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ON THE (2, 3)-CLOSURES OF IDEALS

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Let R be a Noetherian integral domain and let I be an ideal of R. Assume that the integral closure R' of R in its quotient field is a finite R-module. Let $R[t^{-1}, It]$ denote the generalized Rees ring, where t is an indeterminate. In [9], it is shown that if $R[t^{-1}, It]$ is seminormal then I^k is equal to the relevant component $(I^k)^* := \bigcup\{I^{i+k} : I^i ;\ i \geqq 1\}$ for all $k \in N$ (where N denotes the set of integers $\geqq 1$). But the converse statement are not necessarily valid. So it seems natural to ask when $R[t^{-1}, It]$ is seminormal. In this paper, we define the (2, 3)-closure I' of I in R, which has the following properties:

(a) I' contains I,

(b) I is (2, 3)-closed in R (i.e., $a^2 \in I^2$, $a^3 \in I^3$ $(a \in R)$ imply $a \in I$) if and only if $I = I'$,

(c) I' is (2, 3)-closed in R,

(d) I is a reduction of I', that is, $I(I')^n = (I')^{n+1}$ for all large n,

(e) $I \subset I' \subset I_a$, where I_a denotes the integral closure of I.

Our objectives of this paper are to investigate the relations among I, I^* and I', to study the relations between the seminormality of R and (2, 3)-closedness of divisorial ideals and to determine the seminormalization of the (generalized) Rees ring.

Unless otherwise specified, let R be a *Noetherian domain*, let R' denote the integral closure of R in its quotient field K, let I be an ideal of R and let B be an intermediate ring between R and R' which is a *finite* R-module. Our unexplained technical terms are standard and are seen in [4].

1. Definitions and Basic Properties of (2, 3)-Closures of Ideals. Let t denote an indeterminate, and we call $R[It]$ (resp. $R[t^{-1}, It]$) the *Rees ring* (resp. the *generalized Rees ring*) of R with respect to I. The ring $R[t^{-1}, It]$ is a subring of the torus extension $R_R = R[t^{-1}, t]$ of R. After Mirbagheri and Ratliff [7], we call the ideal $I^* := \bigcup\{I^{i+1} : I^i ;\ i \geqq 1\}$ the *relevant component* of I.

Let ${}^B\Delta_\#(I) := \{a \in B ;\ a^2 \in I^2, a^3 \in I^3\}$ and let ${}_BI_\#$ denote an R-submodule of B generated by ${}^B\Delta_\#(I)$. When $I = R$, we denote ${}_BR_\#$, an R-module generated by ${}^B\Delta_\#(R) = \{a \in B ;\ a^2 \in R, a^3 \in R\}$. We claim that ${}_BR_\#$ is an R-subalgebra of B and ${}_BI_\#$ is an ideal of ${}_BR_\#$. Indeed, since ${}_BR_\#$ is an R-module generated by ${}^B\Delta_\#(R)$, for $x, y \in {}^B\Delta_\#(R)$, $x+y \in {}_BR_\#$ is trivial. Next $(xy)^2 =$

$x^2y^2 \in R$ and $(xy)^3 = x^3y^3 \in R$ imply $xy \in {}^B\Delta_*(R)$. Since for any elements $\alpha = \sum r_i x_i$ and $\beta = \sum t_i y_i$ in ${}_BR_*$ with $x_i, y_i \in {}^B\Delta_*(R)$, $r_i, t_i \in R$, $\alpha\beta$ is expressed as a linear combination of $x_i y_j$ over R, the above argument shows that ${}_BR_*$ is an R-algebra. Since ${}_BI_*$ is an R-module generated by ${}^B\Delta_*(R)$, for $x, y \in {}^B\Delta_*(I)$, $x+y \in {}_BI_*$ is trivial. For $z \in {}^B\Delta_*(R)$, $(zx)^2 = z^2x^2 \in {}_BI_*$ and $(zx)^3 = z^3x^3 \in {}_BI_*$. So $zx \in {}^B\Delta_*(I)$. By the same reason as above, we conclude that ${}_BI_*$ is an ideal of ${}_BR_*$. Let ${}^B\Delta_0(I) := {}^B\Delta_*(I)$, ${}_BI_* = {}_BI_0$ and ${}_BR_0 = {}_BR_*$. Once ${}_BR_i$ and ${}_BI_i$ are defined, we put ${}^B\Delta_{i+1}(I) := {}^B\Delta_*({}_BI_i) = \{\alpha \in B\,;\ \alpha^2 \in ({}_BI_i)^2,\ \alpha^3 \in ({}_BI_i)^3\}$ and let ${}_BI_{i+1}$ denote an R-submodule ${}_B({}_BI_i)_*$ of B generated by ${}^B\Delta_{i+i}(I)$. It is clear that ${}^B\Delta_i(I) \subset {}^B\Delta_{i+1}(I)$. Then by induction, we can easily see that ${}_BR_{i+1}$ is an R-subalgebra of B and ${}_BI_{i+1}$ is an ideal of ${}_BR_{i+1}$. Since B is a finite R-module, the ascending chain of R-submodules of B:

