=\Delta_1$.*

(2) *$\Gamma(x_\Omega)\neq 0$ if and only if Γ is a right factor of Ω and if this is the case, $\Gamma(x_\Omega)=x_{\Gamma_0}$ where Γ_0 is a left factor of Ω.*

(3) *$\Lambda(x_\Omega)=x_\Omega$ for all $\Lambda\in P(1)$.*

Proof. (1) If $\Gamma\neq\Delta_1$, then $(\Delta_1/\Gamma)_r\in P_1-P(1)$ and we have a contradiction

$$1=\Delta_1(x_{\Delta_1})=(\Delta_1/\Gamma)_r\Gamma(x_{\Delta_1})=(\Delta_1/\Gamma)_r(1)=0.$$

(2) Assume $\Gamma(x_\Omega)=\Gamma(\Delta_1/\Omega)_\ell(x_{\Delta_1})\neq 0$. Then $\Gamma(\Delta_1/\Omega)_\ell\neq 0$ ($\in P_1$) by Lemma 4.1.(1) and hence $\Gamma_0\Gamma(\Delta_1/\Omega)_\ell=\Delta_1$ for some $\Gamma_0\in P_1$. Thus $\Gamma_0\Gamma=\Omega$. Conversely, if $\Gamma_0\Gamma=\Omega$ for some $\Gamma_0\in P_1$ then $\Gamma_0\Gamma(x_\Omega)=1$ yields $\Gamma(x_\Omega)\neq 0$. Let $\Gamma_0\Gamma(x_\Omega)=1$ for $\Gamma_0\in P_1$. Then

$$1=\Gamma_0\Gamma(x_\Omega)=\Gamma_0\Gamma(\Delta_1/\Omega)_\ell(x_{\Delta_1})=\Delta_1(x_{\Delta_1})$$

implies that $\Gamma_0\Gamma(\Delta_1/\Omega)_\ell=\Delta_1$, by (1) and hence $\Gamma(\Delta_1/\Omega)_\ell=(\Delta_1/\Gamma_0)_\ell$. Therefore $\Gamma(x_\Omega)=\Gamma(\Delta_1/\Omega)_\ell(x_{\Delta_1})=(\Delta_1/\Gamma_0)_\ell(x_{\Delta_1})=x_{\Gamma_0}$.

Since $\Gamma_0\Gamma(x_\Omega)=1=\Gamma_0\Gamma(\Delta_1/\Omega)_\ell(x_{\Delta_1})$, $\Gamma_0\Gamma(\Delta_1/\Omega)_\ell=\Omega(\Delta_1/\Omega)_\ell$ shows that $\Gamma_0\Gamma=\Omega$ and hence Γ_0 is a left factor of Ω

(3) $\Lambda(x_\Omega)=\Lambda(\Delta_1/\Omega)_\ell(x_{\Delta_1})=(\Delta_1/\Omega)_\ell\Lambda(x_{\Delta_1})=(\Delta_1/\Omega)_\ell(x_{\Delta_1})=x_\Omega$.

For $X:=\{x_\Omega:\Omega\in P_1\}$, a monomial of X means a product of these x_Ω.

We put

$$R_0 = \sum_{\Omega_1,\Omega_2,\cdots,\Omega_n \in P_1} Ax_{\Omega_1}x_{\Omega_2}\cdots x_{\Omega_n}$$

a left A-submodule of B generated by the monomials of X over A.

$$R = \sum_{\Omega_1,\Omega_2,\cdots,\Omega_n \in P_1} B_0x_{\Omega_1}x_{\Omega_2}\cdots x_{\Omega_n}$$

a left B_0-submodule of B.

