CYCLE STRUCTURE OF DICKSON PERMUTATION POLYNOMIALS

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1. If $R$ is a commutative ring with identity and $a \in R$, then the Dickson polynomial $D_n(x, a)$ of degree $n$ is defined by

$$D_n(x, a) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n}{n-j} \binom{n-j}{j} (-a)^j x^{n-2j},$$

where $\lfloor \rfloor$ denotes the greatest integer function. Dickson polynomials have been extensively studied over finite fields and over residue class rings of integers as well as over various other rings. For a survey of many properties of Dickson polynomials including applications to cryptography and number theory, see Lidl [3] and for results related to finite fields, see Lidl and Niederreiter [4] and Mullen [6].

If $F_q$ denotes the finite field of order $q$ a prime power, it is well known that $D_n(x, 0) = x^n$ permutes $F_q$ if and only if $n$ and $q-1$ are relatively prime, i.e. if and only if $(n, q-1) = 1$, and for $a \neq 0$, $D_n(x, a)$ permutes $F_q$ if and only if $(n, q^2 - 1) = 1$. Moreover the Dickson permutation polynomials are closed under composition of polynomials if and only if $a = 0, 1,$ or $-1$, see [4, Thm.7.22] for details.

In section 2 we determine the cycle structure of the Dickson permutation polynomials over $F_q$ and in section 3 we consider the analogous problem in the setting of a Galois ring.

2. Finite Fields. We will make use of the following properties. First for $a, x \in F_q$, let $\mu \in F_q^{*}$ be such that $x = \mu + a/\mu$. Then the functional equation for Dickson polynomials indicates that

$$D_n(x, a) = \mu^n + a^n/\mu^n. \quad (1)$$

see [4, Equation (7.8)]. Use will also be made of the easy to prove fact

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that for $a \in F_q$, if $M(a)$ is the subset of $F_q^r$ consisting of all solutions of the $q$ equations of the form $x^2 - rx + a = 0$ with $r \in F_q$, then

$$M(a) = |\mu \in F_q^r| \mu^{q-1} = 1 \text{ or } \mu^{q^2-a} = a|. \quad (2)$$

We now consider the cycle structure of the Dickson permutation polynomials. While the cycle structure for the power polynomial $x^n$ on $F_q^r$ was determined in Ahmad [1], for the sake of completeness we restate the result here. Recall that $n$ belongs to the exponent $m \mod t$ if $m$ is the smallest positive integer such that $n^m \equiv 1 \mod t$. Throughout this paper we let $(a, b) = \gcd(a, b)$.

**Theorem 1.** Let $m$ be a positive integer. Then $x^n$ has a cycle of length $m$ over $F_q^r$ if and only if $q-1$ has a divisor $t$ such that $n$ belongs to the exponent $m \mod t$. Moreover the number $N_m$ of such cycles is

$$mN_m = (q-1, n^m-1) - \sum_{i \mid m, t \mid m} iN_t. \quad (3)$$

**Proof.** We have $x^{n^m} = x$ if and only if $n^m - 1 \equiv 0 \mod t$ where $t$ is the multiplicative order of $x$ so the first part follows. There are $mN_m$ elements that belong to cycles of length $m$ and $(n^m-1, q-1)$ elements of $F_q^r$ which belong to cycles of length $i$ where $i \mid m$.

From (3) with $m = 1$ we can easily deduce that $x^n$ has $(q-1, n-1)+1$ fixed points over $F_q$ where by a fixed point is meant an element $x$ so that $x^n = x$.

**Theorem 2.** Let $m$ be a positive integer and let $D_n(x, 1)$ permute $F_q$. Then $D_n(x, 1)$ has a cycle of length $m$ if and only if $q-1$ or $q+1$ has a divisor $t$ such that $n^m \equiv \pm 1 \mod t$. Moreover the number $M_m$ of such cycles is

$$mM_m = [(q+1, n^m+1)+(q-1, n^m+1)+(q+1, n^m-1)$$

$$+(q-1, n^m-1)]/2 - \varepsilon_1 - \sum_{i \mid m, t \mid m} iM_t. \quad (4)$$

where

$$\varepsilon_1 = \begin{cases} 1 & \text{if } p = 2 \text{ or } p \text{ is odd and } n \text{ is even} \\ 2 & \text{if } p \text{ is odd and } n \text{ is odd}. \end{cases}$$

