ON THE ATTACHING MAP IN THE STEIFEL MANIFOLD OF 2-FRAMES

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0. Introduction. Let $F = R(\text{real}), C(\text{complex})$ or $H(\text{quaternionic})$ and $d = \dim_R F$. Let $\iota_n \in \pi_n(S^n)$ be the identity map, $\eta_n \in \pi_{n+1}(S^n)$ for $n \geq 2$ and $\nu_n \in \pi_{n-2}(S^n)$ for $n \geq 4$ the Hopf maps. Throughout the paper $O_{n,k}(F)$ stands for the Steifel manifold consisting of orthonormal $k$-frames in $F^n$, $Q_{n,k}(F) \subset O_{n,k}(F)$ does for the stunted quasiprojective space and $Q_{2n+1,2}(F) = S^{2n+1} \cup \omega_n(F)e^{2n+1}$, where $\omega_n(R) = 2\iota_{2n-1}$, $\omega_n(C) = \eta_{n-1}$ and $\omega_n(H) = (2n+1)\nu_{n-1}$. We have a cellular decomposition:

$$O_{2n+1,2}(F) = Q_{2n+1,2}(F) \cup \gamma_{n+1}F \cdot e^{(n-1)d-2}.$$

The purpose of the present note is to determine the $(d-k)$-fold suspension $\Sigma^{d-k} \gamma_n(F) \in \pi_{(d+1)n-1-k}(\Sigma^{d-k}Q_{2n+1,2}(F))$ for $0 \leq k \leq d$. We shall freely use the notation and results of [16], [10] and [11]. We shall also use the EHP-sequences and the information about the (relative) Whitehead products [17]. We denote by $\#_a$ the order of $a$. Our result is stated as follows.

Theorem 1. i) $\#_1 \Sigma^{d} \gamma_n(F) = 2$ and $\#_1 \Sigma \gamma_n(C) = 2$. ii) $\#_1 \Sigma \gamma_n(H) = 2$ for $1 \leq k \leq 3$; $\#_1 \Sigma^{k} \gamma_n(H) = 8$ for $n \geq 2$ and $k = 1$ or $2$; $\#_1 \Sigma^{3} \gamma_n(H) = 4$ for $n \geq 2$.

Theorem 2. $\pi_{2n+1,d-3}(X) \cong K|\gamma_n(F)| \oplus \pi_{2n-1d-3}(W)$, where $X = Q_{2n+1,2}(F), W = O_{2n+1,2}(F)$ and $K = Z$ if $d \neq 1$ or $d = n = 1$; $K = Z_8$ if $d = 1$ and $n = 3$ or $n \geq 5$; $K = Z_4$ if $d = 1$ and $n = 2$ or $4$.

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The paper is organized as follows. §1 is devoted to prepare some lemmas due to James and Toda. §2 is to summarize the behavior of the $J$-image of the characteristic element for $O_{2n+1,2}(F)$. §§3–5 are devoted to prove the theorems and to determine the generalized Hopf invariant of $\gamma_n(R)$.

1. Some results of James and Toda. Let $X = S^q \cup e^q$ for $q \leq n-1$ and $B = X \cup e^{n+1}$, where $B$ is regarded as the $q$-sphere bundle over $S^n$ [5].

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Let \( i: S^q \to X \), \( j: (X, \ast) \to (X, S^q) \) be the inclusions and \( p: (X, S^q) \to (S^q, \ast) \) a map collapsing \( S^q \) to the base point. Let \( \kappa = \kappa_n: (CS^{n-1}, S^{n-1}) \to (X, S^q) \) be a characteristic map, where \( CS^n \) is a cone on \( S^n \). By (5.1) of [6] and (3.3) of [2], we have

\[
j \ast \kappa = (-1)^n q [\tau_q, \kappa].
\]

By Lemma 4.4.3 of [1] and by Lemma 2.32 and Corollary 3.6 of [15], we have the following

**Lemma 1.** Let \( \beta \in \pi_{n-1}(SO_{q+1}) \) be the characteristic element for \( B \) and \( \theta \in \pi_{n+q}(S^{q+1}) \) an element obtained from \( \beta \) by the Hopf construction. Then \( \Sigma \kappa = \pm (\Sigma i) \ast \theta \) and \( H(\theta) = \pm \Sigma^{q+1} \alpha \).

We denote by \( \hat{\alpha} \in \pi_{k+1}(CS^n, S^n) \) for \( \alpha \in \pi_k(S^n) \) an element satisfying \( \partial \hat{\alpha} = \alpha \), where \( \partial: \pi_{k+1}(CS^n, S^n) \to \pi_k(S^n) \) is the boundary isomorphism. We denote by \( \Sigma': \pi_t(X, S^n) \to \pi_{t+1}(\Sigma X, S^{q+1}) \) the relative suspension homomorphism [15]. By Theorem 2.1 of [3], we have an exact sequence for \( t = n + 2q + 3k - 2 \) \((k \geq 0)\):

\[
\pi_t(\Sigma X, S^{q+k}) \to \Sigma' p \to \ldots \to \pi_t(\Sigma X, S^{q+k}) \to (\Sigma^{q+1} p) \ast
\]

\[
\pi_r(S^{n+k}) \xrightarrow{H'} \pi_{r-n-k}(S^{q+k}) \xrightarrow{Q} \pi_{r-1}(\Sigma^k X, S^{q+k}) \to \ldots,
\]

where \( H' = (\Sigma^k \alpha) \ast (\Sigma^{-n-k} H) \) and \( Q(\ ) = [\ , (\Sigma^k) \ast \kappa] \).

**Lemma 2.**

1) \( \text{Ker}(\Sigma^k i) = \pi_r(S^{q+k}) \to \pi_r(\Sigma^k X) = (\Sigma^k \alpha) \ast \pi_r(S^{n+k-1}) \)

for \( r = n + q + k - 1 \) if \( k = 0 \) or \( k \geq 2 \).

2) \( \text{Ker}(\Sigma i) = [\tau_{q+1}, \Sigma \kappa] + (\Sigma \alpha) \ast \pi_{n+q}(S^n) \).

