ON SOME SERIES ASSOCIATED WITH
DISCRETE SUBGROUPS OF U(1, n ; ℂ)

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0. Let $F$ be a fuchsian group acting on the unit disk. An element $g_k$ of $F$ is of the form

$$g_k(z) = \frac{a_kz + c_k}{c_kz + a_k}, \quad |a_k|^2 - |c_k|^2 = 1.$$  

It is well-known about the convergence or divergence of the series $\sum_{g_k \in F} |c_k|^{-t}$  
(see [4]). In this paper we show some generalized results on the series associated with discrete subgroups of $U(1,n; \mathbb{C})$.

1. Let us recall some definitions and notation. Let $V = V^{1,n}(\mathbb{C})$  
($n \geq 1$) denote the vector space of $\mathbb{C}^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\Phi(z, w) = -\overline{z_0} w_0 + \sum_{i=1}^n \overline{z_i} w_i.$$  
for $z = (z_0, z_1, \ldots, z_n)$ and $w = (w_0, w_1, \ldots, w_n)$. An automorphism $g$ of $V$, that is a linear bijection of $V$ onto $V$ such that $\Phi(g(z), g(w)) = \Phi(z, w)$ for $z, w \in V$, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,n; \mathbb{C})$. Let $V_- = \{ z \in V \mid \Phi(z, z) < 0 \}$. Obviously $V_-$ is invariant under $U(1,n; \mathbb{C})$. Let $P(V)$ be the projective space obtained by $V$.

We define $H^n(\mathbb{C}) = P(V_-)$. Let $\overline{H^n(\mathbb{C})}$ denote the closure of $H^n(\mathbb{C})$ in projective space $P(V)$ . An element $g$ in $U(1,n; \mathbb{C})$ operates in $P(V)$, leaving $\overline{H^n(\mathbb{C})}$ invariant. Since $H^n(\mathbb{C})$ is identified with the unit ball $B^n(\mathbb{C})$  
$= \{ |\xi| \leq \|\xi\|^2 = \sum_{k=1}^n |\xi_k|^2 < 1 \}$, we can regard discrete subgroups of $U(1,n; \mathbb{C})$ as generalized fuchsian groups (see [2]).

2. Let $g_k = (a^{(k)}_{i,j})_{1 \leq i,j \leq n+1}$ be an element in $U(1,n; \mathbb{C})$. We denote a point of $P^{-1}(0)$ by $0^*$. Let $d$ be the derived metric from $\mathfrak{F}$ (see [2, Proposition 2.4.4]). We easily obtain

**Proposition 2.1.** $|a^{(k)}_{i,i}| = |\Phi(g_k(0*), 0*)||\Phi(0*, 0*)|^{-1}$  
$\quad = \cosh d(0, g_k(0))$.
For the sake of simplicity and brevity, we denote \( 2|a_{1,1}^{(k)}| \) by \( \nu(g_k) \).

**Proposition 2.2.** If \( g \) and \( h \) are elements of \( U(1, n; \mathcal{C}) \), then

1. \( \nu(g^{-1}) = \nu(g) \).
2. \( \nu(gh) \leq \nu(g)\nu(h) \).
3. \( \nu(hgh^{-1}) \leq [\nu(h)]^2\nu(g) \leq [\nu(h)]^4\nu(hgh^{-1}) \).

**Proof.** The first is immediate.

2. Using Proposition 2.1, we have

\[
\nu(g)\nu(h) = 2|\cosh d(0, g(0))|2|\cosh d(0, h(0))| \\
= 2|\cosh d(0, g(0))|2|\cosh d(g(0), gh(0))| \\
\geq \exp|d(0, g(0))| + \exp|d(0, gh(0))| \\
= 2\cosh d(0, gh(0)) \\
= \nu(gh).
\]

3. It follows from (1) and (2) that

\[
\nu(hgh^{-1}) \leq \nu(h)\nu(g)\nu(h^{-1}) \\
= [\nu(h)]^2\nu(g) \\
= [\nu(h)]^2hgh^{-1}h \\
\leq [\nu(h)]^4\nu(hgh^{-1}).
\]

3. Unless otherwise stated, we shall always take \( G \) to be a discrete subgroup of \( U(1, n; \mathcal{C}) \). First we give

**Definition 3.1 (cf. [3, Theorem 5.1]).** For any point \( a \in H^n(\mathcal{C}) \), \( G \) is called of convergence type or divergence type according as \( \sum_{g \in G} (1 - \|g(a)\|)^n \) converges or diverges.

**Theorem 3.2.** \( G \) is of convergence type or divergence type according as \( \sum_{g \in G} |a_{1,1}^{(k)}|^{-2n} \) converges or diverges.