$$I \subset {}_BI_0 \subset \cdots \subset {}_BI_i \subset \cdots$$

terminates, that is, there exists an integer N such that for all $n \geqq N$ ${}_BI_n = {}_BI_{n+1} = \cdots$. Put ${}_BI := {}_BI_n$ for such n. When $I = R$, we employ the notation ${}_BR$ for ${}_BI$. When $B = R'$ (here R' is assumed to be a finite R-module), we use the notation $I^*{}_i$ for ${}_BI_i$, I^* for ${}_BI$ and R^* for ${}_BR$. Moreover *when* $B = R$, we denote ${}_BI_i$ by I_i, ${}_BI$ by I'. We denote also $I_{i+1} = (I_i)_*$. Note here that I_i and I' are ideals of R by definition. Thus we obtain the following lemma.

Lemma 1.1. (i) *${}_BR_i$ is an R-subalgebra of B for all $i \in N$;*
(ii) *${}_BI_i$ is an ideal of ${}_BR_i$ for all $i \in N$;*
(iii) *${}_BI$ is an ideal of an R-algebra ${}_BR$.*

We call ${}_BI$ the *(2, 3)-closure* of I in B and I' the *(2, 3)-closure* of I in R. When $I = {}_BI$ (resp. $I = I'$), we say that I is *(2, 3)-closed in B* (resp. I is *(2, 3)-closed in R*). Since R is an integral domain, the ideal (0) is (2, 3)-closed. More generally it is obvious that any radical ideal is (2, 3)-closed in R by Proposition 1.4 below.

Hereafter in this section, we treat only the case $B = R$.

Proposition 1.2. *The following statements hold:*
(i) *$(I')' = I'$, i.e., I' is (2, 3)-closed;*
(ii) *for any ideal J of R satisfying $I \subset J \subset I'$, we have $J' = I'$;*
(iii) *$(I_0)^n \subset (I^n)_0$ and $(I')^n \subset (I^n)'$ for all $n \in N$.*

Proof. (i) By construction, $I' = I_m$ for all large $m \in N$. Hence $(I')_* = (I_m)_* = I_{m+1} = I'$, which implies that $(I')' = I'$.

(ii) follows from (i).

(iii) Let $a = a_1 \cdots a_n$ be an element in R with $a_i \in \Delta_0(I)$. Then $a^2 = a_1^2 \cdots a_n^2 \in I^{2n}$ and $a^3 = a_1^3 \cdots a_n^3 \in I^{3n}$ and hence $a \in (I^n)_0$. Since I_0 is generated by $\Delta_0(I)$, any element in $(I_0)^n$ is a linear combination of products of n elements in $\Delta_0(I)$ over R. Thus by the preceding argument shows that $(I_0)^n \subset (I^n)_0$. Replace I by I_i in this inclusion and we get $((I_i)_*)^n \subset ((I_i)^n)_*$. So $(I_{i+1})^n = ((I_i)_*)^n \subset ((I_i)^n)_* \subset ((I_i)^n)'$, that is, $(I_{i+1})^n \subset ((I_i)^n)'$. Thus $(I_{i+1})^n \subset ((I_i)^n)' \subset (((I_{i-1})^n)')' \subset ((I_{i-1})^n)'$ by (i), and consequently we have $(I_{i+1})^n \subset ((I_0)^n)' \subset ((I^n)_0)' = (I^n)'$. Since $I_m = I'$ for large m, we have $(I')^n \subset (I^n)'$ for all $n \in N$.

Proposition 1.3. *Let J be an ideal generated by the set $\{a \in R\,;\ a^k \in I^k$ for all large $k \in N\}$. Then $I' \subset J$, $I^* \subset J$ and $\sqrt{I} = \sqrt{I'} = \sqrt{I^*} = \sqrt{J}$.*

Proof. Let J be the ideal generated by $\{a \in R\,;\ a^k \in I^k$ for all large $k \in N\}$. Then it is obvious that $I' \subset J$ because for all large k, $k = 2m+3n$ for some $m, n \in N$. Take $a \in I^*$. Then $a^k \in (I^*)^k = I^k$ for all large k by [8, (2.1)]. Hence $a \in J$. The second assertion follows from $\sqrt{I} = \sqrt{J}$.