Then we have the following

Theorem 5.6. (1) *X is a left (as well as right) linearly independent over B_0.*

(2) *R_0 has systems $\{x_i, y_i\,;\ i = 1, 2, \cdots, t\}$ and $\{z_j, w_j\,;\ j = 1, 2, \cdots, u\}$ such that*

(a) *$x_i, w_j \in X$ and y_i and z_j are monomials of X for $i = 1, 2, \cdots, t$ and $j = 1, 2, \cdots, u$*

(b) *$\sum_{i=1}^{t} \Omega(x_i)y_i = \delta_{\Delta_1,\Omega}$ and $\sum_{j=1}^{u} z_j\Omega(w_j) = \delta_{\Delta_1,\Omega}$ for all $\Omega \in P_1$.*

For these systems $\{x_i, y_i\,;\ i = 1, 2, \cdots, t\}$ and $\{z_j, w_j\,;\ j = 1, 2, \cdots, u\}$

(3) *$R_0 = B_1 = \sum_{i=1}^{t} Ay_i$ and $R = B = \sum_{i=1}^{t} B_0y_i$.*

(4) *If $P(1) = \{1\}$ then $B = \sum_{i=1}^{t} Ay_i = \sum_{j=1}^{u} z_jA$.*

Proof. (1) Let $\alpha = \sum_{\Omega, ht(\Omega)=2} b_\Omega x_\Omega + b_1x_1 = 0$ $(b_\Omega, b_1 \in B_0)$. Then, for any $\Gamma \in P_1$ with $ht(\Gamma) = 2$,

$$\begin{aligned} 0 = \Gamma(\alpha) &= \textstyle\sum_{\Omega, ht(\Omega)=2}(g(\Gamma, \Gamma)(b_\Omega)\Gamma(x_\Omega) + g(\Gamma, 1)(b_\Omega)x_\Omega) \\ &= \textstyle\sum_{\Omega, ht(\Omega)=2} g(\Gamma, \Gamma)(b_\Omega)\Gamma(x_\Omega) = g(\Gamma, \Gamma)(b_\Gamma) \end{aligned}$$

by Lemma 5.5.(2) since Γ is not a right factor of Ω for $\Omega \neq \Gamma$. Thus $b_\Gamma = 0$ and $b_1 = 0$. Assume now $\{x_\Omega\,;\ \Omega \in P_1(m) = P_1 \cap P(m)\}$ is left linearly independent over B_0. Let $\beta = \sum_{\Omega \in P_1(m+1)} b_\Omega x_\Omega = 0$. For any $\Gamma \in P_1$ with $2 \leq ht(\Gamma) \leq m+1$,

$$0 = \Gamma(\beta) = \textstyle\sum_{\Omega \in P_1(m+1)} g(\Gamma, \Gamma)(b_\Omega)\Gamma(x_\Omega).$$

If $\Gamma(x_\Omega) \neq 0$, then $\Gamma(x_\Omega) = x_{\Gamma_0}$ where $\Gamma_0 = (\Omega/\Gamma)_r$ by Lemma 5.5.(2). Moreover $\Gamma(x_\Omega) \neq \Gamma(x_{\Omega_0})$ for $\Omega \neq \Omega_0$. For if $\Gamma(x_{\Omega_0}) \neq 0$, and the equality is hold, then $\Gamma(x_\Omega) = \Gamma(\Delta_1/\Omega)_\ell(x_{\Delta_1}) = \Gamma(x_{\Omega_0}) = \Gamma(\Delta_1/\Omega_0)_\ell(x_{\Delta_1})$ implies $1 = \Gamma_0\Gamma(\Delta_1/\Omega)_\ell(x_{\Delta_1}) = \Gamma_0\Gamma(\Delta_1/\Omega_0)_\ell(x_{\Delta_1})$ where $\Gamma_0 = (\Omega/\Gamma)_r$. Hence $\Delta_1 = \Gamma_0\Gamma(\Delta_1/\Omega)_\ell = \Gamma_0\Gamma(\Delta_1/\Omega_0)_\ell$ implies a contradiction that $(\Delta_1/\Omega)_\ell = (\Delta_1/\Omega_0)_\ell$. Thus $b_\Omega = 0$ for any Ω such that $\Gamma(x_\Omega) \neq 0$ by the assumption. Further, there exists Γ with $2 \leq ht(\Gamma) \leq m+1$ such that $\Gamma(x_\Omega) \neq 0$ for any Ω with $ht(\Omega) \geq$

2. Consequently $\{x_\Omega\,;\ \Omega \in P_1(m+1)\}$ is left linearly independent over B_0. Next, let $\sum_{\Omega\in P_1} x_\Omega b_\Omega = 0$ $(b_\Omega \in B_0)$. Then $0 = \Delta_1(\sum_{\Omega\in P_1} x_\Omega b_\Omega) = b_{\Delta_1}$. The right linear independence of X also can be proved by induction on the height of Ω.

