SOME COMMUTATIVITY THEOREMS FOR PRIME RINGS WITH DERIVATIONS AND DIFFERENTIALLY SEMIPRIME RINGS

To the memory of Takeshi Onodera

YASUYUKI HIRANO and HISAO TOMINAGA

Throughout the present paper, $R$ will represent a ring with center $C$, and $U$ a non-zero ideal of $R$. Let $\sigma$, $\tau$ be ring-automorphisms of $R$, and set $C_{\sigma\tau} = \{ c \in R \mid c \sigma(x) = \tau(x)c \text{ for all } x \in R\}$; in particular, $C_{1,1} = C$. Given $x, y \in R$, we write $[x, y]_{\sigma\tau} = x\sigma(y) - \tau(y)x$; in particular, $[x, y]_{1,1} = [x, y]$, in the usual sense. Let $d : x \to x'$ be a $(\sigma, \tau)$-derivation of $R$, that is an additive map of $R$ satisfying $(xy)' = x'\sigma(y) + \tau(x)y'$ for all $x, y \in R$. We consider the following conditions:

a) $R$ is commutative.
b) $[u', u] = 0$ for all $u \in U$.
c) $[u', u]_{\sigma\tau} = 0$ for all $u \in U$.
d) $[u', u]_{\sigma\tau} \subseteq C_{\sigma\tau}$ for all $u \in U$.
e) $U'$ is commutative.
f) $U' \subseteq C$.

As a generalization of Posner's theorem [6, Theorem 2], the present authors and A. Kaya [3, Theorem 1 (2)], and independently J. H. Mayne [5, Theorem 1], have proved that if $d$ is a non-zero $(1,1)$-derivation of a prime ring $R$ then the conditions a) and d) are equivalent. On the other hand, L. O. Chung and J. Luh [1] have proved that if $d$ is a non-zero derivation of a prime ring $R$ of characteristic not 2 then the conditions a), e) for $U = R$, and f) for $U = R$ are equivalent, and more recently A. Trzepizur [7] has proved a similar result for semiprime rings.

In § 1, we generalize partially Posner's theorem in two directions (Propositions 1, 2), and give a partial generalization of [3, Theorem 1 (2)] (Theorem 1). In § 2, we prove one more generalization of Posner's theorem (Theorem 2). Finally, in § 3, we generalize Trzepizur's theorem for differentially semiprime rings and the result of Chung and Luh for prime rings (Theorem 3).

1. Throughout this section, $R$ will be a prime ring. We begin with
the following partial generalization of [6, Theorem 2].

**Proposition 1.** Let $d$ be a non-zero $(\sigma, 1)$-derivation of a prime ring $R$. Then a) and b) are equivalent.

**Proof.** Linearizing the identity b) on $U$, we obtain

\[(u', v) = [u, v'] \quad \text{for all } u, v \in U.\]

Replacing $v$ by $uv$ in (1), we get

\[[u', uv] = [u, (uv)v'] = [u, u'\sigma(v) + uv'] \quad \text{for all } u, v \in U.\]

Combining this with b) and (1), we have

\[[u, uv] = u[u, v'] = [u, v'] = [u', uv] = u'[u, \sigma(v)] + [u, uv'],\]

and therefore $u'[u, \sigma(v)] = 0$ for all $u, v \in U$, namely $u'[u, \sigma(U)] = 0$ for all $u \in U$. Noting that $\sigma(U)$ is an ideal of $R$, we have $u'\sigma(U)[u, v] = u'[u, \sigma(U)v] = 0$ for all $u, v \in U$. Then we can easily see that either $U' = 0$ or $U$ is commutative. But, as is easily seen, $U' \neq 0$, and hence $U$ is commutative. Now, it is a routine to prove that $R$ is commutative (see [3, Lemma 1 (1)]).

**Proposition 2.** Let $d$ be a non-zero $(\sigma, \tau)$-derivation of a prime ring $R$. Then c) implies a) and $\sigma = \tau$.

