ON PERIODIC P.I. RINGS AND
LOCALLY FINITE RINGS

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An element $x$ of a ring $R$ is called periodic if there exist distinct positive integers $m, n$ for which $x^m = x^n$. Especially, $x$ is called potent if $x^n = x$ for some positive integer $m > 1$. A ring $R$ is called periodic if all elements of $R$ are periodic. It is easily seen that a periodic ring $R$ has the property that every element of $R$ is expressible as a sum of a potent element and a nilpotent element. However it is not known whether a ring $R$ with this property is periodic or not. On the other hand, by a result of the author and H. Tominaga [6], if $R$ is a P. I. ring in which every element is the sum of two idempotents, then $R$ is periodic. In this paper, we shall prove that a P. I. ring $R$ in which every element is expressible as a sum of two periodic elements, is periodic.

We shall next consider the local finiteness of a periodic P. I. ring. A ring $R$ is said to be locally finite if any finitely generated subring of $R$ is a finite ring. Let $R$ be a periodic P. I. ring, and $S$ a finitely generated subring of $R$. We shall show that the additive group of $S$ is finitely generated and that some power of $S$ is a finite ring. Consequently a P. I. ring $R$ is locally finite if and only if $R$ is periodic and the additive group of $R$ is a torsion group. Using this, we shall give a characterization of a locally finite ring.

We begin with the following lemma.

**Lemma 1.** Let $R$ be a ring. Then $R$ is periodic if and only if all prime factor rings of $R$ are periodic.

**Proof.** Suppose that all prime factor rings of $R$ are periodic. For each $x \in R$, let $S(x) = \{x^n - x^{n+1}f(x) | n > 0 \}$ is an integer, $f(t) \in \mathbb{Z}[t] \}$, which is multiplicatively closed. By virtue of [3, Proposition 2], $R$ is periodic if and only if $0 \in S(x)$ for all $x \in R$. Assume, to the contrary, that there exists $a \in R$ such that $0 \notin S(a)$. Then, by Zorn's lemma, we can find an ideal $I$ of $R$ which is maximal with respect to the property that $S(a) \cap I = \emptyset$. It is easy to check that $I$ is a prime ideal of $R$. Hence $R/I$ is periodic by hypothesis. But this contradicts the fact that $S(a) \cap I = \emptyset$.

A ring $R$ is said to be of bounded index (of nilpotence) if there is a positive integer $n$ such that $a^n = 0$ for any nilpotent element $a$ in $R$. The least
such integer is called the index of $R$. We shall show that a periodic ring of bounded index is a P.I. ring. Let $G$ denote the symmetric group of degree $n$. The identity

$$s_n = \sum_{\sigma \in G} \text{sgn}(\sigma)X_{1\sigma}X_{2\sigma} \cdots X_{n\sigma}$$

is called the standard identity of degree $n$.

**Proposition 1.** Let $n$ be a positive integer and let $R$ be a periodic ring of index $n$. Then $R$ satisfies the polynomial identity $(s_{2n})^n$.

**Proof.** Let $J$ denote the Jacobson radical of $R$, and $x$ an element of $J$. Then there exist positive integers $p, q$ such that $x^{p+q} = x^p$. By [4, Theorem 1.2.3] all elements of $J$ are right-quasi-regular. Hence there exists $y \in R$ such that $(-x^p + y + (-x^q)y = 0$. Then $x^p = x^p + x^p(-x^q + y - x^q)y = (x^p - x^{p+q}) + (x^p - x^{p+q})y = 0$. This implies that $J$ is a nil ideal. Let $P$ be a primitive ideal of $R$. By [7, Theorem 2.3] $R/P = M_t(D)$ for some division ring $D$ and some positive integer $t \leq n$. Since $D$ is a periodic division ring, $D$ is commutative by [4, Lemma 3.1.3]. Hence $R/P$ satisfies the standard identity $s_{2n}$ of degree $2n$ by [8, Theorem 1.4.1]. Since $R/J$ can be embedded in the direct product of all primitive factor rings of $R$, $R/J$ also satisfies the identity $s_{2n}$, in other words, $s_{2n}(a_1, a_2, \ldots, a_{2n}) \in J$ for all elements $a_1, a_2, \ldots, a_{2n}$ in $R$. Since $J$ is a nil ideal of index at most $n$, we have that $s_{2n}(a_1, a_2, \ldots, a_{2n})^n = 0$ for all $a_1, a_2, \ldots, a_{2n} \in R$. This completes the proof.