**Proof.**

1) For \( k = 0 \) is just (3.2) of [6]. Recall \( \text{Ker}(\Sigma^k i) = \text{Im} \partial \), where \( \partial: \pi_{r+1}(\Sigma^k X, S^{q+k}) \to \pi_r(S^{q+k}) \) is the connecting map. Since \( \pi_{r+1}(\Sigma^k X, S^{q+k}) \cong \pi_{r-1}(S^{q+k}) \) for \( k \geq 2 \) and \( \partial((\Sigma^k \alpha) \circ \hat{\beta}) = (\Sigma^k \alpha) \circ \beta \) for \( \beta \in \pi_t(S^{n+k-1}) \), we have the assertion for \( k \geq 2 \).

By (2), we have \( \pi_{n+q+1}(\Sigma X, S^{q+1}) = \mathbb{Z} \cdot [\tau_{q+1}, \Sigma \kappa] \oplus \pi_{n+q+1}(S^{q+1}) \).

This leads us to ii) and completes the proof.

As is well known, we have the following

**Remark 1.**

1) Let \( G \) be a group generated by \( \Delta(\tau_{n-1}) = [\tau_{2n-1}, \tau_{2n-1}] \).

Then \( G = 0 \) if \( n = 1 \), 2 or 4 and \( G = \mathbb{Z}_2 \) if otherwise. We have a short exact sequence
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(3) \[ 0 \to G|\Delta(\iota_{n-1})| \to \pi_{n-3}(S^{2n-1}) \to \pi_{n-2}(S^{2n}) \to 0 \]

which is split if \( n = 1, 2, 4 \) or \( n \) is not a power of 2.

ii) We have

(4) \[ \pi_{n-1}(S^{2n}) \cong \mathbb{Z} |\Delta(\iota_{n-1})| \oplus \Sigma \pi_{n-2}(S^{2n-1}) \text{ for } n = 3 \text{ or } n \geq 5. \]

By Propositions 2.7 and 2.2 of [16], \( H(\Delta(\iota_{n-1})) = \pm 2 \iota_{n-1} \) and
\( H(2\iota_{2n} \circ \Delta(\iota_{n+1})) = \Sigma(2\iota_{2n-1} \wedge 2\iota_{2n-1}) \circ H(\Delta(\iota_{n+1})) = \pm 8 \iota_{n-1}. \) So, by (4), we have \( [\iota_{2n}, 2\iota_{2n}] \in (2\iota_{2n})^*\pi_{n-1}(S^{2n}) \). By this and [12], we have the following

Remark 2. \([\iota_{q+1}, \Sigma a] \in (\Sigma a)^*\pi_{n+q}(S^{n}) \) for some \( n \), where \( X = Q_{2n-1, 2}(F), a = \omega_n(F) \) and \( q = 2dn-1 \).

Let \( RP^n \) be the real \( n \)-dimensional projective space and \( RP^n_k = RP^n_{p_k}/RP^{q-1} \) the stunted space.

Lemma 3. \( \pi_{4n+1, d-3}(X, S^{2dn-1}) \cong \pi_{4n+1, d-3}(S^{2n+1, d-1}) \oplus L|\iota_{2dn-1}, \chi| \),
where \( X = Q_{2n+1, 2}(F), \chi = \chi_{2n+1, d-1} \) and \( L = \mathbb{K} \) if \( d \neq 1 \) or \( d = n = 1 \); \( L = \mathbb{K}_2 \) if \( d = 1 \) and \( n = 3 \) or \( n \geq 5 \); \( L = \mathbb{Z}_2 \) if \( d = 1 \) and \( n = 2 \) or 4.

Proof. First we shall give a proof in the real case. By (2), we have an exact sequence for \( n \geq 2 \):

\[ \pi_{n-1}(S^{2n}) \xrightarrow{H'} \pi_{n-1}(S^{2n-1}) \xrightarrow{Q} \pi_{n-2}(X, S^{2n-1}) \xrightarrow{P^*} \pi_{n-2}(S^{2n}) \to 0. \]

By (4), \( \text{Im } H' \cong 4\mathbb{Z} \) for \( n = 3 \) or \( n \geq 5 \). \( \text{Im } H' \cong 2\mathbb{Z} \) for \( n = 2 \) or 4. So we have a short exact sequence

(5) \[ 0 \to L \to \pi_{n-2}(X, S^{2n-1}) \xrightarrow{P^*} \pi_{n-2}(S^{2n}) \to 0. \]

We set \( m = 2n-1 \) and \( a = \Delta(\iota_{m+1}) \). By (2.18) of [15], we have \( \hat{\alpha} = [\iota_m, \hat{\iota}_m] \), where \( \hat{\iota}_m \) coincides with the identity map of \( (CS^n, S^n) \). So, by (2.16–18) of [15] or by (3.4–6) of [2], we have \( x\hat{\alpha} = x\iota_m, \hat{\iota}_m = 2[\iota_m, \chi] = 2[\iota_m, \chi] \). Let \( \beta \in \pi_{n-1}(S^n) \) be an element such that \( \#\beta = \#\Sigma\beta \). Then \( p*(x\circ\hat{\beta}) = \Sigma\beta \) and \( \#(x\circ\hat{\beta}) = \#\Sigma\beta \). Therefore, if (3) is split, so is (5).

Suppose that (3) is not split. Then there exists an element \( \beta \in \pi_{n-1}(S^n) \) such that \( 2\beta = \Delta(\iota_{m+1}) \) and \( \#\Sigma\beta = 2 \). Since \( 2(x\hat{\beta}) = x\hat{\alpha} = 2[\iota_m, \chi] \), we have \( \#\delta = 2 \) and \( p*(\delta) = \Sigma\beta \) for \( \delta = x\hat{\beta} - [\iota_m, \chi] \). So (5) is also split in this case. This leads us to the assertion of the real case except for \( n = 1 \).
We have $X = \mathbb{R}P^2$ and $O_{2g}(\mathbb{R}) = \mathbb{R}P^{3}$ if $d = n = 1$. So, by use of the homotopy exact sequence of a pair $(X, S^1)$, we have $\pi_5(X, S^1) \cong \mathbb{Z}[\gamma_1(\mathbb{R})] \oplus \mathbb{Z}[\iota]$. Since $p\ast \iota = \iota_2$ and $j \ast \gamma_1(\mathbb{R}) = [\iota_1, x]$ by (1), we have the splitting of (5).

