ON A THEOREM OF S. KOSHITANI

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The purpose of this note is to present a theorem which refines the result of S. Koshitani [5, Theorem]. Throughout the present paper, $F$ will represent a field of characteristic $p > 0$, and $G$ a finite $p$-solvable group. Let $B$ be a block ideal of the group algebra $FG$, and $J(B)$ the Jacobson radical of $B$. Recently, in [5], S. Koshitani proved that if $d$ is the defect of $B$, then the least positive integer $t$ such that $J(B)^t = 0$ is greater than or equal to $d(p-1)+1$. In view of [3, IV, Lemma 2.2 and Theorem 4.5], [2, Lemma 4.6] and [6, Lemma 12.9], we see that there exists an irreducible $B$-module a vertex of which is a defect group of $B$. Hence Koshitani’s result follows from the following

**Theorem.** Let $M$ be an irreducible $FG$-module. If a vertex of $M$ has order $p^n$, then the Loewy length of the projective cover of $M$ is greater than or equal to $v(p-1)+1$.

All modules considered here are finitely generated right modules. The following notation will be used in the proof of the theorem. Given an $FG$-module $M$, we denote by $\nu(M)$ a vertex of $M$ and by $L(M)$ the Loewy length of $M$. If $H$ is a subgroup of $G$, then $M|_H$ is an $FH$-module obtained from $M$ by restricting the domain of operators to $FH$. The full matrix ring of degree $m$ over a ring $R$ is denoted by $M_m(R)$. If $n$ is a positive integer, then $\nu(n)$ is the exponent of the highest $p$-power dividing $n$.

**Proof of Theorem.** Let $e$ be a primitive idempotent of $FG$ such that $eFG$ is a projective cover of $M$. If $E$ is a finite extension field of $F$, then it is well known that $J(EG) \cong E \otimes_F J(FG)$, and so $E \otimes_F M$ is a completely reducible $EG$-module. Let $E \otimes_F M = X_1 \oplus \cdots \oplus X_r$ be a decomposition of $E \otimes_F M$ into a direct sum of irreducible $EG$-submodules. Then observing that

$$eEG/eJ(EG) \cong E \otimes_F eFG/eJ(FG) \cong E \otimes_F M,$$

we see that $eEG$ is a projective cover of $E \otimes_F M$, and so $eEG$ is a direct sum of projective covers of $X_i (i = 1, \ldots, r)$. It is clear that $eEG$ has the same Loewy length as $eFG$. Further, by [2, Lemma 4.6], each $X_i$ and $M$ have a vertex in common. Therefore, in order to prove the theorem, we may assume that $F$ contains the cyclotomic field of order $|G|$ over $GF(p)$. Then $F$
is a splitting field for all subgroups of $G$. The proof is by induction with respect to $|G|$ and $\nu(|G|)$. Suppose, if possible, $G$ is a minimal counter-example.

Case 1: Assume that $Q = O_{\rho}(G) \neq \langle 1 \rangle$.

Since $Q \subset \text{Ker } M$, $M$ becomes an $FG$-module, where $\overline{G} = G/Q$, and [4, Lemma 1.3] asserts $\nu_{\overline{G}}(M) \equiv \nu_{\overline{G}}(M)/Q$. Let $\omega(Q)$ be the augmentation ideal of $FQ$. Then $FG/\omega(Q)FG \cong F\overline{G}$ and $eFG/e\omega(Q)FG$ is a projective cover of the $F\overline{G}$-module $M$. Set $v_1 = \nu(|\nu_{\overline{G}}(M)/Q|)$ and $v_2 = \nu(|Q|)$. Then $L(eFG/e\omega(Q)FG) \cong v_1(p-1) + 1$ by induction. Hence $eJ(FG)^{v_1(p-1)}$ is not contained in $e\omega(Q)FG$. Let $\hat{Q} = \sum_{s \in Q}s$ in $FG$. Then we have

$$|x \in eFG | x\hat{Q} = 0| = |x \in FG | x\hat{Q} = 0| \cap eFG$$
$$= F\omega(Q) \cap eFG$$
$$= eFG\omega(Q) = e\omega(Q)FG.$$

Therefore we get $eJ(FG)^{v_1(p-1)}\hat{Q} \neq 0$. Since $L(FQ) - 1 \geq v_1(p-1)$, we see that $\hat{Q}$ is contained in $\omega(Q)^{v_1(p-1)}$. Hence, noting that $\omega(Q) \subset J(FG)$, we get

$$0 \neq eJ(FG)^{v_1(p-1)}\hat{Q} \subset eJ(FG)^{v_1(p-1)}J(FG)^{v_2(p-1)}$$
$$= eJ(FG)^{v_1(p-1)},$$

proving that $L(eFG) \cong v(p-1) + 1$. So this case does not occur.