Proposition 1.4. *The following statements are equivalent:*

(a) $I = I'$ *i.e., I is (2, 3)-closed in R;*

(b) $\Delta_0(I) \subset I$;

(c) $I_0 = I$;

(d) $a^2 \in I^2$, $a^3 \in I^3$ $(a \in R)$ *imply* $a \in I$.

Proof. (b) $\Longleftrightarrow$ (c) is trivial because I_0 is generated by $\Delta_0(I)$ over R.

(a) $\Longrightarrow$ (c) is trivial because $I \subset I_0 \subset I'$.

(c) $\Longrightarrow$ (d) I_0 is an ideal generated by $\Delta_0(I)$. Hence $a^2 \in I^2$, $a^3 \in I^3$ $(a \in R)$ imply $a \in \Delta_0(I) \subset I_0 \subset I$.

(d) $\Longrightarrow$ (c): We have only to show that $\Delta_0(I) \subset I$ since I_0 is an ideal generated by $\Delta_0(I)$. But this is given by the condition in (d).

(c) $\Longrightarrow$ (a): $I_0 = I$ yields that $I_i = I$ for all i by the definition. So $I' = I_i = I$.

Recall that the *integral closure* I_a of I in R is the set of elements x in R that satisfy an equation of the form $x^n + a_1x^{n-1} + \cdots + a_n = 0$, where $a_i \in I_i$ for $i = 1, \cdots, n$. Also recall that I is said to be normal in case $(I^i)_a = I^i$ for all $i \geqq 1$. It is shown in [7, p.34] that $J \subset I \Rightarrow J_a \subset I_a$ and $(I_a)_a = I_a$.

In the following theorem, we summarize some basic properties of (2, 3)-closures of ideals.

Theorem 1.5. *The following statements hold :*

(i) $I' \subset I_a$;

(ii) $(I_P)' = (I')_P$ *for any* $P \in \mathrm{Spec}(R)$;

(iii) *I is a reduction of I', that is,* $I(I')^n = (I')^{n+1}$ *for all large* n ;

(iv) *I is (2, 3)-closed in R if and only if so is I_m in R_m for each maximal ideal m of R.*

Proof. (i) It is clear that $I_i \subset (I_{i-1})_a$ by the definition. Hence $I_n \subset (I_{n-1})_a \subset (I_{n-2}) \subset \cdots \subset I_a$ by the preceding paragraph. Since $I' = I_n$ for all large n, we have $I' \subset I_a$.

(ii) We prove $(I_i)_P = (I_P)_i$ by induction on i. Take $a/s \in \Delta_i(I_P) \subset R_P$ ($a \in R, s \in R \backslash p$). Then $(a/s)^2 \in ((I_P)_{i-1})^2 = ((I_{i-1})_P)^2$ and $(a/s)^3 \in ((I_P)_{i-1})^3 = ((I_{i-1})_P)^3$; so that $a/s \in \Delta_i(I)R_P$. Thus $(I_P)_i = (I_i)_P$. The converse is similar. Since $I_n = I'$ for all large n, we have our conclusion.

(iii) We need to prove I_{i-1} is a reduction of I_i. For this we have only to show that I is a reduction of I_0. Put $I_0 = (a_1, \cdots, a_r)R$ with $a_i \in \Delta_0(I)$ and let $J = (a_1^2, \cdots, a_r^2)R$. Then J is a reduction of $(I_0)^2$ by [8, (2.8.2)], that is, $J(I_0^2)^n = (I_0^2)^{n+1}$ for all large n. Since $J \subset I^2$, we have $I(I_0)^{2n+1} = I_0^{2n+2}$ for all large n. Thus $I(I_0)^m = I_0^{m+1}$ for all large m, which shows that I is a reduction of I_0. Similarly we can prove I_{i-1} is a reduction of I_i for all i, and since $I_n = I'$ for a large n, I is a reduction of I'.

(iv) is clear by (ii).

Proposition 1.6. *Any intersection of ideals which are (2, 3)-closed in R is also (2, 3)-closed in R.*

Proof. We have only to show that $\bigcap J_i$ is (2, 3)-closed in R for (2, 3)-closed ideals J_i. Since $(\bigcap J_i)_0$ is generated by $\Delta_0(\bigcap J_i)$, we need only to prove $\Delta_0(\bigcap J_i) \subset \bigcap J_i$. Take $\alpha \in \Delta_0(\bigcap J_i)$. Then $\alpha^2 \in (\bigcap J_i)^2$ and $\alpha^3 \in (\bigcap J_i)^3$. Since $\bigcap J_i \subset J_i$ implies $(\bigcap J_i)^2 \subset \bigcap J_i^2$ and $(\bigcap J_i)^3 \subset \bigcap J_i^3$, we have $\alpha^2 \in \bigcap J_i^2$ and $\alpha^3 \in \bigcap J_i^3$. Since J_i is (2, 3)-closed in R, $\alpha \in J_i$, that is, $\alpha \in \bigcap J_i$. Thus $(\bigcap J_i)_0 = \bigcap J_i$. By Proposition 1.4, $\bigcap J_i$ is (2, 3)-closed in R.