For $d = 2$ or 4, we have, by (2), a short exact sequence for $r = (4n+1)d - 3$:

$$0 \to \pi_{2dn-1}(S^{2dn-1}) \xrightarrow{Q} \pi_{r}(X, S^{2dn-1}) \xrightarrow{p_*} \pi_{r}(S^{(2n+1)d-1}) \to 0.$$ 

Since $\Sigma : \pi_{r-1}(S^{(2n+1)d-2}) \to \pi_{r}(S^{(2n+1)d-1})$ is isomorphic onto, the sequence is split. This completes the proof.

By (11.8) and Theorem 11.7 of [16], we have the following

**Lemma 4.** There exists a mapping $\delta : \Sigma^{n-1}R^k_{n+k-1} \to S^n$ such that $\ker |\Sigma^k| : \pi_i(S^n) \to \pi_{i+k}(S^{n+k}) = \delta \ast \pi_i(\Sigma^{n-1}R^k_{n+k-1})$ for $i \leq 3n-3$. In the 2-components, the assertion holds for $i \leq 4n-4$.

By Proposition 7.10 of [4], $Q_{n+k}(\mathbb{F})$ is a stable retract of $O_{n+k}(\mathbb{F})$. Especially we have $\Sigma^{2n+1} \gamma_n(\mathbb{F}) = 0$.

Hereafter, by abuse of notation, we often use the inclusion $i$ and the projection $p$ to denote $\Sigma^i_2$ and $\Sigma^p$ for integers $r$ and $s$, respectively.

Let $\sigma_n \in \pi_{n-1}(S^n)$ for $n \geq 8$ be the Hopf map and $\iota$ the identity class of $X = Q_{2n+1,2}(\mathbb{F})$. Then $X \wedge X$ is homotopy equivalent to a mapping cone

$$\Sigma^{2dn-1}X \cup_{\lambda_n(\mathbb{F})} C(S^{(2n+1)d-2}X),$$

where $\lambda_n(\mathbb{F}) = \iota_\wedge \omega_n(\mathbb{F})$.

In the 2-components, stable Toda brackets $\langle 2\iota, \eta, 2\iota \rangle$, $\langle \eta, \nu, \eta \rangle$ and $\langle \nu, 8\iota, \nu \rangle$ consist of single elements $\eta^2$, $\nu^2$ and $8\sigma$, respectively. By this and by Lemma 3.5 and Theorem 3.6 of [16] and by their proofs, we have the following

**Lemma 5.** $\lambda_n(\mathbb{R}) = i(\eta_{n-2}p, \eta_{n}(C)) = 3a\nu_{n-2}p$ and $\lambda_n(H) = 15b\sigma_{2n-2}p$ $-(\Sigma^{n-1}\tilde{\theta})p$ for $n \geq 1$ and odd integers $a$ and $b$, where $\tilde{\theta}$ is a coextension of $\theta = 2\Sigma^3 \omega_n(H)$ with respect to $\omega_n(H)$.

2. The $J$-image of the characteristic element. Let $\gamma_n(\mathbb{F}) \in \pi_{dn+1,2}(O_n(\mathbb{F}))$ be the characteristic map [11], where $O_n(\mathbb{F}) = O_n$, $U_\mathbb{R}$ or $\text{Sp}_n$ according as $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\text{H}$. Let $J : \pi_k(O_n(\mathbb{F})) \to \pi_{k+dn}(S^{dn})$ be the $J$-homomorphism and $j_n(\mathbb{F}) = J(\gamma_n(\mathbb{F})) \in \pi_{2n+1,2}(S^{dn})$. Then $j_n(\mathbb{F})$ is an
element obtained from the characteristic element \( \gamma_\delta'(R) \), \( c \gamma_\delta'(C) \) or \( rc \gamma_\delta'(H) \) by the Hopf construction, where \( r : U_n \rightarrow SO_{2n} \) and \( c : Sp_n \rightarrow SU_{2n} \) are the canonical maps. We recall the following relations: \( j_n(R) = \Delta(\iota_{2n+1}) = \pm \{ \iota_n, \iota_n \}, \Sigma j_n(C) = j_{2n+1}(R), \Sigma^2 j_n(H) = j_{2n+1}(C), H(j_n(C)) = (n-1) \eta_{4n-1} \) and \( H(j_n(H)) = \pm (n+1) \eta_{4n-1} \). By Lemma 1, we have

\[
\Sigma \gamma_\delta'(F) = \pm i \ast j_2(F), \quad \Sigma^2 \gamma_\delta'(F) = i \ast \Delta(\iota_{2, 2n+1})^{-1}
\]

and \( H(j_2(F)) = \pm \Sigma^2 \omega_0(F) \).

By [8], [14] and [16], we have

\[
\Delta(\eta_{2n+1}) \equiv 0 \text{ if and only if } n = 4, 5 \text{ or } n \equiv 3 \text{ mod } 4
\]

\[
\text{ and } n \geq 8 : \Delta(\eta_{4n-1}) \equiv 0 \text{ if and only if } n = 4 \text{ or } n \equiv 0, 1 \text{ mod } 4 \text{ and } n \geq 6.
\]

We denote by \((a, b)\) the greatest common divisor of integers \(a\) and \(b\).

**Lemma 6.** i) In the 2-component, there exists an element \( \lambda \in \pi_{16n-1}(S^{4n-3}) \) such that \( \pm (2n+1) \Delta(\iota_{16n-1}) = 2j_{2n}(H) - \Sigma^2 \lambda \) and \( H(\lambda) = \nu_{16n-7} \). There exists \( \lambda' \in \pi_{16n-3}(S^{4n-5}) \) such that \( 2 \lambda = \Sigma^2 \lambda' \) and \( H(\lambda') \equiv \varepsilon_{16n-11} \text{ mod } \eta_{16n-11} \sigma_{16n-10} \). We set \( \lambda = \nu_s \delta \) and \( \lambda' = \pm \epsilon \) for \( n = 1 \).

ii) \# \mathfrak{j}_{2n}(C) = 2 ; \# \mathfrak{j}_{2n+1}(C) = 4 \) and \( 2 \mathfrak{j}_{2n+1}(C) = \Delta(\eta_{8n+5}) \) for \( n \geq 2 \);

\( \# \mathfrak{S}_j(H) = 8 \) and \( 4 \Sigma \mathfrak{j}_{2n}(H) = \Delta(\eta_{8n+5}) ; \# \mathfrak{j}_{2n}(H) = 24/(3, 2n+1) \) and \( 12/(3, 2n+1) \mathfrak{j}_{2n}(H) = j_{2n}(C) \cap \eta_{8n} \).