(2) Let $ht(\Delta_1) = n$. By Lemma 5.5.(2), $\Omega(x_\Gamma) = \delta_{\Omega,\Gamma}$ for $\Omega, \Gamma \in P_1$ and $ht(\Omega) \geq ht(\Gamma)$. Hence we have $\Omega(x_{\Delta_1}) - \sum_{\Gamma, ht(\Gamma)=n-1} \Omega(x_\Gamma)\Gamma(x_{\Delta_1}) = \delta_{\Delta_1,\Omega}$ for any $\Omega \in P_1$ such that $ht(\Omega) \geq n-1$. Hence we assume that there exist elements $x_1, x_2, \cdots, x_s$ and $y_1, y_2, \cdots, y_s$ such that

(a) $x_i \in X$ and y_i are monomials of X for $i = 1, 2, \cdots, s$

(b) $\sum_{i=1}^s \Omega(x_i)y_i = \delta_{\Delta_1,\Omega}$ for any $\Omega \in P_1$ with $ht(\Omega) \geq m+1$.

Let $\Omega \in P_1$ with $ht(\Omega) \geq m$.

$$\begin{aligned}&\textstyle\sum_{i=1}^s \Omega(x_i)y_i - \sum_{\Gamma, ht(\Gamma)=m} \Omega(x_\Gamma)(\sum_{i=1}^s \Omega(x_i)y_i)\\ &\quad = \textstyle\sum_{i=1}^s \Omega(x_i)y_i = \delta_{\Delta_1,\Omega}, \quad \text{if } ht(\Omega) \geq m+1.\end{aligned}$$

While if $ht(\Omega) = m$, then

$$\begin{aligned}&\textstyle\sum_{i=1}^s \Omega(x_i)y_i - \sum_{\Gamma, ht(\Gamma)=m} \Omega(x_\Gamma)(\sum_{i=1}^s \Gamma(x_i)y_i)\\ &\quad = \textstyle\sum_{i=1}^s \Omega(x_i)y_i - \Omega(x_\Omega)\sum_{i=1}^s \Omega(x_i)y_i\\ &\quad = \textstyle\sum_{i=1}^s \Omega(x_i)y_i - \sum_{i=1}^s \Omega(x_i)y_i = 0.\end{aligned}$$

Further each $\Gamma(x_i)$ is either 0 or $\Gamma(x_i) \in X$ by Lemma 5.5.(2). Hence each $\Gamma(x_i)y_i$ is a monomial of X provided $\Gamma(x_i)y_i \neq 0$. Therefore we can choose $x_1, x_2, \cdots, x_t$ and $y_1, y_2, \cdots, y_t$ such that

(a) $x_i \in X$ and y_i is a monomial of X for all i.

(b) $\sum_{i=1}^s \Omega(x_i)y_i = \delta_{\Delta_1,\Omega}$ for all $\Omega \in P_1$.

Elements $z_1, z_2, \cdots, z_u$ and $w_1, w_2, \cdots, w_u$ can be choose by the similar way.

(3) It is clear that $R_0 \subseteq B_1$ by Lemma 5.5.(3). Since $\Lambda\Delta_1(bx_i) = \Delta_1\Delta(bx_i) = \Delta_1(\Lambda(b)\Lambda(x_i)) = \Delta_1(bx_i)$ for any $b \in B_1$ and $\Lambda \in P(1)$, we have $\Delta_1(bx_i) \in A$. Hence

$$\begin{aligned}R_0 &\supseteq \textstyle\sum_{i=1}^t Ay_i \ni \sum_{i=1}^t \Delta_1(bx_i)y_i\\ &\quad = \textstyle\sum_{\Omega\leq\Delta_1}(\sum_{i=1}^t g(\Delta_1, \Omega)(b)\Omega(x_i)y_i) = g(\Delta_1, \Delta_1)(b) = b\end{aligned}$$

for all $b \in B_1$ since $g(\Delta_1, \Delta_1) = 1$ show that $R_0 = B_1$.