Proposition 1.7. *Let b be an element in R. Then*

(a) *if I is (2, 3)-closed, then* $I :_R b := \{a \in R ; ab \in I\}$ *is (2, 3)-closed* ;

(b) *if the ideal bI is (2, 3)-closed in R, then I is (2, 3)-closed in R.*

Proof. (a) First we prove that $(I :_R b)^2 \subset I^2 :_R b^2$ and $(I :_R b)^3 \subset I^3 :_R b^3$. Let $I :_R b = (x_1, \cdots, x_r)$. Then $(I :_R b)^2$ (resp. $(I :_R b)^3$) is generated by $\{x_i x_j\}$

(resp. $\{x_ix_jx_k\}$). So we have only to show that $x_ix_j \in I^2 :_R b^2$ and $x_ix_jx_k \in I^3 :_R b^3$. Since $x_ix_jb^2 = (x_ib)(x_jb) \in I \cdot I = I^2$ and $x_ix_jx_kb^3 = (x_ib)(x_jb)(x_kb) \in I^3$. Thus $x_ix_j \in I^2 :_R b^2$ and $x_ix_jx_k \in I^3 :_R b^3$. Next take $\alpha \in (I :_R b)_0$. Then we may assume that α belongs to $\varDelta_0(I :_R b)$ since $(I :_R b)_0$ is generated by $\varDelta_0(I :_R b)$. So $\alpha^2 \in (I :_R b)^2$ and $\alpha^3 \in (I :_R b)^3$. By the previous argument, we have $\alpha^2 \in I^2 :_R b^2$ and $\alpha^3 \in I^3 :_R b^3$, and hence $\alpha^2 b^2 \in I^2$ and $\alpha^3 b^3 \in I^3$. Thus $\alpha b \in I$ because I is (2, 3)-closed in R, which implies that $\alpha \in I :_R b$. Therefore $(I :_R b)_0 = I :_R b$. By Proposition 1.4, we conclude that $I :_R b$ is (2, 3)-closed in R.
(b) This is clear because $(bI :_R b) = I$ and (a).

Corollary 1.7.1. *If I is (2, 3)-closed in R, then any isolated primary component q of I is (2, 3)-closed in R.*

Proof. Let $I = q_1 \cap \cdots \cap q_r$ be an irredundant primary decomposition of I. We may assume that $q = q_1$. Put $p = \sqrt{q}$. Then there exists an element $x \in q_2 \cap \cdots \cap q_r \backslash p$. Thus $I :_R x = (q_1 :_R x) = \cap \cdots \cap (q_r :_R x) = q_1 :_R x$. Sinbe $x \notin p$, we have $q_1 :_R x = q_1$. Hence $q = q_1 = I :_R x$ is (2, 3)-closed in R by Proposition 1.7.

Corollary 1.7.2. *Assume that I has no embedded prime divisors. Let $I = q_1 \cap \cdots \cap q_r$ be an irredundant primary decomposition. Then I is (2, 3)-closed in R if and only if q_i is (2, 3)-closed in R for all $i = 1, \cdots, r$.*

Proof. This follows from Proposition 1.6 and Corollary 1.7.1.

2. Seminormal Domains and (2, 3)-Closed Ideals. When R is (2, 3)-closed in B, that is, $\alpha^2, \alpha^3 \in R$ for $\alpha \in B$ implies that $\alpha \in R$, we say that R is (2, 3)-closed in B. When R is (2, 3)-closed in R', we say that R is *seminormal*.

The following proposition asserts that the converse statement of Proposition 1.6(b) holds if R is seminormal.

Proposition 2.1. *If R is seminormal and I is (2, 3)-closed in R, then bI is (2, 3)-closed in R for any $b \in R$.*

Proof. Take $\alpha \in \varDelta_0(bI)$. Then $\alpha^2 \in (bI)^2$ and $\alpha^3 \in (bI)^3$. So $(\alpha/b)^2 \in I^2 \subset R$ and $(\alpha/b)^3 \in I^3 \subset R$. Since R is seminormal, α/b belongs to R. Hence $\alpha \in bI$, which implies that bI is (2, 3)-closed in R.