**Proof.** i) for \( n \geq 2 \) is obtained from Lemma 11.17 and Proposition 11.15 of [16]. For \( n = 1 \), the assertion holds [16].

We recall that \( \pi_4(SO_4) \cong (Z_2)^3 \) or \( (Z_2)^3 \) according as \( n \) is odd or even [7]. Since \( \mathfrak{j}_n(C) = J(r \gamma_n'(C)) \) and \( H(j_n(C)) = \eta_{8n-1} \), we have the first of ii).

Since \( \pi_0(SO_4) = 0 \), we have \( j_2(C) = 0 \). We consider an anti-commutative diagram between exact sequences for \( n \geq 2 \):

\[
\begin{array}{ccc}
\pi_{4n+3}(S^{4n+2}) & \xrightarrow{\partial} & \pi_{4n+2}(SO_{4n+2}) \\
\downarrow \Sigma^4n+3 & & \downarrow J \\
\pi_{8n+6}(S^{4n+5}) & \xrightarrow{\Delta} & \pi_{8n+4}(S^{4n+2}) \\
\end{array}
\]

By [7], \( \pi_{4n+3}(SO_{4n+2+k}) \equiv \mathbb{Z}_{2k-2} \) for \( k = 0 \) or 1 and \( 2r \gamma_{2n+1}'(C) = \partial \eta_{2n+2} \).

So we have \( 2 \mathfrak{j}_{2n+1}(C) = \Delta(\eta_{8n+5}) \). By (7), we have the second of ii).

We recall that \( \pi_{8n+3}(SO_{8n}) = \{ \gamma_{8n}(R) \circ \nu_{8n-1}, rc \gamma_{8n}(H) \} \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_8 \) and \( \pi_{8n+2}(SO_{8n+1}) = \mathbb{Z}_8 \lor r'c \gamma_{8n+1}(H) \) [11]. \( J(\gamma_{8n}(R) \circ \nu_{8n-1}) = j_{8n}(R) \circ \nu_{8n-1} = \pm \Delta(\nu_{16n-1}) \), \( J(rc \gamma_{8n}(H)) = j_{8n}(H) \) and \( J(r'c \gamma_{8n}(H)) = \Sigma j_{8n}(H) \). By Theorem 4
of \([11]\), \(12/(3, 2n+1)\) \(j_2n(H) = j_4n(C) \circ \eta_{\delta_{2n}}\) and \(12/(3, 2n+1)\) \(\Sigma j_2n(H) = \Sigma j_4n(C) \circ \eta_{\delta_{2n+1}} = \Delta(\eta_{\delta_{2n+3}})\). By (7) and an EHP-sequence

\[
\pi_{16n+5}(S^{16n+3}) \xrightarrow{\Delta} \pi_{16n+3}(S^{8n+1}) \xrightarrow{\Sigma} \pi_{16n+4}(S^{8n+2}),
\]

we have the rest of ii). This completes the proof.

Let \(G_k\) be the \(k\)-th stable homotopy group of spheres. By \([14]\) and Lemma 6, we have the following

**Remark 3.**

i) \(\pi_{\delta_{2n}}(S^{4n}) \cong G_{\delta_{2n}} \oplus Z_{2^k} \{\Delta(\eta_{\delta_{2n+1}})\} \oplus Z_2 \{j_{2n}(C)\}\); \(\pi_{\delta_{2n-1}}(S^{4n-1}) \cong G_{\delta_{2n}} \oplus Z_{2^k} \{H(\delta) = \eta_{\delta_{2n-3}}\} \oplus \Sigma\delta = \Delta(\eta_{\delta_{2n-1}})\).

ii) \(\pi_{16n+4}(S^{8n+2}) \cong G_{\delta_{2n+2}} \oplus Z_8 \{j_{2n+1}(C)\} ; \pi_{16n+3}(S^{8n+1}) \cong G_{\delta_{2n+2}} \oplus Z_8 \{j_{2n+1}(C)\} \oplus (Z_8 \oplus Z_2) \{\Delta(\nu_{16n+1}) , j_{2n}(H)\}\); \(\pi_{16n+3-\kappa}(S^{8n-\kappa}) \cong G_{\delta_{2n+2}} \oplus Z_8 \{\Sigma^{k-\kappa}\} \) for \(k = 1, 2\).

3. The complex or quaternionic case. Hereafter we set \(X = Q_{2n+1,2}(F)\) and \(\gamma = \gamma_{F_{\kappa}}(F)\).

**Proposition 7.** \(\# \Sigma \gamma = \# \Sigma^2 \gamma = 2\) for \(F = C\).

**Proof.** By (6), \(\Sigma^2 \gamma = i_2 \Delta(\delta_{2n-3})\). Assume that \(\Sigma^2 \gamma = 0\). Then, by Lemma 2, there exists an element \(\beta \in \pi_{\delta_{2n-1}}(S^{4n})\) satisfying \(\Delta(\delta_{2n-3}) = \eta_{\delta_{2n+1}} \circ \Sigma^2 \beta\). So we have \(j_{2n}(C) = \eta_{\delta_{2n}} \circ \Sigma^2 \beta + \alpha \Delta(\eta_{\delta_{2n+1}})\) for \(a = 0, 1\). Apply \(H\) to this relation. Then \(\eta_{\delta_{2n-3}} = 0\) and this is a contradiction. Therefore \(\Sigma^2 \gamma \neq 0\) and \(\# \Sigma^2 \gamma = 2\).