Case 2: Assume that $O_{\rho}(G) = \langle 1 \rangle$.

Let $H = O_{\rho}(G)$. Suppose that $M$ belongs to the block ideal $B$ of $FG$. Let $N$ be an irreducible component of $M|_H$ and let $T$ be the inertial group of $N$ in $G$:

$$T = \{g \in G | N \otimes_{FG} g \cong N \text{ as } FH\text{-modules}\}.$$  

At first, suppose that $G \neq T$. Then, by [5, Lemma 1], there exists a block ideal $b$ of $FT$ with block idempotent $f$ and the $F$-algebra isomorphism $\phi: B \cong \text{End}(FGf_{fr})$ given by $[\phi(x)](y) = xy$, $x \in B$, $y \in FG$. Further, the map sending $X$ to $X^c = X \otimes_{FG} F$ is a one to one correspondence between irreducible $b$-modules and irreducible $B$-modules ([3, V, Theorem 2.5]). We set $t = [G: T]$ and let $[g_1, g_2, \ldots, g_t]$ be a right transversal of $T$ in $G$. Then $[f, g_1^{-1}f, \ldots, g_t^{-1}f]$ is a basis for the free $FTf$-module $FGf$. We denote by $\psi$ the isomorphism $\text{End}(FGf_{fr}) \cong M_t(FTf)$ defined naturally with respect to this basis. Now let $X$ and $Y$ be irreducible $b$-modules. Then the above together with Frobenius reciprocity theorem implies that
\[
\dim_r \text{Hom}_{Fr}(Y, X^e|_r) = \dim_r \text{Hom}_{Fr}(Y^e, X^e) = \begin{cases} 1 & \text{if } Y \cong X, \\ 0 & \text{if } Y \not\cong X. \end{cases}
\]

Hence we see that the socle of \( X^e|_r \) is isomorphic to a direct sum of \( X \) and irreducible \( FT \)-modules which belong to blocks different from \( b \). Therefore, noting that \( X \) is isomorphic to a direct summand of \( X^e|_r \), we get \( X^e f \cong X \). We may assume that \( X \) is a minimal right ideal of \( B \), and so we may identify \( X^e \) with a right ideal of \( B \) generated by \( X \). Then we have
\[
[\phi(X^e)](g_i^{-1}f) = (X^e)g_i^{-1}f = (X^e g_i^{-1})f = X^e f = X.
\]
for all \( i, 1 \leq i \leq t \). Thus we get
\[
\psi \phi(X^e) = \begin{pmatrix} X & X & \cdots & X \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subseteq M_t(FTf).
\]

Now, we may assume that \( M \cong X^e \). Then from the above we get
\[
eFG \cong \begin{pmatrix} P & P & \cdots & P \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},
\]
where \( P \) is a projective cover of \( X \), and hence we have \( L(eFG) = L(P) \). Since \( G \neq T \), noting that \( \varpi_e(M) \cong \varpi_r(X) \) (\([1, \text{ Theorem } 19.16]\)), we get
\[
L(eFG) = L(P) \cong \nu(p-1)+1
\]
by induction. Next, suppose that \( G = T \). Set \( \bar{G} = G/H \). Then \([7, \text{ Theorem } 2]\) asserts that there exists a finite group \( \bar{G} \) and a short exact sequence
\[
\langle 1 \rangle \longrightarrow Z \longrightarrow \bar{G} \longrightarrow \bar{G} \longrightarrow \langle 1 \rangle
\]
where \( Z \) is a cyclic \( p' \)-subgroup in the center of \( \bar{G} \), and there exists a block ideal \( \bar{B} \) of \( FG \) such that \( B \cong M_n(F) \otimes_r \bar{B} \) \((n = \dim_r X)\). This asserts that there is an irreducible \( \bar{B} \)-module \( \bar{M} \) such that \( M \cong I \otimes_r \bar{M} \), where \( I \) is an irreducible \( M_n(F) \)-module. So we have \( eFG \cong I \otimes_r \bar{P} \), where \( \bar{P} \) is a projective cover of \( \bar{M} \), and so we get \( L(eFG) = L(\bar{P}) \). Since \( G \) is \( p \)-solvable and \( \nu(|G|) \geq 1 \), it is clear that \( O_p(\bar{G}) \neq \langle 1 \rangle \). Hence, noting that \( \varpi_e(M) \cong \varpi_{\bar{G}}(\bar{M}) \), we have
\[ L(eFG) = L(\bar{P}) \geq v(p-1) + 1 \]

by Case 1 applied for $\bar{G}$. So this case does not occur either, and the theorem is proved.

REFERENCES


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