Next, for $b \in B$, $\Delta_1(bx_i) \in B_0$, and so

$$\begin{aligned}R &\supseteq \textstyle\sum_{i=1}^t B_0y_i \ni \sum_{i=1}^t \Delta_1(bx_i)y_i\\ &\quad = \textstyle\sum_{\Omega\leq\Delta_1}(\sum_{i=1}^t g(\Delta_1, \Omega)(b)\Omega(x_i)y_i) = g(\Delta_1, \Delta_1)(b) = b\end{aligned}$$

show that

$$R = \textstyle\sum_{i=1}^t B_0y_i.$$

(4) Assume $P(1)=\{1\}$. Then $B_1=B$ and so $B=\sum_{i=1}^{t}Ay_i$. Moreover

$$\begin{aligned}\textstyle\sum_{j=1}^{u}z_jA \ni \sum_{j=1}^{u}z_j\Delta_1(w_jb) &= \textstyle\sum_{\Omega\leq\Delta_1}(\sum_{j=1}^{u}z_jg(\Delta_1,\Omega)(w_j)\Omega(b))\\ &= \textstyle\sum_{j=1}^{u}z_j\Delta_1(w_j)b=b\end{aligned}$$

for all $b\in B\ (=B_1)$ show that $B=\sum_{j=1}^{t}z_jA$.

6. The case of algebras. In this section we assume that P satisfies conditions (i) and (ii) of §5, A is a commutative ring and B is an A-algebra.

Let B and B' be A-algebras. For finite posets $P\subseteq \mathrm{End}(B_A)$ and $P'\subseteq \mathrm{End}(B'_A)$, $P\otimes P':=\{\Omega\otimes\Omega':\ \Omega\in P,\Omega'\in P'\}$ becomes a finite poset of End $((B\otimes_A B)_A)$ by $(\Omega\otimes\Omega')(\sum b\otimes b')=\sum(\Omega(b)\otimes\Omega'(b'))$ where the order $\Omega_1\otimes\Omega_1'\geq\Omega_2\otimes\Omega_2'$ is defined by $\Omega_1\geq\Omega_2$ and $\Omega_1'\geq\Omega_2'$. Assume $\Omega\otimes\Omega'=0$ only if $\Omega=0$ or $\Omega'=0$. If P and P' satisfy (A.1)-(A.4), then $P\otimes P'$ also satisfies the conditions. Since

$$\begin{aligned}(\Omega\otimes\Omega')(xy\otimes x'y') &= \Omega(xy)\otimes\Omega'(x'y')\\ &= \textstyle(\sum_{\Gamma\leq\Omega}g(\Omega,\Gamma)(x)\Gamma(y))\otimes(\sum_{\Gamma'\leq\Omega'}g(\Omega',\Gamma')(x')\Gamma'(y'))\\ &= \textstyle(\sum_{\Gamma\leq\Omega,\Gamma'\leq\Omega'}(g(\Omega,\Gamma)\otimes g(\Omega',\Gamma'))(x\otimes x')(\Gamma\otimes\Gamma')(y\otimes y'),\end{aligned}$$

we put $g(\Omega\otimes\Omega',\Gamma\otimes\Gamma')=g(\Omega,\Gamma)\otimes g(\Omega',\Gamma')$ for $\Gamma\otimes\Gamma'\leq\Omega\otimes\Omega'$. Then $(\Omega\otimes\Omega')((x\otimes x')\ (y\otimes y'))=\sum_{\Gamma\otimes\Gamma'\leq\Omega\otimes\Omega'}g(\Omega\otimes\Omega',\ \Gamma\otimes\Gamma')\ (x\otimes x')\ (\Gamma\otimes\Gamma')\ (y\otimes y')$.

Thus $P\otimes P'$ becomes a r.s.h if $\Omega\otimes\Omega'=0$ implies $\Omega=0$ or $\Omega'=0$. Moreover P and P' satify (A.5) and (A.6) so does $P\otimes P'$.

Let B/A be a P-Galois extension. Then B_A is a progenerator, and hence $B^*(B)=A$ [see [3]]. Since $B^*=j(\Delta\cdot B)$ by Theorem 3.4, we can choose an element $x\in B$ such that $\Delta(x)=1$. Hence $B_A\oplus>A_A$ by Corollary 3.10. Thus, if B/A is a P-Galois extension then A_A is a direct summand of B_A.

