GALOIS SUBRINGS OF INDEPENDENT AUTOMORPHISM GROUPS OF COMMUTATIVE RINGS ARE QUORITE

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Let $R$ be a commutative ring, $G$ a finite group of automorphisms, let $A = R^G$ be the Galois subring, let $G^{ex}$ denote the canonical extension of $G$ to the quotient ring $Q = Q_{co}(R)$, and $F = Q^{co}$. It is easy to see ([1]) that $F$ is the partial quotient ring of $A$ with respect to the multiplicatively closed subset $S$ of $A$ consisting of all $a \in A$ that are regular in $R$, that is, $S = A \cap R^*$, and thus that $G$ is quorite in the sense that $Q_{co}(A) = F$ iff $R$ is torsion free over $A$ in the sense that $A^* \subseteq R^*$. Sufficient ring-theoretical conditions for this are: (1) $R$ is reduced ( = semiprime, or non-singular); (2) $R$ is flat over $A$. As stated, (1) happens if $R$ is semihereditary, and (2) when $G$ is a Galois group.

The purpose of this short note is to report another useful sufficient condition. A group $G$ of automorphisms of $R$ is independent provided that the elements of $G$, considered as functions $R \to R$, are linearly independent over $R$ ([2]).

**Theorem.** If $G$ is an independent finite automorphism group of $R$, then $G$ is quorite.

**Proof.** Assume the above notation. Let $a \in A^*$, and $I$ the annihilator ideal of $a$ in $R$. Let $T_0(x)$ denote the trace of any $x$ in $R$ under $G$. Since $I \cap A = 0$, evidently $T_0(x) = 0$ for any $x$ in $I$, and moreover, if $r$ is any element of $R$, we have then that $xr \in I$, whence $T_0(xr) = 0$, that is,

$$\sum_{g \in G} g(x)g(r) = 0$$

and therefore

$$\sum_{g \in G} g(x) = 0 \text{ on } R.$$  

Since $G$ is independent, it follows that $x = 0$, which proves the theorem.

**Example.** The converse of the theorem fails. Let $R$ be the direct product of three fields $F_1 \times F_2 \times F_3$, with $F_1 \simeq F_2$, and let $g$ denote the extension of this isomorphism to an automorphism of $R$ with Galois subring
$R^g \supseteq F$. Since $R^g$ contains an ideal, then by [2] the group $(g)$ is dependent, but quorite since $R$ is reduced.

**Question.** Let $G$ be a group of automorphisms of a non-commutative ring. If $G$ is finite and independent, is $G$ quorite?

If $R$ is an integral domain, then the answer is yes by [3] without assumption of independence, a result generalized by [4] to any ring $R$ such that $R^g$ is semiprime. See [6] for related results.

The assumption that $G$ is independent is the basis of a number of theorems for what are called strictly Galois extensions in [8], and e.g., [7].

Let $Q = Q_{\text{max}}^r(R)$ denote the maximal right quotient ring of $R$, and $G^{\text{ex}}$ now denote the group of automorphisms of $Q$ which extends $G$ to $Q$. The theorem of Kitamura [5] states

$$Q^{g^{\text{ex}}} = Q_{\text{max}}^r(R)$$

that is, $G$ is maximally quorite, if the trace function $R \to R^g$ is non-degenerate, that is, does not vanish on any nonzero right ideal.

**Corollary.** If $R$ is commutative, and $G$ independent, then $G$ is maximally quorite.

**Proof.** As stated in [2], if $G$ is independent, then the trace function is non-degenerate.

In fact, Theorem 8.2 of [2] states that a torsion group $G$ is dependent over $R$ iff for some $g \neq 1$ in $G$ either the fixring $R^g$ contains a nonzero ideal of $R$, or else the $g$-trace function is degenerate.

**References**


Galois Subrings of Independent Automorphism Groups


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