If $R$ is a periodic ring, each element $x$ in $R$ can be expressed in the form $y + w$, where $y^n = y$ for some $n = n(y) > 1$ and $w$ is nilpotent (e.g., see [2, Lemma 1]). However it is not known whether this property characterizes a periodic ring. On the other hand, by [6, Theorem 2], if $R$ is a P.I. ring in which every element is the sum of two idempotents then, for any $x \in R$, $x^3 - x$ is nilpotent. Hence $R$ is periodic by [3, Proposition 2]. We shall now prove the following

**Theorem 1.** Let $R$ be a P.I. ring. If every element of $R$ is expressed as a sum of two periodic elements, then $R$ is periodic.

**Proof.** By virtue of Lemma 1, we may assume that $R$ is a prime ring. Then, by [5, Theorem 1.4.2] the center $C$ of $R$ is nonzero. We claim that $C$ is periodic. Let $c$ be a nonzero element of $C$. Then, by hypothesis, there
exist \( x, y \in R \) such that \( c = x + y \), \( x^n = x^n \) for some \( m > n > 0 \), and \( y^p = y^q \) for some \( p > q > 0 \). Then \((c - y)^m = (c - y)^n\), and so \( c^m - c^n = zy \) for some \( z \in C[y](\subseteq R) \). If \( c^m - c^n \) is nilpotent, then \( c^m = c^n \), because \( C \) is an integral domain. Assume now that \( c^m - c^n \) is not nilpotent. Then \( e = y^{p-q} \) is a nonzero idempotent and \( y^q e = ey^q = y^q \). Therefore we have that \( (c^m - c^n)^q(ae - a) = 0 \) for all \( a \in R \). Let us put \( L = \{ ae - a \mid a \in R \} \). Then \( L \) is a left ideal of \( R \), and as seen above, \( (c^m - c^n)^q L = 0 \). Since \( (c^m - c^n)^q = 0 \) and since \( R \) is a prime ring, we obtain \( L = 0 \), that is, \( e \) is a right identity of \( R \). We can similarly prove that \( e \) is a left identity of \( R \). Hence \( e \) is the identity of \( R \). We shall now prove that the characteristic of \( R \) is nonzero. Assume, to the contrary, that the characteristic of \( R \) is zero. Then we may assume that \( R \) contains the ring \( Z \) of integers as a subring. By hypothesis, there exist two periodic elements \( v, w \in R \) such that \( 3 = v + w \). Obviously the subring \( S = Z[v, w] \) of \( R \) generated by \( v \) and \( w \) over \( Z \) is a commutative ring which is integral over \( Z \). By [1, Theorem 5.10] there exists a prime ideal \( P \) of \( S \) such that \( P \cap Z = 0 \). Consider now the factor ring \( \overline{S} = S/P \). Then \( \overline{S} \) is an integral domain which is integral over \( Z \). So, without loss of generality, we may assume that \( \overline{S} \) is a subring of the field \( C \) of complex numbers. In general, if \( a \) is a periodic element of \( C \), then the absolute value \( |a| \) of \( a \) is either 0 or 1. Hence we have \( 3 = |v + w| \leq |v| + |w| \leq 2 \), which is a contradiction. Therefore the characteristic of \( R \) is nonzero. Let \( F \) denote the prime field of \( C \). Since \( x \) and \( y \) are integral over \( F \), \( c = x + y \) is integral over \( F \). Hence \( c \) generates a finite subring of \( C \), and so \( c \) is periodic. Therefore we proved that \( C \) is a periodic field. By [8, Corollary 1.6.28], \( R \) is a simple P. I. ring. Hence, by Kaplansky's theorem [8, Theorem 1.5.16], \( R \) can be identified with the matrix ring \( M_d(D) \) over a division ring \( D \) which is finite dimensional over \( C \). Then \( D \) is also periodic, and hence \( D \) is commutative. Thus we get \( C = D \). Therefore \( R = M_d(C) \) is periodic.