Corollary 2.1.1. *Assume that R is seminormal. If I is (2, 3)-closed in R,* *any ideal J of R which is R-isomorphic to I is (2, 3)-closed in R.*

Proof. Since $\mathrm{Hom}_R(I, J) \subset \mathrm{Hom}_R(I, J)_R \otimes K = \mathrm{Hom}_K(K, K) = K$ (where K denotes the field of fractions of R), any R-isomorphism of I into J is the multiplication of an element α in K, that is, $\alpha I = J$. Put $\alpha = c/d$ with $c, d \in R$. Then $cI = dJ$. Since I is (2, 3)-closed in R, $cI = dJ$ is (2, 3)-closed in R by Proposition 2.1. Hence J is (2, 3)-closed in R by Proposition 1.7(b).

Remark. It is known and is not hard to see that $R = R'$ if and only if any principal ideal of R is integrally closed, i.e., $(bR)_a = bR$ for all $b \in R$.

The next proposition shows that the similar argument in the above Remark is valid for (2, 3)-closedness.

Proposition 2.2. *The following statements are equivalent:*
(a) *R is seminormal;*
(b) *$(aR)' = aR$ for any non-unit a in R;*
(c) *$(aR)' = aR$ for any element a in R;*
(d) *$(aR)'$ is a principal ideal for any element a of R.*

Proof. Note first that $aR = (aR)' \Longleftrightarrow aR = (aR)_0$ by Proposition 1.4.

(a) $\Longrightarrow$ (b): Since R itself is (2, 3)-closed in R, aR is (2, 3)-closed in R by Proposition 2.1.

(b) $\Longrightarrow$ (a): Take $\alpha \in R'$ with $\alpha^2 \in R$ and $\alpha^3 \in R$. Put $\alpha = b/a$ with a, $b \in R$. If a is a unit in R, then $\alpha \in R$. Suppose a is not a unit in R. Then $b^2 \in a^2R$ and $b^3 \in a^3R$, so that $b \in aR$ by the assumption. Hence $\alpha = b/a \in R$. We conclude that R is seminormal.

(c) $\Longleftrightarrow$ (b) and (b) $\Longrightarrow$ (d) are trivial.

(d) $\Longrightarrow$ (b): By Theorem 1.5, we may assume that R is a local domain with the maximal ideal m. Let $(aR)' = bR$ and let $x_1, \cdots, x_r$ be elements in $\Delta_0(aR)$ which generates $(aR)'$ over R. Then there exist x_i such that $(aR)' = x_iR$. Indeed, take $x_i \in (aR)' \backslash (aR)'m$. Then the image x_i in $(aR)'/(aR)'m \simeq bR/bm \simeq R/m$ is a basis over the field R/m. From this, $x_iR + bm = bR$ and hence $x_iR = bR = (aR)'$ by Nakayama's lemma. Put $x = x_i$. Since $a \in bR = xR$, we have $a = xc$ for some $c \in R$. Since $x \in \Delta_0(aR)$, we have $x^2 = a^2r \in a^2R$ for some $r \in R$. Thus $x^2 = x^2c^2r$ implies that c is a unit in R. Thus $(aR)' = xR = aR$.

Relating to Proposition 1.7 and Propositions 2.1 and 2.2, we refer to the example raised in [9].

Example. Let $R = k[X^2, X^3]$ be a subring of a polynomial ring $k[X]$ over a field k and let I be an ideal X^2R. It is easy to see that R is not seminormal and that $(X^3)^2 \in I^2$ and $(X^3)^3 \in I^3$; hence $X^3 \in I'$ but X^3 does not belong to I. Since $I^* = I$, we have $I^* \neq I'$. Moreover it is clear that $(I^n)' \neq I^n$ for all $n \in N$ because $X^{2n+1} \notin I^n$ but $X^{2n+1} \in (I^n)'$.

Corollary 2.2.1. *Assume that I is invertible and R is seminormal. Then $(I^n)' = I^n$ for all $n \in N$.*

Proof. Note that R is seminormal if and only if R_P is seminormal for each $P \in \mathrm{Spec}(R)$ and that I being invertible imples that I is locally principal. So our conclusion follows from Theorem 1.5(ii) and Proposition 2.2.

Corollary 2.2.2. *Assume that R is a Dedekind domain. Then any ideal of R is (2, 3)-closed in R.*

We denote by K the field of fractions of R, and $R :_R \alpha$ denotes the denominator ideal $\{a \in R;\ a\alpha \in R\}$ of $\alpha \in K$.

Lemma 2.3. *Assume that R is seminormal. Then for any element α in K, the ideal $R :_R \alpha$ is (2, 3)-closed in R.*

Proof. Put $\alpha = c/d$ with $c, d \in R$ $(d \neq 0)$. Then $R :_R \alpha = dR :_R cR$. Since R is seminormal, dR is (2, 3)-closed in R by Proposition 2.2. Hence by Proposition 1.6, $R :_R \alpha = dR :_R cR$ is (2, 3)-closed in R.

