ON A THEOREM OF Y. TSUSHIMA

KAORU MOTOSE

Let \( p \) be a fixed prime number, let \( G \) be a finite \( p \)-solvable group with a \( p \)-Sylow subgroup \( P \) of order \( p^a \) (\( a \geq 1 \)) and let \( t(G) \) be the nilpotency index of the radical of a group algebra of \( G \) over a field of characteristic \( p \). Recently, Y. Tsushima [3] has proved that if \( t(G) = a(p-1)+1 \) and \( P \) is regular then \( P \) is elementary abelian. Unfortunately his proof is correct only when \( p \) is not a Fermat prime. A cause of his mistake is in the part of an application of [1, Theorem A (ii)]. It should be noted that the first part of [1, Theorem A (ii)] used essentially in his paper easily follows from [1, Theorem B]. At this point of view we shall present the next proposition which shall give a refinement of his theorem and a generalization of [2, Corollary 13]. Moreover this proof shall give an improvement of his proof.

**Proposition.** Assume that \( P \) is non-abelian and regular. If \( t(G) = a(p-1)+1 \) then \( p \) is a Fermat prime and a \( 2 \)-Sylow subgroup of \( G/O_{p'}(G) \) is non-abelian.

**Proof.** We argue by induction on \( |G| \). We may assume \( O_{p'}(G) = 1 \) by the inequality \( t(G) \geq t(G/O_{p'}(G)) \geq a(p-1)+1 \) (see [4]). We set \( U = O_p(G) \neq 1 \). By the inequality \( t(G) \geq t(G/U) + t(U) - 1 \geq (p-1)+1 \) (see [4]), \( U \) is elementary abelian and it may be assumed by induction that \( P/U \) is abelian. Since \( P \) is regular, it follows from this that \( (xy)^p = x^py^p \) for all \( x, y \in P \) and so \( p \) is odd as \( P \) is non-abelian. For all \( y \in U \) and \( x \in P \), we have

\[
y^{x^{p-1}+\ldots+x+1} = y^{x^{p-1}}\ldots y^{x} = x^{-\rho}(xy)^p = 1
\]

where \( y^{x^g} = x^{-g}yx^g \) and \( x^{p-1}+\ldots+x+1 \) is the sum of endomorphisms \( x^{p-1}, \ldots, x, 1 \) of \( U \). Since \( G/U \) is a subgroup of \( GL(U) \) (see [1, Lemma 1.2.5]), Hall-Higman's theorem [1, Theorem B] together with the last equation yields that \( (X-1)^{p-1} \) is the minimal polynomial on \( U \) of an element of order \( p \) in \( P/U \) and this implies the result as \( p \) is odd.

**References**


DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE
OKAYAMA UNIVERSITY
OKAYAMA 700, JAPAN

(Received October 24, 1983)