We shall next consider the finitely generated subrings of a periodic P. I. ring. Clearly a periodic P. I. ring need not be locally finite. For example, the subring

\[
\begin{pmatrix}
0 & Z \\
0 & 0
\end{pmatrix}
\]

of \( M_d(Z) \) is a finitely generated periodic commutative ring, but this is not a finite ring. We shall prove the following:

**Theorem 2.** Let \( R \) be a periodic P. I. ring and let \( S \) be a finitely gener-
ated subring of $R$. Then the additive group $S^*$ of $S$ is a finitely generated abelian group. Moreover there exists a positive integer $n$ such that $S^n$ is a finite ring. In particular, if $S$ has an identity, then $S$ is finite.

Proof. Let $t(S)$ denote the torsion submodule of the $\mathbb{Z}$-module $S$. Then $t(S)$ is an ideal of $S$ and $S/t(S)$ is torsion-free. Let $x$ be an element of $S/t(S)$. Then $x^{m+n} = x^m$ for some positive integers $m, n$. Then we can easily see that $x^m$ is an idempotent. Since $(2x^{m+n})^{p+q} = (2x^{m+n})^p$ for some positive integers $p$ and $q$, we obtain a positive integer $h$ such that $hx^{m+n} = 0$. Since $S/t(S)$ is torsion-free, we conclude that $x^{m+n} = 0$. Thus $S/t(S)$ is a nil ring. Since $S/t(S)$ is also a finitely generated P. I. ring, there exists a positive integer $n$ such that $(S/t(S))^n = 0$ by [8, Proposition 1.6.34]. Hence we have $S^n \subseteq t(S)$. Let $c_1, c_2, \ldots, c_m$ generate the subring $S$. Then $A = \{c_{i_1}c_{i_2} \cdots c_{i_r}| 1 \leq i_j \leq m\}$ is a finite set, and hence there exists a positive integer $k$ such that $kA = 0$. Hence we have $kS^n = 0$. Let $B$ denote the set $\{c_{i_1}c_{i_2} \cdots c_{i_p}| 1 \leq i_j \leq m, 1 \leq p \leq n\}$. Then we can easily see that

$$kS = \sum_{b \in B} \mathbb{Z}kb.$$

Hence $kS$ is a finitely generated $\mathbb{Z}$-module. Let $S'$ denote the ring $S/kS$ and let us write $k = \Pi_{i=1}^l p_i^{k_i}$ where the $p_i$ are distinct primes and $k_i > 0$ for all $i$. Then, for each $i$, $S'_i = \{a \in S'| p_i^{k_i}a = 0\}$ is a subring of $S'$ and $S'$ is the direct sum of $S'_1, S'_2, \ldots, S'_l$. We shall show that $S'$ is finite. To show it, it suffices to prove that $S'_i$ is finite for each $i = 1, 2, \ldots, l$. Hence, without loss of generality, we may assume that $k = p^h$ for some prime $p$ and some positive integer $h$. Let us set $I = pS'$. Then $I^h = 0$ and $p^{h-1}I = 0$. Then the ring $S'/I$ is a finitely generated periodic algebra over $\mathbb{Z}/p\mathbb{Z}$ satisfying a polynomial identity. Hence $S'/I$ is a finite dimensional algebra over $\mathbb{Z}/p\mathbb{Z}$ by [4, Theorem 6.4.3]. Let $S'/I = \{a_0 + I, a_1 + I, \ldots, a_d + I\}$ where $a_0 = 0$, $a_1, a_2, \ldots, a_d$ are elements of $S'$. Then we can choose elements $b_1, b_2, \ldots, b_f$ of $I$ such that $a_i, a_2, \ldots, a_d, b_1, b_2, \ldots, b_f$ generate $S'$. For any $i, j$ with $1 \leq i, j \leq d$, we have a unique integer $t(i, j)$ with $1 \leq t(i, j) \leq d$ such that $a_i a_j \equiv a_{t(i, j)}$ modulo $I$. Similarly we have a unique integer $s(i, j)$ such that $a_i + a_j \equiv a_{s(i, j)}$ modulo $I$. Let us now set $x_{ij} = a_i a_j - a_{s(i, j)}$ and $y_{ij} = a_i + a_j - a_{t(i, j)}$ for each $1 \leq i, j \leq d$. Let $J$ denote the subring of $S'$ generated by $x_{\alpha\beta}, y_{\gamma\nu}, b_\lambda, a_\gamma x_{\alpha\beta}, a_\gamma y_{\mu\nu}, a_\beta b_\lambda, x_{\alpha\beta} y_{\mu\nu}, y_{\mu\nu} y_{\lambda\gamma}, b_\lambda a_\gamma$ for $1 \leq \alpha, \beta, \gamma \leq d, 1 \leq \mu, \nu \leq d, \text{ and } 1 \leq \lambda \leq f$. Then $J$ is a finitely generated subring of $I$. Since $I^h = 0$ and $p^{h-1}I = 0$, $J$ must be finite. We can now easily see that each element $x$ of $S'$ can be uniquely expressed in the form $a_i + z$, where $0 \leq i \leq d$ and $z \in J$. This implies that $I = J$. Therefore $S'$ is a finite ring. Conse-
quently $S$ is a finitely generated $\mathbb{Z}$-module. Since the additive group of $S^n$ is a torsion group, $S^n$ is a finite ring. In particular, if $S$ has an identity, then $S^n = S$, and hence $S$ is finite.