**Proof.** From (2) let

$$M_1(a) = |\mu \in F_q^r| \mu^{q^2-a} = a|. \quad M_1(a) = |\mu \in F_q^r| \mu^{q-a} = 1|.$$
If \( w \) is a primitive element of \( F_q \) then
\[
M_1(1) = \{ w^{q-1}\tau^r \mid r = 0, 1, \ldots, q \}, \quad M_2(1) = \{ w^{q+1+s} \mid s = 0, 1, \ldots, q-2 \}.
\]
We note that \( \mu \in M_1(1) \cap M_2(1) \) if and only if \( \mu = \pm 1 \). Let \( N_1(1) = \{ 1 \} \) if \( p = 2 \) and \( N_1(1) = \{ \pm 1 \} \) if \( p \) is odd and let \( N_1(1) = M_1(1) \setminus N_1(1) \) and \( N_2(1) = M_2(1) \setminus N_3(1) \). We note that \( M(1) \) is the disjoint union \( N_1(1) \cup N_1(1) \cup N_3(1) \). Finally if \( \mu \) is a solution of \( z^2 - \rho z + 1 = 0 \), so is \( \mu^{-1} \), and \( \mu = \mu^{-1} \) if and only if \( \mu^2 = 1 \) so that \( \mu \in N_3(1) \).

Let \( D_m^n(x, 1) \) denote the \( m \)-th iterate of \( D_m(x, 1) \) under composition. Using the functional equation (1), an element \( x = \mu + \mu^{-1} \) has the property that \( D_m^n(\mu + \mu^{-1}, 1) = \mu + \mu^{-1} \) if and only if \( \mu^n + \mu^{-n} = \mu + \mu^{-1} \), i.e. if and only if
\[
(\mu^{n-1} - 1)(\mu^{n-1} - 1) = 0.
\]
(5)
Since a solution \( v \) of (5) is a solution of both \( \mu^{n+1} = 1 \) and \( \mu^{n-1} = 1 \) if and only if \( v \in N_3(1) \), the number of solutions to (5) on \( M(1) \) is the sum of the number of solutions of \( \mu^{n+1} = 1 \) and \( \mu^{n-1} = 1 \) on \( N_1(1) \) and \( N_3(1) \) plus the number of solutions of (5) on \( N_3(1) \).

Now \( v \in M_1(1) \) is a solution of \( \mu^{n+1} = 1 \) if and only if \( r(n^m + 1) \equiv 0 \mod (q + 1) \). This congruence has \( (q + 1, n^m + 1) \) solutions. Similarly \( \mu^{n+1} = 1 \) has exactly \( (q - 1, n^m + 1) \) solutions on \( M_1(1) \). \( \mu^{n-1} = 1 \) has exactly \( (q + 1, n^{m-1}) \) solutions on \( M_1(1) \) and \( \mu^{n-1} = 1 \) has exactly \( (q - 1, n^{m-1}) \) solutions on \( M_1(1) \). We also note that (5) has exactly one solution if \( p = 2 \) or \( p \) is odd and \( n \) is even and it has exactly two solutions when \( p \) is odd and \( n \) is odd. Thus (5) has exactly \( \epsilon_1 \) solutions on \( N_3(1) \). Noting that \( \mu \) is a solution to (5) if and only if \( \mu^{-1} \) is a solution, the proof is complete.

It is worth remarking that for \( m = 1 \) Theorem 2 holds for any \( n \geq 1 \), not just those for which \( D_m(x, 1) \) permutes \( F_q \). Theorem 2 thus determines the number of fixed points of \( D_m(x, 1) \) over \( F_q \).