**Proof.** It is easy to see that $U' \neq 0$. Linearizing the identity c) on $U$, we have

\[u'\sigma(v) - \tau(u)v' = \tau(v)u' - v'\sigma(u) \quad \text{for all } u, v \in U.\]

Replacing $v$ by $uv$ in (2), we get

\[[u', u] + \tau(u)[v', u] = \tau(u)\tau(v)u' + u'\sigma(v)\sigma(u) = 0.\]

Combining this with c) and (2), we have

\[0 = -\tau(u)[u', v] + \tau(u)\tau(v)u' + u'\sigma(v)\sigma(u) = -\tau(u)u'\sigma(v) + u'\sigma(v)\sigma(u),\]

and therefore $u'\sigma([v, u]) = 0$, namely $u'\sigma([U, u]) = 0$ for all $u \in U$. Hence, $u'\sigma(U)\sigma([x, u]) = 0$ for all $u \in U$ and $x \in R$. Since $\sigma(U)$ is a non-zero ideal of $R$, we see that either $U \subseteq C$ or $U' = 0$. Since $U' \neq 0$, we have $U \subseteq C$, and hence $R$ is commutative. Thus, for any $u \in U$ we have
$0 = [u', u]_{\sigma \tau} = u'(\sigma(u) - \tau(u))$. and so we conclude that $\sigma(u) = \tau(u)$ for all $u \in U$. By [3, Lemma 1 (2)], this proves that $\sigma = \tau$.

We conclude this section with the following partial generalization of [3, Theorem 1 (2)].

**Theorem 1.** Let $d$ be a non-zero $(\sigma, \tau)$-derivation of a prime ring $R$ of characteristic not 2. Then c) and d) (and therefore a)) are equivalent.

**Proof.** Suppose $R$ satisfies d). Let $u$ be an arbitrary element of $U$. Then, by repeated use of d), we have

$$[(u^3)', u^2]_{\sigma \tau} = [u', u]_{\sigma \tau} \sigma(u^2) - \tau(u^2)[u', u]_{\sigma \tau}$$

$$+ 2\tau(u)u\sigma(u^2) - 2\tau(u^2)\tau(u)u'$$

$$= 2\tau(u)[u'\sigma(u)\sigma(u) - \tau(u)\tau(u)u']$$

$$= 2\tau(u)[u', u]_{\sigma \tau} \sigma(u) + \tau(u)[u', u]_{\sigma \tau}$$

$$= 4\tau(u^2)[u', u]_{\sigma \tau}.$$

Hence, $\tau(u^2)[u', u]_{\sigma \tau} \in C_{\sigma \tau}$, and therefore for any $x \in R$ we have

$$0 = \tau(u^2)[u', u]_{\sigma \tau} \sigma(x) - \tau(x)\tau(u^2)[u', u]_{\sigma \tau} = \tau([u^2, x])[u', u]_{\sigma \tau}.$$

This proves that either $u^2 \in C$ or $[u', u]_{\sigma \tau} = 0$. Suppose $u^2 \in C$. Then, again by d), $[u', u]_{\sigma \tau} \sigma(u^2) - \tau(u^2) = 0$. If $\sigma(u^2) \neq \tau(u^2)$ then it is easy to see that $[u', u]_{\sigma \tau} = 0$. On the other hand, if $\sigma(u^2) = \tau(u^2)$ ($\in C$) then for any $x \in R$

$$0 = ([u^2, x])' = (u^2)'\sigma(x) + \tau(u^2)x' - x'\sigma(u^2) - \tau(x)(u^2)'$$

$$= (u^2)'\sigma(x) - \tau(x)(u^2)'$$

which says that $(u^2)' = u'\sigma(u) + \tau(u)u'$ is in $C_{\sigma \tau}$. Combining this with d), we get $2\tau(u)u' \in C_{\sigma \tau}$, and hence $\tau(u)[u', u]_{\sigma \tau} = 0$, which implies $[u', u]_{\sigma \tau} = 0$. We have thus shown that $[u', u]_{\sigma \tau} = 0$ in either case.

2. In this section too, we restrict ourselves to a prime ring $R$ with non-zero derivation $d : x \to x'$. Let $[R] = \{x \in R \mid [x, x] \in C\}$, and $(R) = \{x \in R \mid (x, x) = xx + x^2 \in C\}$. We say that $d$ is semicentralizing if $R = [R] \cup (R)$. In particular, if $R = [R]$ then $d$ is centralizing.