By (6) and Lemma 6, \(2 \Sigma \gamma = 2 i_2 j_{2n}(C) = 0\). So we have \(\# \Sigma \gamma = 2\). This completes the proof.

Hereafter in this section, we shall deal with the quaternionic case.

**Lemma 8.** \(\# \Sigma \gamma = \# \Sigma^2 \gamma = 8\Sigma \gamma = 0\).

**Proof.** In an EHP-sequence

\[
\pi_{16n+4}(\Sigma(X \wedge \Sigma X)) \xrightarrow{\Delta} \pi_{16n+2}(\Sigma X) \xrightarrow{\Sigma} \pi_{16n+3}(\Sigma^2 X),
\]

the left group is isomorphic to \(\pi_{16n+4}((S^{16n+1} \cup i_{16n+1} e_{16n+5}) \cup S^{16n+5}) \cong Z_{(24,2n+1)} \{\nu_{16n+1}\}\) by Lemma 5. Hence \(\Sigma\) is homomorphic if \(2n+1 \equiv 1, 2\) mod 3 and so is on the 2-component if \(2n+1 \equiv 0\) mod 3. By Lemma 6, \(\# \Sigma j_{2n}(H) = 8\) and \(8 j_{2n}(H) = 0\) if \(2n+1 \equiv 0\) mod 3. Therefore we have \(8 \Sigma \gamma = 0\).
This completes the proof.

**Proposition 9.**  

i) $\#\Sigma^0\gamma = 2$.

ii) $\#\Sigma^k\gamma = 2$ for $n = 1$ and $1 \leq k \leq 3$.

iii) $\#\Sigma^2\gamma = 4$ for $n \geq 2$.

**Proof.** By use of the homotopy exact sequence of a pair $(\Sigma^0X, S^{8n+4})$, we have $\pi_{16n+7}(\Sigma^5X) \cong \mathbb{Z} \langle i\Delta(\iota_{16n+7}) \rangle \oplus K$, where $K$ is a finite abelian group. In an EHP-sequence

$$
\begin{array}{c}
\pi_{16n+7}(\Sigma^5X) \xrightarrow{H} \pi_{16n-7}(\Sigma(\Sigma^4X \wedge \Sigma^4X)) \xrightarrow{\Delta} \pi_{16n+5}(\Sigma^4X).
\end{array}
$$

We have

$$Z \langle i\Delta(\iota_{16n+7}) \rangle \cong \pi_{16n+7}(S^{16n+7})$$

$H(i\Delta(\iota_{16n+7})) = \pm 2i\iota_{16n+7}$. So we have $\#(i\Delta(\iota_{16n+7})) = 2$. So, by (6), we have i).

By i) of Lemma 6, $2j_2(H) = 3\nu_8 \circ \sigma_1 \pm 3\Delta(\nu_{17})$. So, by Lemma 8 and its proof, $2i^*j_2(H) = 0$ and $2\Sigma^k\gamma = 0$ for $n = 1$ and $1 \leq k \leq 3$. So, by i), we have ii).

By Lemmas 6, 8 and i), $\#\Sigma^2\gamma = 2$ or 4. Assume that $\Sigma^2\gamma = 2i^*\Sigma^2j_2(H) = 0$. Then, by (4) and Lemma 2, there exists an element $\alpha \in \pi_{16n-3}(S^{8n-1})$ satisfying $2\Sigma^2j_2(H) = (2n+1)\nu_{8n+1} \circ \Sigma^2\alpha$. So $2\Sigma^2j_2(H) \equiv (2n+1)\nu_{8n+1} \circ \Sigma^2\alpha \mod \Delta(\iota_{16n+3}) = 2\Sigma^2j_2(H)$. Therefore $\pm 2j_2(H) = (2n+1)\nu_{8n} \circ \Sigma^2\alpha \circ x \Delta(\nu_{16n+1})$ for an integer $x$. Since $2(2n+1)\nu_{16n-1} = 2H(j_2(H)) = \pm 2x\nu_{16n-1}$, we have $x \equiv \pm (2n+1) \mod 12$. By Lemma 6, we have $\Sigma^2\lambda = \pm (2n+1)\nu_{8n} \circ \Sigma^2\alpha$ since $4H(j_2(H)) \neq 0$. By use of the EHP-sequences, we have $\pm \lambda \equiv (2n+1)\nu_{8n-3} \circ \Sigma^2\alpha \mod \Delta(\nu_{16n-5})$. Applying $H$ to this relation, we have $\nu_{16n-7}^2 \equiv 0 \mod H(\Delta(\nu_{16n-5})) = 2\iota_{16n-7} \circ \nu_{16n-7}^2 = 0$. This is a contradiction and hence we have iii). This completes the proof.

**Proposition 10.** $\#\Sigma^4\gamma = 8$ if $n \geq 2$ and $k = 1$ or 2.

**Proof.** By Lemma 8, it suffices to work in the 2-components and to prove the assertion for $k = 2$. By Lemma 8 and Proposition 9, $\#\Sigma^2\gamma = 4$ or 8. Assume that $4\Sigma^2\gamma = 4i^*\Sigma^2j_2(H) = 0$. Then, by Lemmas 2 and 6, there exists an element $\alpha \in \pi_{16n}(S^{8n+1})$ satisfying $\Sigma^6\lambda = \nu_{8n+1} \circ \Sigma^3\alpha$. By (7), $\Delta(\iota_{16n-3}) \neq 0$ and $\Delta(\iota_{16n-5}) \neq 0$. So, by use of the EHP-sequences, there exists an element $\beta \in \pi_{16n-4}(S^{8n-3})$ satisfying $a = \Sigma^4\beta$. By an EHP-sequence
\[ \pi_{16n+4}(S^{16n+1}) \xrightarrow{\Delta} \pi_{16n-2}(S^{4n}) \xrightarrow{\Sigma} \pi_{16n+3}(S^{4n+1}), \]