We now consider the case where \( a = -1, n \) is odd, and since \( D_n(x, -1) = D_n(x, 1) \) if \( p = 2 \), we may assume the characteristic \( p \) of \( F_q \) is odd. Let \( \nu_p(m) \) denote the highest power of \( p \) dividing \( m \) for \( m \neq 0 \) and set \( \nu_p(0) = \infty \). Then clearly \( \nu_p((a, b)) = \min \{ \nu_p(a), \nu_p(b) \} \), \( \nu_p(ab) = \nu_p(a) + \nu_p(b) \) and if \( a \mid b \), then \( \nu_p(b/a) = \nu_p(b) - \nu_p(a) \) for integers \( a \) and \( b \). We can now prove

**Theorem 3.** Let \( m \) be a positive integer. If \( n \) and \( q \) are odd then
$D_n(x, -1)$ has a cycle of length $m$ if and only if $q - 1$ or $q + 1$ has a divisor $t$ such that $n^m \equiv 1 \mod t$ or $2(n^m + 1) \equiv 0 \mod t$. Moreover the number $K_m$ of such cycles is

$$mk_m = [(a_1(n^m + 1, 2(q + 1)) + a_2(n^m + 1, q - 1) + a_3((n^m - 1)/2, q + 1) + (n^m - 1, q - 1)]/2 - \varepsilon_{-1} - \sum_{i=1, j< m} iK_i,$$

where

$$a_1 = \begin{cases} 1 & \text{if } v_2(n^m + 1) = v_2(q + 1) \\ 0 & \text{otherwise}, \end{cases}$$

$$a_2 = \begin{cases} 1 & \text{if } v_2(n^m + 1) < v_2(q + 1) \\ 0 & \text{otherwise}, \end{cases}$$

$$a_3 = \begin{cases} 1 & \text{if } v_2(n^m - 1) > v_2(q + 1) \\ 0 & \text{otherwise}, \end{cases}$$

$$\varepsilon_{-1} = \begin{cases} 2 & \text{if } n^m \equiv 1 \mod 4 \text{ and } q \equiv 1 \mod 4 \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** We first note that if $w$ is a primitive element of $F_{q^2}$, then

$$M_1(-1) = |w^{(q-1)r^2}| r = 1, 3, \ldots, 2q+1|,$$

$$M_2(-1) = |w^{(q+1)q}| s = 0, 1, \ldots, q-2|.$$

For $i = 0, 1$ let $\mu_i = w^{(q^2-1)q^i}$. For $\mu \in M_1(-1) \cap M_2(-1)$ we have $\mu^{q+1} = -1$ and $\mu^{q-1} = 1$ so that $\mu^2 = -1$ and $\mu \notin |\mu_0, \mu_i|$. If $q = 4t+1$ then for $i = 0, 1$, $\mu^{q+1} = w^{(q^2-1)q^i} = -1$ and $\mu^{q-1} = 1$ for $i = 0, 1$ so $\mu \in M_1(-1) \cap M_2(-1)$. If $q = 4t+3$ then $\mu^{q+1} = 1$ and $\mu^{q-1} = -1$ and so $\mu \notin M_1(-1)$ and hence $\mu \notin M_1(-1) \cap M_2(-1)$.

Let $N_1(-1) = |\mu_0, \mu_1| \text{ if } q \equiv 1 \mod 4$ and let $N_3(-1) = \phi$ if $q \equiv 3 \mod 4$ and for $j = 1, 2$ let $N_j(-1) = M_j(-1) \backslash N_3(-1)$. Then $M(-1)$ is the disjoint union $M_1(-1) \cup M_2(-1) \cup N_3(-1)$. Finally note that $z^2 - \rho z - 1$ has solutions $\mu$ and $\mu^{-1}$ and $\mu = -\mu^{-1}$ only when $\mu^2 = -1$ so that $\mu \in N_3(-1)$.

If $x = \mu - \mu^{-1}$ satisfies $D_n(x, -1) = x$ then by (1) we have

$$(\mu^{n^{x+1}} + 1)(\mu^{n^{x-1}} - 1) = 0.$$  \hspace{1cm} (6)

Since a solution $\mu$ of (6) is a solution of both $\mu^{n^{x-1}} = -1$ and $\mu^{n^{x+1}} = 1$ if and only if $\mu \in N_3(-1)$, the number of solutions of (6) which are in $M(-1)$ is the sum of the number of solutions of the equations in the sets
$N_1(-1)$ and $N_2(-1)$ plus the number of solutions of (6) which are in $N_3(-1)$. 