The purpose of this section is to generalize [6, Theorem 2] as follows:

**Theorem 2.** If a prime ring $R$ has a non-zero semicentralizing derivation $d$, then $R$ is commutative.
For the proof of Theorem 2, we need the following four lemmas.

**Lemma 1** ([3, Lemma 2]). *Let d be semcentralizing.*

1. Let \( x, y \in [R] \) (resp. \((R)\)). Then \( x + y \in [R] \) (resp. \((R)\)) if and only if \( x - y \in [R] \) (resp. \((R)\)).
2. If \( y \in (R) \), then \([y', y]^2 = [y, (y')^2] = 0\).

**Lemma 2.** *Let d be semcentralizing, and let R be of characteristic not 2.*

1. If \( y \in [R] \), then \((y')^4 = 0\) and \((y')^4 = 0\).
2. If \( C \) is not zero then \( d \) is centralizing.

**Proof.** (1) Since \((y')^4 = (y', y) \in C\) and \([y', y]^2 = 0\) (Lemma 1 (2)), we have

\[
[(y^2+y'), y^2+y] = [(y^2-y'), y^2-y] = [y', y] \in C,
\]

which means that \( y^2+y \in [R] \) and \( y^2-y \in [R] \). Then, by Lemma 1 (1),

\[
(y^2+y)-(y^2-y) = 2y \in [R]
\]

shows that \( 2y^2 = (y^2+y)+(y^2-y) \in (R) \), and so \( y^2 \in (R) \). Hence, \( 2(y')^2 = ((y')^2, y') \in C \), i.e., \((y')^2+y^2 \in C\). Furthermore, by Lemma 1 (2),

\[
0 = (y')^2[(y^2+y), (y^2+y)^2] = 2(y')^2[y', y]
\]

implying \((y')^2+y^2 = 0\). Noting here that \((y')^4 \in C\), we get

\[
(y', y) = (y')^4 = 0 \quad \text{and} \quad (y', y)+(y', y') = (y', y') = 0.
\]

Since \( y^2+y \in [R] \), we can apply (3) to see that \( 2y'y^2 = (y', y^2+y) = ((y'+y'), y^2+y) = 0 \), and so

\[
y'y^2 = y^2y' = 0 \quad \text{and} \quad y^2y' = (y^2y')' = 0.
\]

If \( y' \in [R] \), then \((y')^2y' = 0 \) by (4). Since \([y, (y')^2] = 0\) (Lemma 1 (2)), by (3) we have \( 2(y')^4 = (y')^2((y', y)+(y', y')) = 0 \), i.e., \((y')^4 = 0\). Thus, we assume henceforth that \( y' \in [R] \). Then, by Lemma 1 (1), either \( y+y' \in [R] \) or \( y-y' \in [R] \). We assume first that \( y+y' \in [R] \). Then, by (3) we have

\[
(y', y') = (y+y', (y+y')') = 0.
\]

Since \([y', y'] \in C\), (5) proves that \( y'y \in C \) and \( y'y' \in C \). Hence, by (3) and (4), we get

\[
y'(y'y+(y')^2) = (y')^2y' + (y')^2 = (y^2 + (y')^2 + (y', y))(y'+y')
\]
= (y + y')(y + y')' = 0.

Obviously, if $y'y = 0$ then $(y')^2 = 0$. On the other hand, if $y'y' \neq 0$ then $y'y(y'y' + (y')^2) = 0$ gives $y'y + (y')^2 = 0$, and so $y'y(y' + y') = 0$, whence it follows that $y' + y' = 0$. This together with (5) implies $(y')^2 = 0$. Also, in case $y - y' \in [R]$, we can see that $(y')^2 = 0$.

(2) This is [3, Lemma 4 (3)].

**Lemma 3** ([2, Lemma 1]). Let $f$ be a non-trivial idempotent of $R$. If $(f + fx - fzf)' = 0$ for all $x \in R$, then $d = 0$.

**Lemma 4.** Let $Q$ be the Martindale quotient ring of $R$. Let $p$, $q$, $r$ be elements of $Q$. If $puqr = 0$ for all $u \in U$, then one, at least, of $p$, $q$, $r$ is zero.

**Proof.** If $x$, $y$ are elements of $Q$ such that $xUy = 0$, then $x$ or $y$ is zero. By making use of this fact, we can prove the lemma in the same way as in the proof of [6, Lemma 2].