Theorem 6.1. *Let* $m_{\Delta_1}=m_{\Delta_1'}=1$, $g(\Delta_1,\Delta_1)=1$ *and* $g(\Delta_1',\Delta_1')=1$. *If* B/A *is a* P*-Galois extension and* B'/A *is a* P'*-Galois extension, then* $P\otimes P'$ *is a r.s.h. for* $B\otimes_A B'/A$ *and* $B\otimes_A B'/A$ *is a* $P\otimes P'$*-Galois extension.*

Proof. B/B_0 (resp. B'/B_0') is a P_1 (resp. P_1')-Galois extension by Theorem 4.2 and B_0/A (resp. B_0'/A) is a $P(1)$ (resp. $P_1'(1)$)-Galois extension by Theorem 5.2. Assume $0\neq\Omega\in P$. Then there exists an element $x_\Omega\in B$ such that $\Omega(x_\Omega)=1$. Hence $\Omega\otimes\Omega'=0$ only if $\Omega'=0$ for $\Omega'\in P'$. Thus $P\otimes P'$ is a r.s.h.

Let $x_{\Delta_1} \in B$ be $\Delta_1(x_{\Delta_1}) = 1$. For $b \otimes b' \in (B \otimes_A B')^{P \otimes P'}$,

$$\begin{aligned} B_0 \otimes B' \ni \Delta_1(x_{\Delta_1} b) \otimes b' &= \textstyle\sum_{\Gamma \in P} g(\Delta_1, \Gamma)(x_{\Delta_1}) \Gamma(b) \otimes b' \\ &= \textstyle\sum_{\Gamma \in P} (g(\Delta_1, \Gamma) \otimes 1)(x_{\Delta_1} \otimes 1)(b \otimes b') = b \otimes b'. \end{aligned}$$

By the same way, we can also see that $b \otimes b' \in B \otimes B'_0$. Noting that $B \otimes_A B'_0$ and $B_0 \otimes_A B'$ are direct summands of $B \otimes_A B'$, we have $b \otimes b' \in B_0 \otimes B'_0$.

Next, let $y \in B_0$ be an element such that $T(y) = 1$. Then

$$\begin{aligned} A \otimes_A B' \ni T(yb) \otimes b' &= \textstyle\sum_{\Lambda \in P(1)} \Lambda(y) \Lambda(b) \otimes b' \\ &= \textstyle\sum_{\Lambda \in P(1)} (\Lambda \otimes 1)(y \otimes 1)(\Lambda \otimes 1)(b \otimes b') \\ &= \textstyle\sum_{\Lambda \in P(1)} (\Lambda(y) \otimes 1)(b \otimes b') = T(y)(b \otimes b') \\ &= b \otimes b'. \end{aligned}$$

We have $b \otimes b' \in B \otimes_A A$ by the similar way. Therefore $b \otimes b' \in A$. Further this is true for $\sum_j b_j \otimes b'_j \in (B \otimes_A B')^{P \otimes P'}$. For a P-Galois system $\{x_i, y_i ; i = 1, 2, \cdots, t\}$ for B/A and a P'-Galois system $\{x'_i, y'_i ; i = 1, 2, \cdots, t'\}$ for B'/A, $\{(x_i \otimes x'_j), (y_i \otimes y'_i) ; i = 1, 2, \cdots, t$ and $j = 1, 2, \cdots, t'\}$ forms a $P \otimes P'$-Galois system for $(B \otimes_A B')/A$.

Finally, we assume that B/A is a commutative P-Galois extension.