An element $v \in M_1(-1)$ is a solution of $\mu^{n^{m+1}} = -1$ if and only if

$$r(n^m+1) \equiv q+1 \mod (2(q+1)).$$

(7)

This is solvable if and only if $v_2(n^m+1) \leq v_2(q+1)$. Let $d = (n^m+1, 2(q+1))$ so that $v_2(d) = \min|v_2(n^m+1), v_2(q+1)+1| = v_2(n^m+1)$. If $\alpha$ and $\beta$ are integers with

$$\alpha(n^m+1) + 2\beta(q+1) = d.$$ 

(8)

then all solutions of (7) are given by

$$\frac{\alpha(q+1)}{d} + \frac{2(q+1)i}{d}, \ i = 0, 1, \ldots, d-1.$$

If $\alpha$ is even, $v_2(d) \geq \min|v_2(n^m+1)+1, v_2(q+1)+1| > v_2(n^m+1)$ so that by considering the highest power of 2 in (8), we have a contradiction so that $\alpha$ must be odd. Now $v_2((q+1)/d) = v_2(q+1) - v_2(n^m+1)$ so that $(q+1)/d$ is odd if and only if $v_2(q+1) = v_2(n^m+1)$. Since $d\mid(q+1), 2(q+1)/d$ is even. Finally (7) has a solution $r$ with $r$ odd if and only if $v_2(n^m+1) = v_2(q+1)$ and in this case it has $(n^m+1, 2(q+1))$ solutions each of which is odd.

Now $v \in M_2(-1)$ is a solution of $\mu^{n^{m+1}} = -1$ if and only if $w^{(q+1)\cdot n^{m+1}} = -1$, i.e. if and only if $s(n^m+1) \equiv (q-1)/2 \mod (q-1)$ which is solvable if and only if $v_2(n^m+1) < v_2(q-1)$ in which case it has $(n^m+1, q-1)$ solutions. Similarly $v \in M_1(-1)$ is a solution of $\mu^{n^{m-1}} = 1$ if and only if

$$r(n^{m-1}) \equiv 0 \mod (2(q+1)).$$

(9)

Let $d = (n^m-1, 2(q+1))$ so that all solutions of (9) are given by $2(q+1)/d$ for $i = 0, 1, \ldots, d-1$. Moreover $2(q+1)/d$ is odd if and only if $v_2(n^m-1) > v_2(q+1)$. Hence (9) is solvable if and only if $v_2(n^m-1) \geq v_2(q+1)$ and then it has $((n^m-1)/2, q+1)$ odd solutions.

We note that $\mu^{n^{m-1}} = 1$ has exactly $(n^m-1, q-1)$ solutions in $M_1(-1)$. The set of all solutions of (9) on $N_3(-1)$ is the set of all solutions of $\mu^{n^{m-1}} = 1$ on $N_3(-1)$ which is empty if $q \equiv 3 \mod 4$. It is also empty if $q \equiv 1 \mod 4$ and $n^m \equiv 3 \mod 4$ and it is equal to $|\mu_0, \mu_1|$ if $q \equiv n^m \equiv 1 \mod 4$. Hence $\varepsilon_{-1}$ is determined. To complete the proof we note that $\mu$ is a solution of $\mu^{n^{m+1}} = -1$ (resp. $\mu^{n^{m-1}} = 1$) if and only if $-\mu^{-1}$ is also a solution.

We note that if $m = 1$, the above results reduce to those of Nöbauer.
[7] for the number of fixed points of $D_n(x, a)$ where by a fixed point is meant an element $x \in F_q$ with the property that $D_n(x, a) = x$.

3. Galois Rings. If $p$ is a prime and $r, s \geq 1$ are integers $GR(p^r, s)$ will denote the Galois ring of order $p^{rs}$ which can be obtained as a degree $s$ Galois extension of $\mathbb{Z}/(p^r)$, the residue class ring of integers mod $p^r$. Thus as special cases we have $GR(p^r, 1) = \mathbb{Z}/(p^r)$ and $GR(p, s) = F_{p^s}$, the finite field of order $p^s$. Numerous properties of Galois rings can be found in Chapter XVI of McDonald [5].

In Gomez-Calderon and Mullen [2, Thm.3] it was shown that if $a \in GR(p^r, s)$ is a unit, then $D_n(x, a)$ permutes $GR(p^r, s)$ with $r > 1$ if and only if $(n, p^{rs} - 1) = (n, p) = 1$ while in Theorem 4 of that same paper, it was shown that the Dickson permutation polynomials with a unit, are closed under composition if and only if $a = \pm 1$. It is thus sufficient to consider the cycle structure of $D_n(x, a)$ over $GR(p^r, s)$ for $a = 0, \pm 1$. We consider only those cases where $(n, p) = 1$.