An R-submodule J of K is called *fractional* if $rJ \subset R$ for some $r \in R\backslash\{0\}$. Any ideal of R is a fractional ideal of R. We say that a fractional ideal J of R is *divisorial* if $R :_K (R :_K J) = J$. It is known that J is divisorial if and only if J is the intersection of principal fractional ideals of R. For $\alpha \in K$, the denominator ideal $R :_R \alpha$ is divisorial ideal of R. Indeed, it is obvious that $R :_R \alpha = \alpha^{-1}R \cap R$ if $\alpha \neq 0$ and $R :_R \alpha = R$ if $\alpha = 0$.

We now extend Proposition 2.2 as follows.

Theorem 2.4. *The following statements are equivalent:*

(i) *R is seminormal;*

(ii) *any principal ideal aR $(a \in R)$ is (2, 3)-closed in R;*

(iii) *any denominator ideal $R :_R \alpha$ $(\alpha \in K)$ is (2, 3)-closed in R;*

(iv) *any divisorial ideal of R is (2, 3)-closed in R.*

Proof. (ii) $\Longrightarrow$ (i) is shown in Proposition 2.2.

(iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) is clear because any principal ideal aR $(a \neq 0)$ is $R :_R a^{-1}R$ and $R :_R a$ is divisorial for any $\alpha \in K$ by the preceding argument.

(i) $\Longrightarrow$ (iv): Let I be a divisorial ideal of R $(I \subset R)$. Then $I = \bigcap(\alpha^{-1}R \cap R)$ for some α's $\in K$. We may assume that $I \neq (0)$ because (0) is (2, 3)-closed in R. Since $\alpha^{-1}R \cap R = R :_R \alpha$, $\alpha^{-1}R \cap R$ is (2, 3)-closed in R by Lemma 2.3. Hence $I = \bigcap(\alpha^{-1}R \cap R)$ is (2, 3)-closed in R by Proposition 1.5.

Proposition 2.5. *Let A be an integral domain containing R and let I be an ideal of R. Assume that A is faithfully flat over R. If IA is (2, 3)-closed in A, then I is (2, 3)-closed in R.*

Proof. We have only to show that $I_0 = I$ by Proposition 1.4. For this we must show that $\Delta_0(I) \subset I$. Take $\alpha \in \Delta_0(I)$. Then $\alpha^2 \in I^2 \subset (IA)^2$ and $\alpha^3 \in I^3 \subset (IA)^3$ and hence $\alpha \in IA$ because IA is (2, 3)-closed in A. Since A is faithfully flat over R, $\alpha \in IA \cap R = I$.

We close this section by showing what happen when $(I^k)' = I^k$ for some $k \in N$.

Proposition 2.6. *Assume that $I^k = (I')^k$ for some $k \in N$. Then*

(i) *$(I')^n = (I^*)^n = I^n$ for all large n.*

(ii) *$I' \subset I^*$.*

Proof. (i) By Theorem 1.5 (iii), I is a reduction of I'. So for any $i \in N$ with $1 \leqq i \leqq k$ we have $I^i(I')^r = (I')^{r+i}$ for all large r. For a large m, consider the case $r = mk$. Then $(I')^{km+i} \subset I^i(I')^{km} \subset I^i((I^k)')^m = I^i(I^k)^m = I^{km+i}$, where we use Proposition 1.2 in the second inclusion. Since $I \subset I'$ implies $I^{km+i} \subset (I')^{km+i}$, we have $(I')^{km+i} = I^{km+i}$ $(1 \leqq i \leqq k)$ for all large m. Thus $(I')^n = I^n$ for all large n.

(ii) Since $(I')^n = (I^*)^n$ for all large $n \in N$ by (i), we have $I' \subset I^*$ by [8, (2.1)].

Corollary 2.6.1. *If I^k is (2, 3)-closed, i.e., $I^k = (I^k)'$ for some $k \in N$, then $I' \subset I^*$.*

Proof. By Proposition 1.2, we have $(I')^k \subset (I^k)'$. So $I^k \subset (I')^k \subset (I^k)' = I^*$, and consequently $I' = I^*$ by Proposition 2.6 (ii).

3. Generalized Rees Rings and (2, 3)-Closed Ideals. Throughout this section, we assume that the integral closure R' of R is a finite R-module.