We are now ready to complete the proof of Theorem 2.

**Proof of Theorem 2.** By [6, Theorem 2], it suffices to show that $d$ is centralizing, and so we may assume that $R$ is of characteristic not 2. In view of Lemma 2 (2), we may further assume that $C = 0$. Then $R$ satisfies the non-trivial differential identity $[x^2, x'] = 0$. By [4, Corollary 5], the central closure $S$ of $R$ is a primitive ring with non-zero socle. According to [2, Lemma 4], we can extend $d$ in a unique way to a derivation of $S$, which is also denoted by $d : x \to x'$. Now, let $e$ be an arbitrary idempotent in $S$. Then there exists a non-zero ideal $A$ of $R$ such that $eA \subseteq R$ and $Ae \subseteq R$. For any $a \in A$, we have either $ea(ea)' = (ea)'ea$ or $ea(ea)' = -(ea)'ea$. In either case, we have $e(ea)'ea = (ea)'ea$. Hence, we see that $(ee' - e'aea = 0$ for all $a \in A$, and so $ee' = e'$ by Lemma 4. Similarly, we can show that $e'e = e'$. We see therefore that $e' = (e')' = ee' + e'e = 2e'$, that is, $e' = 0$. Noting here that $f + fx - fzf$ is an idempotent for every idempotent $f \in S$ and every $x \in S$ and that $d$ is non-zero, we see that $S$ has no non-trivial idempotents (Lemma 3). Hence $S$ has to be a division ring, and so $R$ is a domain. Now, by Lemma 2 (1), we conclude that $d$ is centralizing.

3. Throughout this section, $d$ will represent a derivation of $R$, and
$U$ a differential ideal of $R$ with $l(U) = 0$. If $R$ is a prime ring of characteristic not 2 and $d$ is non-zero, then we can prove that the conditions a), e) and f) are equivalent (see Corollary 1 below).

We say that $R$ is differentially prime (abbr. $d$-prime) if one of the following equivalent conditions is satisfied:

1) If $I$ is a non-zero differential ideal of $R$ and $xy^{k+1} = 0$ ($x, y \in R$) for all $k \geq 0$ then $x = 0$ or $y = 0$.

2) If $I$ is a non-zero differential ideal of $R$ and $xy^{k+1} = 0$ ($x, y \in R$) for all $k \geq 0$ then $x = 0$ or $y = 0$.

3) If $I, J$ are differential ideals of $R$ and $IJ = 0$ then $I = 0$ or $J = 0$.

As is easily seen, if $R$ is $d$-prime then $R$ is either of prime characteristic or torsion free. A differential ideal $P$ of $R$ is said to be $d$-prime if the factor ring $R/P$ is $d$-prime. The intersection of all $d$-prime ideals of $R$ is called the $d$-prime radical of $R$. We say that $R$ is differentially semiprime (abbr. $d$-semiprime) if the $d$-prime radical of $R$ is zero. It is a routine to verify the equivalence of the following conditions:

i) $R$ is $d$-semiprime.

ii) $R$ contains no non-zero nilpotent differential ideals.

iii) $R$ is differentially isomorphic to a subdirect sum of $d$-prime rings.

A little care is needed here. If $R$ is $d$-semiprime then $l(U) = 0$ shows that the intersection of all $d$-prime ideals not including $U$ is zero. Needless to say, every semiprime (resp. prime) differential ring is $d$-semiprime (resp. $d$-prime). If $R$ is $d$-prime, "$l(U) = 0"$ becomes "$U \neq 0$".

**Lemma 5.** Suppose $d$ is non-zero.

1) If $R$ is $d$-prime then $U' \neq 0$.

2) If $R$ is $d$-semiprime and $2R = R$ then $U'' \neq 0$.

**Proof.** (1) Suppose, to the contrary, that $U' = 0$. Then, for any non-zero $u \in U$ and $x \in R$, we have $0 = (ux)' = ux'$. Hence $0 = u(xy)' = uy'x'$, whence it follows that $uRx^{k+1} = 0$ for all $k \geq 1$. Hence $x' = 0$ for all $x \in R$. But this is a contradiction.