For $a = 0$ we have by [2, Cor. 15(a)] that $D_n(x, a) = x^n$ permutes $GR(p^r, s)$ if and only if $n = 1$ or $r = 1$ and $(n, p^{s} - 1) = 1$. For $a = \pm 1$ we make use of the following results of [2]. The first result generalizes the well known result concerning lifting solutions over $\mathbb{Z}/(p^r)$.

**Lemma 5.** Let $f(x)$ be a monic polynomial with coefficients in $GR(p^r, s)$. Assume $r \geq 2$ and let $t$ be a solution of the equation $f(x) = 0$ in the Galois ring $GR(p^{r-1}, s)$.

(a) Assume $f'(t) \not= 0$ over the field $GR(p, s)$. Then $t$ can be lifted in a unique way from $GR(p^{r-1}, s)$ to $GR(p^r, s)$.

(b) Assume $f'(t) = 0$ over the field $GR(p, s)$. Then we have two possibilities:

(b.1) If $f(t) = 0$ over $GR(p^r, s)$, $t$ can be lifted from $GR(p^{r-1}, s)$ to $GR(p^r, s)$ in $p^s$ distinct ways.

(b.2) If $f(t) \neq 0$ over $GR(p^r, s)$, $t$ cannot be lifted from $GR(p^{r-1}, s)$ to $GR(p^r, s)$.

The next technical lemma is proved as Corollary 6 of Gomez-Calderon and Mullen [2]. The structure of the group $U(p^r, s)$ of units of $GR(p^r, s)$ is given in McDonald [5, p.322–323].

**Lemma 6.** For $p$ odd and $q = p^s$, let $w = fp^r$ denote a positive integer
with \((f, p) = 1\). The group \(U(p^r, 2s)\) of units can be written as a product of a cyclic group \(G\) of order \(q^r - 1\) and \(2s\) cyclic groups \(H_i\) each of order \(p^{r-1}\). Let \(H_i\) denote the subgroup of \(H_i\) of order \((p^r, p^{r-1})\) for \(i = 1, \ldots, 2s\). Let \(C_1\) and \(C_2\) denote the groups \(C_1 = H_1 \times \cdots \times H_s\) and \(C_2 = H_{s+1} \times \cdots \times H_{2s}\) where \(H_i = \langle \beta_i \rangle\) and

\[
\sigma(\beta_i) = \begin{cases} 
\beta_i & \text{if } 1 \leq i \leq s \\
\beta_i^{-1} & \text{if } s < i \leq 2s,
\end{cases}
\]

where \(\sigma\) denotes a generator of the Galois group for \(GR(p^r, 2s)/GR(p^r, s)\).

Then

(a) Assume \(\mu \in GR(p^r, s)\). Then

(a.1) \(|\mu|\mu^w = 1| = A_1 \times C_1\) where \(A_1\) denotes the group of \(G\) of order \((w, q-1)\).

(a.2) \(|\mu|\mu^w = -1| = \begin{cases} 
\phi & \text{if } w/(w, (q-1)/2) \text{ is even} \\
|ac| a \in G, a^{w(q-1)/2} = -1, c \in C_1 & \text{otherwise.}
\end{cases}\)

(b) Assume \(\mu \in GR(p^r, 2s)\). Then

(b.1) \(|\mu|\mu^w = 1, \mu\sigma(\mu) = 1| = A_2 \times C_2\) where \(A_2\) denotes the subgroup of \(G\) of order \((w, q+1)\).

(b.2) \(|\mu|\mu^w = -1, \mu\sigma(\mu) = -1| = \begin{cases} 
\phi & \text{if } w/(w, q+1) \text{ or } (q+1)/(w, q+1) \text{ is even} \\
|ac| a \in G, a^{w(q+1)} = -1, c \in C_1 & \text{otherwise.}
\end{cases}\)

(c) Assume \(w\) is even and \(\mu \in GR(p^r, 2s)\). Then

\(|\mu|\mu^w = 1, \mu\sigma(\mu) = -1| = \phi\).