Let $C = \sum_{n\in N} C_n$ be a graded domain with integral closure C' in the domain $S^{-1}C$, where S donotes the set of all non-zero homogeneous elements in C. Then the integral closure C' is a graded domain $\sum_{n\in N} C'_n$. After D. F. Anderson [1], we say that C is *almost seminormal* if whenever $x^2, x^3 \in C$ for homogeneous $x \in C'$ with $\deg x > 0$, then $x \in C$. It is known that the canonical homomorphism $\mathrm{Pic}(C_0) \to \mathrm{Pic}(C)$ is an isomorphism if and only if C is almost seminormal (cf. [1]). We also say that *a* $\boldsymbol{Z}$-graded domain $L = \sum_{n\in Z} L_n$ is almost seminormal if whenever $x^2, x^3 \in L$ for homogeneous $x \in L'$ with $\deg x \neq 0$, then $x \in L$, where $L' = \sum_{n\in Z} L_n$ is its integral closure in the domain $T^{-1}L$ with T the set of all non-zero homogeneous elements in L. It is shown that C (resp. L) is seminormal if and only if any homogeneous element $a \in C'$ (resp. L') with a^2, $a^3 \in C$ (resp. L) belongs to C (resp. L) (cf. [2]).

Proposition 3.1. *The generalized Rees ring $R[t^{-1}, It]$ is (2, 3)-closed in the torus extension $R_R = R[t, t^{-1}]$ if and only if $(I^n)' = I^n$ for all $n \in N$.*

Proof. In order to prove that $R[t^{-1}, It]$ is (2, 3)-closed in the torus extension $T_R = R[t, t^{-1}]$, we have only to show that any homogeneous element x in $R[t, t^{-1}]$ with $x^2, x^3 \in R[t^{-1}, It]$ belongs to $R[t^{-1}, It]$. Take a homogeneous element x in $R[t, t^{-1}]$ with $x^2, x^3 \in R[t^{-1}, It]$ whose degree is s. Then $x^2 \in I^{2s}t^{2s}$ and $x^3 \in I^{3s}t^{3s}$. Put $x = yt$ with $y \in R$. Since $y^2 \in I^{2s}$ and $y^3 \in I^{3s}$, $y \in (I^s)_0 \subset (I^s)' = I^s$. Hence $x = yt \in It \subset R[t^{-1}, It]$. Conversely take $y \in \Delta_i(I^n)$ for a fixed large i such that $(I^n)_i = (I^n)'$. Then $(yt^n)^2 \in ((I^n)_{i-1})^2 t^{2n}$, $(yt^n)^3 \in ((I^n)_{i-1})^3 t^{3n}$. By induction we may assume that $(I^n)_{i-1} t^n \subset R[t^{-1}, It]$. Hence $yt^n \in R[t^{-1}, It]$, which implies that $y \in I^n$. Thus $(I^n)' \subset I^n$.

Remark. It is not hard to see that the statement in Proposition 3.1 is valid for the Rees ring $R[It]$ in the polynomial ring $R[t]$.

By use of [2,Th.3], we know the following: the Rees ring $R[It]$ is seminormal if and only if R is seminormal and $R[It]$ is almost seminormal.

Proposition 3.2. *The following statements are equivalent:*

(a) *The generalized Rees ring $R[t^{-1}, It]$ is almost seminormal;*

(b) *$R[t^{-1}, It]$ is seminormal;*

(c) *R is seminormal and $(I^n)' = I^n$ for all $n \in N$.*

Proof. (a) $\Longleftrightarrow$ (c) follows from the fact that we have only to consider all homogeneous elements in the integral closure of $R[t^{-1}, It]$ as remarked above.

(a) $\Longleftrightarrow$ (b): We need to prove that R is seminormal. Since any homogeneous component of negative degree is the form Rt^{-s} $(s > 0)$, almost seminormality of $R[t^{-1}, It]$ implies that R is (2, 3)-closed in R'. The converse implication is shown by the similar argument to that in the proof of Proposition 3.1.

The following Corollary is established for the Rees ring $R[It]$ in [2].

Corollary 3.2.1. *If R is seminormal and I is an invertible ideal of R, then the generalized Rees ring $R[t^{-1}, It]$ is seminormal.*

Proof. This follows from Corollary 2.2.1 and Theorem 3.2.

The seminormalization of R in B was defined by Traverso to be

$$ {}_B{}^{+}R = \{x \in B\,;\ x/1 \in R_P + J(B_P) \text{ for all } P \in \operatorname{Spec}(R)\}, $$

where J denote the Jacobson radical. Equivalently, ${}_B{}^{+}R$ is the largest subring C of B containing R such that (i) $\operatorname{Spec}(C) \to \operatorname{Spec}(R)$ is injective and (ii) for all $Q \in \operatorname{Spec}(C)$ the canonical map of residue class fields $k(Q \cap R) \to k(Q)$ is an isomorphism. R is called seminormal in B if $R = {}_B{}^{+}R$, and R is called seminormal if it is seminormal in its integral closure R'. It is known that R is seminormal in this sense if and only if R is (2, 3)-closed in R' (cf.[3]). So our definition of seminormality by use of (2, 3)-closedness is equivalent to the one defined here.