\[ \Sigma^5 \lambda - \nu_{8n} \circ \Sigma^5 \beta = a \Delta(\nu_{16n+1}) \] for an integer \( a \). Applying \( H \) to this relation, we have \( \pm 2a \nu_{16n-1} = 0 \) and \( a = 4b \) for an integer \( b \). By Lemma 6, \( 4b \Delta(\nu_{16n+1}) = -2b \Sigma^5 \lambda \). So we have \((1+2b) \Sigma^5 \lambda = \nu_{8n} \circ \Sigma^5 \beta \). Since \( \Sigma : \pi_{16n+k}(S^{8n-2+k}) \to \pi_{16n+k+1}(S^{8n-1+k}) \) is monomorphic for \( k = 0 \) or 1, \((1+2b) \Sigma^5 \lambda = \nu_{8n-2} \circ \Sigma^5 \beta \). We set \( m = 8n-5 \). By Lemma 4, there exists a mapping \( \delta : \Sigma^{m-1}RP_m^{m+1} \to S^m \) such that \( \text{Ker} \{ \Sigma^3 : \pi_{2m+7}(S^m) \to \pi_{2m+10}(S^{m+3}) \} = \delta \circ \pi_{2m+7}(\Sigma^{m-1}RP_m^{m+1}) \). \( RP_m^{m+2} = \Sigma^{m-3}RP^3 \) and \( \pi_{2m+7}(\Sigma^{m-1}RP_m^{m+1}) = \pi_{11}(RP^3) \) (the stable group). Therefore we have

\[
(8) \quad (1+2b) \lambda - \nu_m \circ \Sigma^5 \beta \in \delta \circ \pi_{11}(RP^3).
\]

Recall \( RP^3 = (S^3 \cup Z_{1\epsilon}e^4) \cup Z_{1\epsilon}e^5 \). By \( [9] \), \( \pi_{8}^5(RP^3) = Z_2[8 \sigma] \oplus Z_4[i \eta \sigma] \oplus Z_2[i \epsilon] \). By use of a cofibre sequence starting with \( i_1 \), we have an exact sequence

\[ Z_{16} \sigma \xrightarrow{(i \eta)_*} \pi_{11}^5(RP^3) \xrightarrow{i^*} \pi_{11}^5(RP^3) \xrightarrow{p^*} Z_2 \nu \to 0, \]

where \( i : \Sigma^5RP^3 \to RP^3 \) and \( p : RP^3 \to S^3 \) are the canonical maps. Let \( \tilde{\nu} \) be an element of the Toda bracket \( \langle i^*, i \eta, \nu \rangle \subset \pi_{11}^5(RP^3) \). Then \( 2 \tilde{\nu} \nu = \langle i^*, i \eta, \nu^3 \rangle \circ 2 \epsilon = -i^* \langle i \eta, \nu^3 \rangle \) and \( \tilde{\nu} \nu = i^* \langle i \eta, \nu \rangle \) mod \( i^* \eta \sigma = 0 \), where \( i^* = i^* \circ i : S^3 \to RP^3 \). So we have \( 2 \tilde{\nu} \nu = i \epsilon \) and \( \pi_{11}^5(RP^3) = Z_2[i \tilde{\nu} \sigma] \oplus Z_4[i \tilde{\nu} \nu] \).

On the other hand, \( H(\delta) \in [\Sigma^{2m-4}RP^3, S^{2m-1}] \equiv [RP^3, S^3] \). We recall that \( [RP^3, S^3] = Z_2[i \eta \rho '] \) and \( [RP^3, S^3] = Z_4[i \eta] \), where \( \eta \) is an extension of \( \eta \). By use of the above cofibre sequence, we have an exact sequence

\[ 0 \to Z_2[i \eta \rho '] \xrightarrow{i^*} [RP^3, S^3] \xrightarrow{\rho^*} Z_2[i \eta^2 \nu] \xrightarrow{(i \eta)_*} Z_4[i \eta]. \]

Let \( \rho \in [RP^3, S^4] \) be an extension of \( p \) with respect to \( i \eta \). Then \( [RP^3, S^4] = Z_2[i \eta \rho \eta \eta] \). \( \eta \rho \circ i \tilde{\delta} \sigma = \eta \rho \tilde{\delta} \sigma = 8 \eta \sigma = 0 \) and \( \eta \rho \circ \tilde{\nu} \nu \in \eta \rho \circ \nu \subset \eta \circ G \times \nu \subset \eta \circ G \times \nu = 0 \). So we have \( (\eta \rho) \circ \pi_{11}^5(RP^3) = 0 \). Applying \( H \) to \( (8) \), we get \( (1+2b \beta) H(\lambda) \in H(\delta) \cap \pi_{11}^5(RP^3) \subset (\eta \rho) \circ \pi_{11}^5(RP^3) = 0 \). By Lemma 6, \( H(\lambda) \equiv \epsilon_{2m-1} \mod \eta_{2m-1} \sigma_{2m} \). This is a contradiction and completes the proof.

4. The real case. We set \( Y = \Sigma^{2n-3}RP^2 \) for \( n \geq 2 \), \( X = Q_{2n+1,3}(R) = \Sigma^{2n-3}RP^2 \) and \( \gamma = \gamma_n(R) \) for \( n \geq 1 \).

Proposition 11. \#(\Sigma \gamma = 2 \) for \( n \geq 1 \).
Proof. By (6), $\Sigma \gamma = i*\Delta(\iota_{4n+1})$. Assume that $\Sigma \gamma = 0$. Then, by Lemma 2, we have $\Delta(\iota_{4n+1}) \subseteq 2\Delta(\iota_{4n+1}) + 8(\iota_{4n+1})*\eta_{4n-1}(S^{2n})$. Applying $H$ to this relation, we have $\pm 2\iota_{4n-1} \subseteq 4\iota_{4n-1} + 16\iota_{4n-1}$. This is a contradiction and completes the proof.

By Propositions 7, 9, 10 and 11, we have completed the proof of Theorem 1.

We recall that $\pi_{a}(\Sigma(Y \wedge Y)) = \mathbb{Z}_{4}\{i\}$ and $2^{i} = i^{*}\eta_{4n-3}$ for $n \geq 2$, where $i: S^{4n-3} \hookrightarrow \Sigma(Y \wedge Y)$ is the inclusion [10].

**Lemma 12.** Let $n \geq 2$. Then $2\gamma = \pm \Delta(\iota_{4}^{*}\iota_{3})$, $\# \gamma = 4$ for even $n$ and $\# \gamma = 8$ for odd $n$.