(d) Assume \(w\) is odd and \(\mu \in GR(p^r, 2s)\). Then

\(|\mu|\mu^w = 1, \mu\sigma(\mu) = -1| = \phi\).

We are now ready to prove

**Theorem 7.** Let \(r, s, m \geq 1, p\) be an odd prime and \(q = p^s\). Let \(e, E, k, K\) denote nonnegative integers such that \(m^m - 1 = ep^k\) and \(n^m + 1 = Ep^k\) with \((e, p) = (E, p) = 1\). Let \(C_{2, m}\) denote the number of cycles of \(D_n(x, 1)\) of length \(m\) over \(GR(p^r, s)\) consisting of elements \(x \equiv \pm 2 \mod p\). Then

\[
mC_{2, m} = [A(e, q) - \beta]q^{-m+1} + [B(E, q) - \beta]q^{-m+1} - \sum_{\delta \leq m < \infty} iC_{2, \delta}. \tag{10}
\]
where \( A(e, q) = [(e, q-1)+(e, q+1)]/2 \) and \( B(E, q) = [(E, q-1)+(E, q+1)]/2 \) and \( \beta = 1 \) if \( n \) is even and \( \beta = 2 \) if \( n \) is odd.

Proof. Let \( D^n_m(x, 1) \) denote the \( m \)-th iterate of \( D_n(x, 1) \) under composition. Let \( x \in GR(p^r, s) \) with \( x \equiv \pm 2 \mod p \). Then \( x = \mu + 1/\mu \) for some \( \mu \in GR(p^r, 2s) \). Then

\[
D^n_m(\mu + 1/\mu, 1) = \mu^n + 1/\mu^n = \mu + 1/\mu
\]

if and only if \((\mu^{n-1} - 1)(\mu^{n+1} - 1) = 0\).
If \( \mu^{n-1} - 1 \equiv \mu^{n+1} - 1 \equiv 0 \mod p \), then \( \mu \equiv \pm 1 \mod p \) so that \( x \equiv \pm 2 \mod p \), a contradiction. Hence \( D^n_m(\mu + 1/\mu, 1) = \mu + 1/\mu \) if and only if

\[
\mu^{n-1} = 1 \text{ or } \mu^{n+1} = 1.
\tag{11}
\]

Moreover by Lemma 5, \( x = \mu_1 + 1/\mu_1 = \mu_2 + 1/\mu_2 \) with \( \mu_1, \mu_2 \in GR(p^r, 2s) \) if and only if \( \mu_1 = \mu_2 \) or \( \mu_1 \mu_2 = 1 \).

By Lemma 6 the number of elements \( x \equiv \pm 2 \mod p \) with \( D^n_m(x, 1) = x \) is given by

\[
(1/2)[(e, q-1) + (e, q+1) - \alpha]q^{mn/[r-1, k]}
+ (1/2)[(E, q-1) + (E, q+1) - \alpha]q^{mn/[r-1, k]}
\]

where \( \alpha = 2 \) if \( n \) is even and \( \alpha = 4 \) if \( n \) is odd. By subtracting the number of elements whose cycle length divides \( m \), we complete the proof.

Let \( C_m \) be the number of cycles of \( D_n(x, 1) \) of length \( m \).

Corollary 8. Let \( r, s, m \geq 1 \). \( p \) be an odd prime and \( q = p^s \). If \( n^{2m} \equiv 1 \mod p \) then

\[
mC_m = [(n^m - 1, q-1) + (n^m - 1, q+1) + (n^m + 1, q-1) + (n^m + 1, q+1)]/2 - \varepsilon - \sum_{i=0}^{m-1} i C_i.
\]

where \( \varepsilon = 1 \) if \( n \) is even and \( \varepsilon = 2 \) if \( n \) is odd.

Proof. Let \( f(x) = D^n_m(x, 1) - x \) so that \( f(2) \equiv 0 \mod p \) and

\[
f(-2) \equiv 0 \mod p \text{ if } n \text{ is odd}
\]

\[
f(-2) \equiv 4 \mod p \text{ if } n \text{ is even}.
\]

Also \( f'(-2) = D^n_{m'}(-2, 1) - 1 = (-1)^{n-1} n^{2m} - 1 \equiv 0 \mod p \) by hypothesis. There are \( \varepsilon \) fixed points \( x \) with \( x \equiv \pm 2 \mod p \).