Lemma 3.3. *Let ${}_B{}^{+}R$ denote the seminormalization of R in B. Then ${}_BR \subset {}_B{}^{+}R$.*

Proof. By induction on i, we shall show the following;

(1) The canonical map $\operatorname{Spec}({}_BR_{i-1}) \to \operatorname{Spec}({}_BR_{i-2})$ is injective (where ${}_BR_{-1} := R$);

(2) For $P \in \operatorname{Spec}({}_BR_{i-1})$, the canonical homomorphism of residue class fields $k(P \cap {}_BR_{i-2}) \to k(P)$ is an isomorphism.

For this, we have only to prove the following special case:

(1′) The canonical map $\operatorname{Spec}({}_BR_0) \to \operatorname{Spec}(R)$ is injective;

(2′) For $P \in \operatorname{Spec}({}_BR_0)$, the canonical homomorphism of residue class fields $k(P \cap R) \to k(P)$ is an isomorphism.

(1′): We may assume that $R \neq {}_BR_0$. Suppose $P \cap R = Q \cap R$ for $P \not\subset Q \in \operatorname{Spec}({}_BR_0)$. There exists $\alpha \in P$, $\alpha \notin Q$ and $\alpha^n \in R$ (Indeed, we can take such α in ${}^B\Delta_0(R)$). Hence $\alpha^n \in P \cap R = Q \cap R$; so that $\alpha \in Q$, contradiction. Thus $P \subset Q$. Similarly we get $Q \subset P$.

(2′): Since $R/P \cap R \to {}_BR_0/P$ is injective, the map $k(P \cap R) \to k(P)$ is injective. Take a non-zero element $a' \in ({}_BR_0/P)_P$. We may assume that a preimage a of a' in ${}_BR_0$ is an element in ${}^B\Delta_0(R)$. Then $a^n \in R$ for all large n. Since $a \notin P$ implies $a^n \notin P \cap R$ for all n. Thus $a' = a'^{n+1}/a'^n \in (R/P \cap R)_{P\cap R}$. Thus $({}_BR_0/P)_P \subset (R/P \cap R)_{P\cap R}$, which yields that $k(P) = k(P \cap R)$. Thus after repeating the above argument, we conclude that ${}_BR$ satisfies the condition (i) and (ii) mentioned above. Since ${}_B{}^+R$ is the largest subring of R' satisfying the same conditions (i) and (ii), we obtain that $R \subset {}_BR \subset {}_B{}^+R$.

Traverso [10] shows that ${}_B{}^+R$ has no proper subrings containing R and seminormal in B.

Proposition 3.4. *${}_BR$ is the seminormalization of R in B, that is, ${}_BR = {}_B{}^+R$.*

Proof. By the above remark by Traverso, we have only to show that ${}_BR$ is seminormal in B because ${}_BR \subset {}_B{}^+R$ by Lemma 3.3. By the definition, there exists $n \in N$ such that ${}_BR_n = {}_BR_{n+1} = \cdots = {}_BR$. If $a^2, a^3 \in {}_BR = {}_BR_n$ for $a \in B$, then $a \in {}^B\Delta_{n+1}(R)$. So $a \in {}_BR_{n+1} = {}_BR$, as was to be shown.

We close this paper by determining the seminormalization of $R[t^{-1}, It]$ and $R[It]$.

Theorem 3.5. *Let R be a Noetherian domain and let I be an ideal of R. Assume that R' is a finite R-module. Then the seminormalization of $R[It]$ (resp. $R[t^{-1}, It]$) in the integral closure $R'[t, t^{-1}]$ (resp. in $R'[t]$) is $R^*[t^{-1}, \{(I^i)^* t^i ;\ i > 0\}]$ (resp. $R^*[\{I^i)^* t^i ;\ i > 0\}]$).*

Proof. It is clear that the integral closure of $R[t^{-1}, It]$ (resp. $R[It]$) is $R'[t, t^{-1}]$ (resp. $R'[t]$). By Lemma 3.3, $R^*[t^{-1}, \{(I^i)^* t^i ;\ i > 0\}]$ (resp. $R^*[\{(I^i)^* t^i ;\ i > 0\}]$ is contained in the seminormalization of $R[t^{-1}, It]$ (resp. $R[It]$) in $R'[t, t^{-1}]$ (resp. $R'[t]$). By the same way as in the proof of Proposition 1.2, we can see that $(I^i)^*$ is (2, 3)-closed in R' for all $i \in N$. So $R^*[t^{-1}, \{(I^i)^* t^i ;\ i > 0\}]$ (resp. $R^*[\{(I^i)^* t^i ;\ i > 0\}]$) is (2, 3)-closed in $R'[t, t^{-1}]$ (resp. $R'[t]$). Thus by the Traverso's remark mentioned above, we get our conclusion.

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