Corollary 6.2. *Let $B^P = A$ and $g(\Delta_1, \Delta_1) = 1$. Then the following conditions are equivalent.*

(1) *B/A is a P-Galois extension.*

(2) *B/B_0 is a P_1-Galois extension and B_0/A is a $P(1)$-Galois extension.*

(3) *B/B_1 is a $P(1)$-Galois extension and B_1/A is a P_1-Galois extension.*

Proof. (1) $\Longrightarrow$ (2). Let $\{x_i, y_i ; i = 1, 2, \cdots, s\}$ be a P-Galois system for B/A. Then, for each $\Omega_1 \in P_1$, $\sum_{i=1}^{s} x_i g(\Delta_1, \Omega_1)(y_i) = \sum_{i=1}^{s} x_i g(\Delta, \Omega_1)(y_i) = \delta_{1,\Omega_1}$ shows that $\{x_i, y_i ; i = 1, 2, \cdots, s\}$ is also a P_1-Galois system for B/B_0, and hence, B/B_0 is a P_1-Galois extension. Moreover, B_0/A is a $P(1)$-Galois extension by Theorem 5.2.(1).

(2) $\Longrightarrow$ (3). B_0 has a $P(1)$-Galois system and it is also that for B/B_1. Thus B/B_1 is a $P(1)$-Galois extension. Next, if B/B_0 is a P_1-Galois extension and B_0/A is a $P(1)$-Galois extension, then there exist $x \in B$ and $b_0 \in B_0$ such that $\Delta_1(x) = 1$ and $T(b_0) = 1$. Then $T\Delta_1(xb_0) = T(\sum_{\Gamma \leq \Delta_1} g(\Delta_1, \Gamma)(x)\Gamma(b_0)) = T(b_0) = 1$. Since $T(xb_0) \in B_1$ and $T\Delta_1 = \Delta_1 T$, there exists $y \in B_1$ such that $\Delta_1(y) = 1$. Hence there exists a system $\{u_i, v_i ; i = 1, 2, \cdots, t\}$ in B_1 such that $\sum_{i=1}^{t} u_i \Omega(v_i) = \delta_{\Delta_1,\Omega}$ for all $\Omega \in P_1$ by Theorem 5.6.(2). Then this system $\{u_i, v_i ; i$

$= 1, 2, \cdots, t\}$ is a P_1-Galois system for B_1/A. For, any $b \in B$,

$$\textstyle\sum_{i=1}^t u_i \Delta_1(v_i b) = \sum_{i=1}^t u_i \Delta_1(bv_i) = \sum_{i=1}^t u_i (\sum_{\Gamma \in P_1} g(\Delta_1, \Gamma)(b)\Gamma(v_i)) = b.$$

Hence $\sum_{i=1}^t u_i(\sum_{\Gamma \neq 1} g(\Delta_1, \Gamma)(v_i)\Gamma(b)) = 0$ for all $b \in B$, and this means that

$$\textstyle\sum_{i=1}^t u_i g(\Delta_1, \Gamma)(v_i) \cdot \Gamma = 0 \quad \text{for any } \Gamma \neq 1, \quad \text{and so,}$$
$$\textstyle\sum_{i=1}^t u_i g(\Delta_1, \Gamma)(v_i) = 0 \qquad \text{for any } \Gamma \neq 1.$$

(3) $\Longrightarrow$ (1). $B^P = A$ is clear. Let $\{x_i, y_i\,;\ i = 1, 2, \cdots, s\}$ be a $P(1)$-Galois system for B/B_1 and let $\{u_j, v_j\,;\ j = 1, 2, \cdots, t\}$ be a P_1-Galois system for B_1/A. Let $\Gamma = \Lambda\Gamma_1$ for $\Lambda \in P(1)$ and $\Gamma_1 \in P_1$. Then

$$\textstyle\sum_{i=1}^s x_i(\sum_{j=1}^t u_j g(\Delta, \Gamma)(v_j y_i)) = \sum_{i=1}^s x_i(\sum_{j=1}^t u_j g(\Delta_1, \Gamma_1)(\Lambda(v_j)\Lambda(y_i)))$$
$$= \begin{cases} 0 & \text{if } \Lambda \neq 1 \\ \sum_{j=1}^t u_j g(\Delta_1, \Gamma_1)(v_j) & \text{if } \Lambda = 1. \end{cases}$$

Further $\sum_{j=1}^t u_j g(\Delta_1, \Gamma_1)(v_j) = \delta_{1,\Gamma_1}$. Consequently, we have

$$\textstyle\sum_{i=1}^s x_i(\sum_{j=1}^t u_j g(\Delta, \Gamma)(v_j y_i)) = \delta_{1,\Gamma}$$

and this shows that the existence of a P-Galois system for B/A.

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