In an analogous way for \( a = -1 \) we may prove
Theorem 9. Let \( r, s, m \geq 1 \), \( p \) be an odd prime and \( q = p^s \). Let \( e, E, k, K \) denote nonnegative integers with \( n^{m-1} = ep^k \) and \( n^m + 1 = Ep^k \) where \( (e, p) = (E, p) = 1 \). Let \( E_{2,z,m} \) denote the number of cycles of \( D_{n}(x, -1) \) of length \( m \) over \( GR(p^r, s) \) consisting of elements \( x \) with \( x^2 \equiv -4 \mod p \). Assume \( n \) is odd. Then

\[
mE_{2,z,m} = A - \sum_{l|m, l < m} iE_{2,z,l},
\]

where \( A \)

\[
= \frac{(n^m - 1, q - 1) + ((n^m - 1)/2, q + 1) - 4}{2} q^{\min_ir - 1, k1} + \frac{(n^m + 1, (q - 1)/2) + (n^m + 1, q + 1) - 4}{2} q^{\min_ir - 1, k1} 
\]

\[
= \frac{(n^m - 1, q - 1) + \varepsilon_1}{2} q^{\min_ir - 1, k1} \quad \text{if } n^m - 1 \equiv q - 1 \equiv 0 \mod 4,
\]

where

\[
\varepsilon_1 = \begin{cases} 
0 & \text{if } (q + 1)/(n^m - 1)/2, q + 1 \text{ is even} \\
((n^m - 1)/2, q + 1) & \text{if } (q + 1)/(n^m - 1)/2, q + 1 \text{ is odd}
\end{cases}
\]

\[
= \frac{(n^m + 1, q - 1) - 2}{2} q^{\min_ir - 1, k1} + \frac{\varepsilon_2}{2} q^{\min_ir - 1, k1} \quad \text{if } n^m + 1 \equiv q - 1 \equiv 0 \mod 4,
\]

where

\[
\varepsilon_2 = \begin{cases} 
0 & \text{if } (n^m + 1)/(n^m + 1, (q - 1)/2) \text{ is even} \\
(n^m + 1, (q - 1)/2) & \text{if } (n^m + 1)/(n^m + 1, (q - 1)/2) \text{ is odd}
\end{cases}
\]

\[
= \frac{(n^m + 1, q - 1)}{2} q^{\min_ir - 1, k1} \quad \text{if } n^m + 1 \equiv q + 1 \equiv 0 \mod 4.
\]

Corollary 10. Let \( r, s, m \geq 1 \), \( p \) be an odd prime and \( q = p^s \). Let \( n \) be an odd positive integer with \( n^m \equiv \pm 1 \mod p \). If \( E_m \) denotes the number of cycles of \( D_{n}(x, -1) \) of length \( m \), then

\[
mE_m = B - \sum_{l|m, l < m} iE_l,
\]

where \( B \)
\[
\begin{align*}
&= \frac{(n^n - 1, q - 1) + \left( (n^n - 1)/2, q + 1 \right) + (n^n + 1, (q - 1)/2) + (n^n + 1, q + 1) - 4}{2} \\
&= \frac{(n^n - 1, q - 1) + \varepsilon_1}{2} \quad \text{if } n^n - 1 \equiv q - 1 \equiv 0 \mod 4. \\
&= \frac{(n^n + 1, q - 1) - 2 + \varepsilon_1}{2} \quad \text{if } n^n - 1 \equiv q + 1 \equiv 0 \mod 4. \\
&= \frac{(n^n + 1, q - 1)}{2} \quad \text{if } n^n + 1 \equiv q - 1 \equiv 0 \mod 4. \\
&= \frac{n^n + 1}{2} \quad \text{if } n^n + 1 \equiv q + 1 \equiv 0 \mod 4.
\end{align*}
\]

If \( x^2 \not\equiv 4a \mod p \) then \( x \in GR(p^r, s) \) can be written as \( x = \mu + a/\mu \) for some \( \mu \in GR(p^r, 2s) \). However if \( x^2 \equiv 4a \mod p \), then it may not be possible to write \( x \) in the above form. In this case the above argument becomes much more complicated and so to keep this paper to a reasonable length, we omit discussion of this more complicated case.

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