**Proof.** Noting that \( 1 - \|g_k(0)\|^2 = 1 - \sum_{j=1}^{n+1} |a_{1,1}^{(k)}|^2 |a_{1,1}^{(k)}|^{-2} = |a_{1,1}^{(k)}|^{-2} \), we see

\[
(1/2)(1 - \|g_k(0)\|)^{-1} \leq |a_{1,1}^{(k)}|^2 \leq (1 - \|g_k(0)\|)^{-1}.
\]

Therefore we have

\[
\sum_{g_k \in G} (1 - \|g_k(0)\|)^n \leq \sum_{g_k \in G} |a_{1,1}^{(k)}|^{-2n} \leq 2^n \sum_{g_k \in G} (1 - \|g_k(0)\|)^n.
\]
Thus our proof is complete.

By using (3) in Proposition 2.2, we obtain

**Corollary 3.3 (Theorem 5.9).** For any element $h$ in $U(1,n; G)$, the conjugate group $hGh^{-1}$ is of the same type as $G$.

Next we shall make the estimate of $\sum_{g \in G, r < r} [\nu(g)]^{-t}$ as $r \to \infty$. From now on we assume that $G \sigma = \{\text{identity}\}$.

We now state our results.

**Theorem 3.4.** Let $r > 2$ and $t$ any real number. Then

$$\sum_{g \in G, r < r} [\nu(g)]^{-t} = \begin{cases} O(1) & \text{as } r \to \infty \text{ if } t > 2n; \\ O(\log r) & \text{as } r \to \infty \text{ if } t = 2n; \\ O(r^{2n-t}) & \text{as } r \to \infty \text{ if } t < 2n. \end{cases}$$

**Theorem 3.5.** Let $D_0$ be a fundamental polyhedron with respect to $0$ for $G$. If $\text{vol}(D_0)$ is finite, then there exist positive numbers $m_1$ and $m_2$ such that

$$m_1 \log r \leq \sum_{g \in G, r < r} [\nu(g)]^{-2n} \leq m_2 \log r,$$

and if $t < 2n$, then

$$m_1 r^{2n-t} \leq \sum_{g \in G, r < r} [\nu(g)]^{-t} \leq m_2 r^{2n-t}.$$

**Remark 3.6.** When $n = 1$, $G$ is a fuchsian group acting on the unit disk. Noting that the radii of isometric circles are bounded, we see that Theorems 3.4 and 3.5 yield some familiar classical results (see [4]).

For proving the above theorems, we need two lemmas.

**Lemma 3.7 (Theorem 4.1).** For $0 \leq r < 1$, the following inequality is satisfied.

$$n(r, a) \leq B(1-r)^{-n},$$

where $B$ is a constant independent of $a \in H^n(G)$.

**Lemma 3.8 (Theorem 4.4).** Let $D_0$ be a fundamental polyhedron with respect to $0$ for $G$. Suppose $\text{vol}(D_0) < \infty$. Let $a \in D_0$ and $\|a\| < \rho$
< 1. Then there exists \( r_0 \) such that the following inequality is satisfied for \( r_0 \leq r < 1 \).

\[
A(1-r)^{-n} \leq n(r,a) \leq B(1-r)^{-n},
\]

where \( A \) is a constant which depends on \( \rho \) and \( B \) is a numerical constant.

We shall prove Theorems 3.4 and 3.5 in the same manner as in the proof of [1, Theorems 2 and 3].

**Proof of Theorems 3.4 and 3.5.** Let \( \chi_0(r) = \# \{ g \in G \mid \nu(g) < r \} \).

By Lemma 3.7, we have

\[
\chi_0(r) = \# \{ g \in G \mid \| g(0) \| < 1 - (4/r^2)^{1/2} \}
= n(1 - (4/r^2)^{1/2}, 0)
\leq B(1 - (4/r^2)^{1/2})^{-n} \leq 2^{-n} Br^n. \tag{1}
\]

For each real number \( t \), we define

\[
\chi_t(r) = \sum_{s \in D_t, \nu(g) < r} [\nu(g)]^{-t}.
\]

If \( r > 2 \), then

\[
\chi_t(r) = \int_2^r \frac{d\chi_0(s)}{s^t} = \frac{\chi_0(r)}{r^t} + t \int_2^r \frac{\chi_0(s)}{s^{t+1}} ds. \tag{2}
\]

Using this equation, together with the inequality (1), we obtain Theorem 3.4. Lemma 3.8 establishes

\[
\chi_0(r) \geq A(1 - (4/r^2)^{1/2})^{-n} \geq 2^{-2n} Ar^n. \tag{3}
\]

By (2) and (3), we complete our proof of Theorem 3.5.

Theorems 3.2 and 3.5 lead to

**Corollary 3.9 (3, Theorem 5.4).** If \( vol(D_0) < \infty \), then \( G \) is of divergence type.

**References**


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