As an immediate consequence of this theorem, we obtain the following:

**Corollary 1.** Let $R$ be a P. I. ring. Then $R$ is locally finite if and only if $R$ is periodic and the additive group of $R$ is a torsion group.

A ring $R$ is said to be of locally bounded index if every finitely generated subring of $R$ is of bounded index. Combining Corollary 1 with Proposition 1, we obtain the following characterization of a locally finite ring.

**Corollary 2.** A ring $R$ is locally finite if and only if $R$ is a periodic ring of locally bounded index and the additive group of $R$ is a torsion group.

The following example due to Golod and Shafarevitch shows that a finitely generated periodic ring with torsion additive group need not be finite.

**Example 1.** Let $p$ be a prime number. By [4, Theorem 8.1.3], there exists an infinite dimensional nil algebra $A$ over $\mathbb{Z}/p\mathbb{Z}$ generated by three elements. Clearly $A$ is generated by those three elements as a ring. Note that those elements generate infinite subsemigroup of the multiplicative semigroup of $R$.

As another corollary of Theorem 2, we obtain the following

**Corollary 3.** Let $R$ be a P. I. ring. Then the following statements are equivalent:

1. $R$ is periodic.
2. For any finitely generated subring $S$ of $R$, there exists a positive integer $n$ such that $S^n$ is a finite subring.
3. For any finitely generated subring $S$ of $R$, there exists a finite ideal $I$ of $S$ such that $S/I$ is a nilpotent ring.
4. The ideal $t(R) = \{a \in R | na = 0 \text{ for some positive integer } n\}$ is locally finite and $R/t(R)$ is a nil ring.

**Proof.** The implication (1) $\Rightarrow$ (2) follows from Theorem 2 and (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Let $x$ be an element of $R$, and $S$ denote the subring of $R$
generated by $x$. Then there exists a finite ideal $I$ of $S$ such that $S/I$ is nilpotent. This implies that some power of $x$ generates a finite subring. Hence there exist distinct positive integers $m, n$ such that $x^m = x^n$.

(1)$\iff$(4). Assume that $R$ is periodic. By Corollary 1 $t(R)$ is locally finite. We also know that $R/t(R)$ is a nil ring by the proof of Theorem 2.

Conversely, suppose that (4) holds, and let $x$ be an element of $R$. Then some power of $x$ generates a finite subring of $R$, and hence $x$ is periodic.

A ring $R$ is periodic if and only if each subsemigroup of $R$ generated by a single element is finite. If $R$ is a commutative periodic ring, then all finitely generated subsemigroups of $R$ are finite. However Example 1 shows that this does not remain valid for noncommutative periodic rings. Thus we have the following

**Conjecture.** Let $R$ be a periodic P. I. ring. Then all finitely generated subsemigroups of $R$ are finite.

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**References**


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