**Proof.** First we shall show $\# \gamma = 4$ for $n = 2$ or 4. We consider a commutative diagram between exact sequences:

\[
\begin{array}{ccc}
\pi_{6}(S^{3}) & \xrightarrow{i^{*}} & \pi_{6}(X) \\
\parallel & & \parallel \\
\pi_{6}(S^{3}) & \xrightarrow{i^{*}} & \pi_{6}(V_{5,2}) \\
\end{array}
\]

By (2) and Lemma 3, we have $\pi_{6}(X, S^{3}) = \mathbb{Z}_{4}\{[\iota_{2}, x]\} \oplus \mathbb{Z}_{2}\{x \eta_{2}\}$ and $\pi_{7}(X, S^{3}) = \mathbb{Z}_{2}\{[\eta_{2}, x]\} \oplus \mathbb{Z}_{2}\{x \nu_{3}\}$. Then $\varnothing x = 2[\iota_{2}, \iota_{3}] = 0$, $\varnothing(x \eta_{3}) = 2\iota_{3} \circ \eta_{3} = 0$, $\varnothing[x, x] = [\eta_{3}, x] = 2\iota_{3} = 0$ and $\varnothing(x \nu_{3}) = 2\iota_{3} \circ \nu_{3} = 2\nu_{3}$. So we have $\text{Im} \ i* \cong \mathbb{Z}_{2}$ and $j*$ is epimorphic. We recall that $\pi_{6}(V_{5,2}) \cong \mathbb{Z}_{2}[13]$ and $\partial \nu_{3} = 2\iota_{3} \circ \nu_{3} = 0$. So we have $\pi_{6}(V_{5,2}) = \mathbb{Z}_{2}\{i^{*}\eta_{2} \nu_{3}\}$ and $i^{*} \nu_{3} = 0$. Therefore we have $i^{*} \nu_{3} = a\gamma$ for $a = 1$ or 2. By (1) and Lemma 3, we have $0 = a\gamma = 2\gamma$ and hence we have $a = 2$, $2\gamma = i^{*} \nu_{3}$ and $\pi_{6}(X) = \mathbb{Z}_{4}\{\gamma\} \oplus \mathbb{Z}_{2}\{\eta_{3}\}$. So we have $2\gamma = i^{*} \nu_{3} = 2\nu_{3}$ and $\pi_{6}(V_{5,2}) \cong \mathbb{Z}_{2}[13]$. By (2) and Lemma 3, we have $\pi_{14}(X, S^{7}) = \mathbb{Z}_{4}\{[\iota_{2}, x]\} \oplus \mathbb{Z}_{2}\{x \nu_{3}\}$ and $\pi_{15}(X, S^{7}) = \mathbb{Z}_{2}\{[\eta_{2}, x]\} \oplus \mathbb{Z}_{2}\{x \nu_{3}\}$. The connecting map $\partial$ is trivial except for the following: $\partial(x \nu_{3}) = 2\iota_{3} \circ \nu_{3} = 2\nu_{3}$. So, by a parallel argument to the above, we have $2\gamma = i^{*} \nu_{3}$ and $\pi_{14}(X) = \mathbb{Z}_{4}\{\gamma\} \oplus \mathbb{Z}_{2}\{\nu_{3}\}$ for $n = 4$. We note that $\pi_{14}(V_{5,2}) \cong \mathbb{Z}_{2}[13]$.

By Proposition 11 and an EHP-sequence

\[
\pi_{4n}(\Sigma(X \wedge X)) \xrightarrow{\Delta} \pi_{4n-2}(X) \xrightarrow{\Sigma} \pi_{4n-1}(\Sigma X),
\]

we have $2\gamma = a\Delta(\iota_{4}^{*}\iota_{3})$ for an integer $a$. If $a$ is even, $2\gamma = (a/2)i^{*}\Delta(\eta_{4n-1})$. So we have $2\gamma = 0$ for $n = 2$ or 4 and $2[\iota_{2n-1}, x] = 2j \gamma = 0$ for $n = 3$ or $n \geq 5$ by (1). This contradicts the above and Lemma 3. Hence we have the first assertion.
By (7), $\Delta(\eta_{n-1})$ is trivial for even $n$ and nontrivial for odd $n$. So

$$2\Delta(\Sigma^2) = i_*\Delta(\eta_{n-1}) = 0$$

and $4\gamma = 0$ for even $n$. By (1) and Lemma 3, $2\gamma = 0$. This leads us to the second assertion.

For odd $n$, it suffices to show $i_\#\Delta(\eta_{n-1}) = 0$. By [10], we have

$$i_\#\Delta(\eta_{n-1}) = i_\#\nu_\eta \eta_{i^*} = 0.$$  Assume that it is trivial for $n \geq 5$. Then, by Lemma 2, there exists an element $\beta \in \pi_{n-2}(S^{2n-1})$ satisfying $\Delta(\eta_{n-1}) = 2i_{2n-1}\circ \beta$. By (7), $\Delta(\eta_{n-3}) = 0$ for odd $n \geq 5$. So we have $\beta = \Sigma\beta'$ for some $\beta' \in \pi_{2n-3}(S^{2n-2})$. Therefore $\Delta(\eta_{n-1}) = 2\Sigma\beta'$ and $\Delta(\eta_{i^*n-1}) = 2\Sigma\beta' \circ \eta_{i^*n-2} = 0$. By (7), $\Delta(\eta_{i^*n-1}) = 0$ for odd $n \geq 5$. This is a contradiction and completes the proof.