(2) It suffices to prove the case that $R$ is $d$-prime. Suppose, to the contrary, that $U'' = 0$. Then $2u'v' = (uv)' - u''v' - uv'' = 0$, and hence $u'v' = 0$ for all $u, v \in U$. The relation $u'v' = (uv)'u' = 0$ gives $u'Uu = 0$ for all $k \geq 1$, whence $U' = 0$ follows. This contradicts (1).

**Lemma 6.** If $R$ is $d$-semiprime then e) implies f). If, furthermore,
$2R = R$, then e) and f) are equivalent.

Proof. It suffices to prove the case that $R$ is d-prime. In case $U = 0$, there is nothing to prove. We may therefore assume that $U \neq 0$.

Since $u^*[v, w] = [u^*v, w] = [(u^*v), w] = 0$ ($u, v, w \in U$), we have $u^*[v, w] = u^*[vx, w] - u^*[v, w]x = 0$, and therefore $u^*U[x, w]^k = 0$ for all $k \geq 0$ ($u \in U$, $x \in R$). Hence $U^* = 0$ or $U \subseteq C$, and so $U^* \subseteq U \subseteq C$. Suppose now that $2R = R$. We shall show that f) implies e).

Obviously, $[v', u'] = 0$ and

$$u^*[v', u'] = [u^*v', u'] = [(u^*v'), u'] - 2[u^*v', u'] - [u', u']v^* = 0.$$ 

Hence $u^*R[v', u]^k = 0$ for all $k \geq 0$ ($u, v \in U$). Then, either $U^* = 0$ or $U$ is commutative. If $U^* = 0$ then

$$u^*[v', u'] = [(uv'), u'] - 2[u', u']v - [uv^*, u'] = 0,$$

and hence $u^*R[v', u]^k = 0$ for all $k \geq 0$. Noting here that $U^* \neq 0$ by Lemma 5 (2), we get e), again.

Careful scrutiny of the proof of Proposition 1 shows the following

**Lemma 7.** Let $R$ be a d-prime ring, and $d \neq 0$. Then b) implies a).

We are now ready to prove the following principal theorem of this section.

**Theorem 3.** Let $R$ be a d-semiprime ring with $2R = R$. If $K = \{x \in R | x = 0\}$ is commutative then the conditions a), e) and f) are equivalent.

Proof. In view of Lemma 6, it remains only to prove that e) implies a).

We claim first that $U \subseteq C$. To see this, we may assume that $R$ is d-prime. As was shown in the proof of Lemma 6, either $U^* = 0$ or $U \subseteq C$. If $U^* = 0$ then $U = 0$ by Lemma 5 (2), and therefore $U^* \subseteq C$ in either case.

Now, let $\bigcap_{\lambda \in A} P_\lambda = 0$ with d-prime ideals $P_\lambda \not\supset U$. Put $A_1 = \{\lambda \in A \mid P_\lambda \not\supset U\}$ and $A_2 = \{\lambda \in A \mid P_\lambda \not\supset U\}$. Let $D$ be the commutator ideal of $R$. Then, Lemma 7 shows that $D \subseteq P_\lambda$ for all $\lambda \in A_2$. Hence $D'U \subseteq (DU) + DU' \subseteq P_\lambda$ for all $\lambda \in A$, and therefore $D' \subseteq \bigcap_{\lambda \in A} P_\lambda = 0$, namely $D \subseteq K$. By hypothesis, $D$ is then a commutative ideal. Now, let $\mu \in A_1$. Then $\bar{R} = R/P_\mu$ is a prime ring. (Note that $R'U \subseteq (RU)' + RU' \subseteq P_\mu$
implies $R' \subseteq P'$. If $D \nsubseteq P'$ then $\overline{D}$ is a non-zero commutative ideal of the prime ring $\overline{R}$. Hence, by [3, Lemma 1 (1)], $\overline{R}$ is commutative, which contradicts $\overline{D} \neq 0$. We have thus seen that $D \subseteq P_\lambda$ for all $\lambda \in \Lambda$, namely $D = 0$, which proves the commutativity of $R$.

Careful scrutiny of the proof of Theorem 3 shows the following

**Corollary 1.** Let $R$ be a prime ring of characteristic not 2. If $d \neq 0$ or $K$ is commutative then the conditions a), e) and f) are equivalent.

**References**


**Department of Mathematics**

**Okayama University**

*(Received June 4, 1984)*