We set $X = Q_{2n+1,2}(F)$, $W = O_{2n+2}(F)$, $r = 2dn-1$ and $s = 2r+d-1 = (4n+1)d-3$. We consider a commutative diagram among exact sequences for $n \geq 2$:

$$\pi_s(W, X) \xrightarrow{\Sigma^*} \pi_s(S^r)$$

$$\downarrow \partial^* \quad \downarrow q$$

$$\pi_{s-1}(S^r) \xrightarrow{i_*} \pi_{s-1}(X) \xrightarrow{j_*} \pi_{s-1}(X, S^r) \xrightarrow{\partial} \pi_{s-2}(S^r)$$

$$\| \downarrow i_*^\pi \quad \downarrow p_*^\pi \|$$

$$\pi_{s-1}(S^r) \xrightarrow{i_*^\pi} \pi_{s-1}(W) \xrightarrow{p_*^\pi} \pi_{s-1}(S^{r+d}) \xrightarrow{\partial'} \pi_{s-2}(S^r)$$

$$\downarrow \quad \downarrow$$

$$0 \quad 0$$

where $\Sigma^* = \Sigma^{-(-r+d-1)} \circ p_\pi^* \pi$ for the canonical map $p^\pi : (W, X) \to (S^s, \pi)$. By (1), Lemmas 3 and 12, we have Theorem 2.

5. Determination of $H(\gamma_n(R))$. We shall show $H(\gamma) = \pm \tilde{\gamma}$, where $\gamma = \gamma_n(R)$ and $n \geq 2$. We set $Y = \Sigma^{2n-1}RP^2$ and $X = SY$.

Lemma 13. Let $n \geq 2$. Then we have

i) $i_*\Delta(\eta_{n-3}) = 0$;

ii) $\text{Im} |\Sigma': \pi_{n-3}(Y, S^{2n-2}) \to \pi_{n-2}(X, S^{2n-1})| \cong \pi_{4n-2}(S^{2n})$.

Proof. By (7), $\Delta(\eta_{n-3}) = 0$ for $n = 2$ or 4. So we have i) for $n = 2$ or 4. It suffices to prove $H(\gamma) \equiv 0$ for $n = 3$ or $n \geq 5$ since $2\tilde{\gamma} = H(c\gamma)$ for $c = 1$ or 2 implies $i_*\Delta(\eta_{n-3}) = \Delta(2\tilde{\gamma}) = \Delta(H(c\gamma)) = 0$.

We consider an anti-commutative diagram:
\[ \pi_{4n-2}(Y) \xrightarrow{\Sigma} \pi_{4n-2}(X) \xrightarrow{H} \pi_{4n-2}(\Sigma(Y \wedge Y)) \]
\[ \downarrow j_* \quad \downarrow j_* \]
\[ \pi_{4n-3}(Y, S^{2n-2}) \xrightarrow{\Sigma'} \pi_{4n-3}(X, S^{2n-1}) \]
\[ \downarrow p_* \quad \downarrow p_* \]
\[ \pi_{4n-3}(S^{2n-1}) \xrightarrow{\Sigma} \pi_{4n-3}(S^{2n}) \]

Assume that \( H(\gamma) = 0 \). Then there exists an element \( \beta \in \pi_{4n-3}(Y) \) such that \( \gamma = \Sigma \beta \). Therefore, by (1), \([\iota_{2n-1}, \alpha] = j_* \gamma = -\Sigma' (j_* \beta)\). By Theorem 2.1 of [3], we have an exact sequence for \( n \geq 3 \):
\[ \pi_{2n-1}(S^{2n-2}) \xrightarrow{Q} \pi_{4n-3}(Y, S^{2n-2}) \xrightarrow{p_*} \pi_{4n-3}(S^{2n-1}) \rightarrow 0, \]
where \( Q(\ ) = [\ , \alpha'] \) and \( \alpha' = (\Sigma')^{-1} \alpha \) is a generator of \( \pi_{2n-1}(Y, S^{2n-2}) \cong \mathbb{Z} \). By the above diagram and Lemma 3, \( \Sigma(p_* j_* \beta) = -p_* \Sigma' (j_* \beta) = 0 \). So we have \( p_* j_* \beta = a \Delta(\iota_{2n-1}) \) for \( a = 0 \) or \( 1 \) and \( p_* (2j_* \beta) = 0 \). Therefore we have \( 2j_* \beta = bQ(\eta_{2n-2}) \) for \( b = 0 \) or \( 1 \). By [15], \( \Sigma'(2j_* \beta) = 0 \) and hence we have \( 2[\iota_{2n-1}, \alpha] = 0 \). This contradicts Lemma 3 and completes the proof of i).

By the lower square of the above diagram, \( p_* \), \( \Sigma \) are epimorphic and \( p_* \) is a split epimorphism. This leads us to ii) and completes the proof.

**Proposition 14.** \( H(\gamma) = \pm \overline{\iota}' \) for \( n \geq 2 \).

**Proof.** It suffices to prove that \( \Sigma: \pi_r(Y) \rightarrow \pi_{r+1}(X) \) for \( r = 4n-4 \) is monomorphic. We consider the suspension homomorphism between exact sequences up to sign:
\[ \pi_{r+1}(Y, S^{2n-2}) \xrightarrow{\partial} \pi_r(S^{2n-2}) \xrightarrow{i_*} \pi_r(Y) \xrightarrow{j_*} \pi_r(Y, S^{2n-2}) \]
\[ \downarrow \Sigma' \quad \downarrow \Sigma \quad \downarrow \Sigma \]
\[ \pi_{r+1}(X, S^{2n-1}) \xrightarrow{\partial'} \pi_{r+1}(S^{2n-1}) \xrightarrow{i_*} \pi_{r+1}(X) \xrightarrow{j_*} \pi_{r+1}(X, S^{2n-1}). \]

By Theorem 2.1 of [3], \( \pi_r(Y, S^{2n-2}) \cong \mathbb{Z}[\iota_{2n-2}, \alpha'] \oplus \pi_r(S^{2n-1}) \) and \( \pi_{r+1}(X, S^{2n-1}) \cong \pi_{r+1}(S^{2n}) \) for \( n \geq 2 \). Since \( \pi_r(Y) \) is finite, \( j_* \alpha \) for \( \alpha \in \pi_r(Y) \) belongs to the second direct summand. The left \( \Sigma \) has the kernel \( \Delta \pi_{r+2}(S^{2n-2}) = \{ \Delta(\eta_{2n-3}) \} \) and \( \partial'[\iota_{2n-1}, \alpha] = 2[\iota_{2n-1}, \iota_{2n-1}] = 0 \). So, by chasing the diagram and using Lemma 13, we conclude that \( \Sigma: \pi_r(Y) \rightarrow \pi_{r+1}(X) \) is monomorphic. This completes the proof.
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