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ON THE BLUM-HANSON THEOREM IN L_p

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1. Introduction and the theorem. In this paper the following will be proved.

Theorem. *Let* $(X, \mathscr{F}, \mu)$ *be a finite measure space and* $1<p<p_1 \leq\infty$. *Let* T *be a positive linear operator on* $L_p(\mu)=L_p(X, \mathscr{F}, \mu)$ *and simultaneously on* $L_{p_1}(\mu)$ *such that*

$$\sup_n \|T^n\|_p<\infty \quad \textit{and} \quad \sup_n \|T^n\|_{p_1}<\infty.$$

Then the two following assertions are equivalent:

(a) T^n *converges weakly in* $L_p(\mu)$.

(b) *For any uniformly regular matrix* (a_{ni}), $A_n=\sum_i a_{ni}T^i$ *converges strongly in* $L_p(\mu)$.

(We recall that a matrix (a_{ni}) is *uniformly regular* if $\sup_n \sum_i |a_{ni}|<\infty$, $\lim_n \sum_i a_{ni}=1$ and $\lim_n \max_i |a_{ni}|=0$.)

Corollary. *Let* $(X, \mathscr{F}, \mu)$ *be a finite measure space and* $1<p<\infty$. *Let* T *be a positive linear operator on* $L_p(\mu)$ *such that there exists a strictly positive function* h *in* $L_p(\mu)$ *and a constant* M *satisfying* $T^n h\leq Mh$ *for all* $n\geq 1$. *Then the assertions* (a) *and* (b) *in the Theorem are equivalent.*

For an operator T induced by a measure preserving point transformation on $(X, \mathscr{F}, \mu)$, the Theorem is due to Blum and Hanson [3]. Later, Akcoglu and Sucheston [2] extended the Blum-Hanson theorem as follows: For any positive contraction T on $L_p(\mu)$, (a) and (b) are equivalent. Akcoglu and Sucheston [1] also showed that (a) and (b) are still equivalent, with $p=1$, if T is a (not necessarily positive) contraction on $L_1(\mu)$. Recently, Millet [5] examined a positive operator T on $L_1(\mu)$ such that $\sup_n \|T^n\|_1<\infty$, and showed that for such a T the equivalence of (a) and (b) holds, with $p=1$. His method of proof chiefly depends upon Akcoglu-Sucheston's theorem [1] and a decomposition theorem due to Sucheston [9]. (A similar argument appears in Sato [8].) In this paper we shall apply Millet's technique together with Hölder's inequality to obtain the above Theorem. The Corollary generalizes Proposition 1 in Millet [5].

Here I would like to express my hearty thanks to the referee for helpful comments; in particular, Lemma 1 is due to him.

2. Proofs. Throughout this section we may and will assume without loss of generality that $\mu(X)=1$. To prove the Theorem we need two lemmas; the first one is somewhat general.

Lemma 1. *Let each S_n be a linear operator on $L_p(\mu)$ and simultaneously on $L_{p_1}(\mu)$ for some $p_1>p$ such that $\sup_n \|S_n\|_{p_1}<\infty$. Then for $f\in L_{p_1}(\mu)$ the sequence S_nf converges in $L_p(\mu)$ if (and only if) S_nf converges in measure.*

Proof. Under the convergence in measure, the convergence of S_nf in $L_p(\mu)$ is equivalent to the uniform p-integrability (see [6], Chapter II), i. e. $\sup_n \|S_nf\|_p<\infty$ and

$$\sup_n \|1_E\cdot S_nf\|_p \longrightarrow 0 \text{ as } \mu(E)\longrightarrow 0.$$

But by Hölder's inequality we have

$$\begin{aligned}\|1_E\cdot S_nf\|_p &\leq \mu(E)^{\frac{p_1-p}{p_1p}}\|S_nf\|_{p_1}\\ &\leq \mu(E)^{\frac{p_1-p}{p_1p}}\|f\|_{p_1}\cdot\sup_k \|S_k\|_{p_1}.\end{aligned}$$

Lemma 2. *Let T be as in the Theorem. If $T1=1$, then* (a) *implies* (b).

Proof. By [7] X is decomposed into two disjoint measurable sets Y and Z such that

(i) $Y=\{x: e(x)>0\}$ for some $0\leq e\in L_q(\mu)$ with $T^*e=e$, where $1/p+1/q=1$,

(ii) $L_p(Z,\mu)$ is invariant for T and $\frac{1}{n}\sum_{i=0}^{n-1}T^i$ converges strongly to 0 on $L_p(Z,\mu)$.

Using this decomposition we devide the proof into several steps.

Step I. *A_nf converges to 0 in $L_p(\mu)$ for $f\in L_{p_1}(Z,\mu)$.*

First of all, we remark that $L_{p_1}(Z,\mu)$ is invariant for T, as well. Since T is positive,

$$\frac{1}{n}\sum_{i=0}^{n-1}\|T^i f\|_1 \leq \|\frac{1}{n}\sum_{i=0}^{n-1}T^i|f|\|_p \longrightarrow 0$$

by (ii). Then there exists a subsequence $T^{k_n}f$ such that $\|T^{k_n}f\|_1 \longrightarrow 0$ as $k_n \longrightarrow \infty$; hence $T^{k_n}f$ converges in measure, and we can apply Lemma 1 with $S_n = T^{k_n}$ to obtain that $\|T^{k_n}f\|_p \longrightarrow 0$. This yields

$$\limsup_n \|T^n f\|_p \leq \sup_n \|T^n\|_p \cdot \limsup_n \|T^{k_n}f\|_p = 0,$$

that is, $T^n f$ converges to 0 in $L_p(Z, \mu)$. Then clearly $A_n f$ converges to 0 in $L_p(Z, \mu)$.

Step II. *Suppose That T^n converges weakly in $L_p(\mu)$. Then $1_Y \cdot A_n f$ converges in $L_p(\mu)$ for $f \in L_{p_1}(Y, \mu)$.*

To see this, let $R(eg) = eTg$ for $g \in L_p(Y, \mu)$. Since $L_p(Y, \mu)$ is dense in $L_1(Y, \mu)$, and by (i)

$$\begin{aligned} \|R(eg)\|_1 &\leq \int eT|g|d\mu \\ &= \int |g|T^* e\, d\mu = \|eg\|_1, \end{aligned}$$

R is well defined and is uniquely extended to a positive contraction on $L_1(Y, \mu)$. Since T^n converges weakly in $L_p(\mu)$, and by definition

$$R^n(eg) = eT^n g,$$

R^n converges weakly in $L_1(Y, \mu)$. Therefore by [1], if $f \in L_{p_1}(Y, \mu)$ and if (a_{ni}) is a uniformly regular matrix, then the sequence

$$B_n(ef) := \sum_i a_{ni} R^i(ef)$$

converges in $L_1(Y, \mu)$. Since

$$B_n(ef) = eA_n f,$$

this implies the measure convergence of the sequence $1_Y \cdot A_n f$. Now we apply Lemma 1 with $S_n = 1_Y \cdot A_n$ to conclude that $1_Y \cdot A_n f$ converges in $L_p(Y, \mu)$.

Step III. *Suppose that $T1 = 1$. If $f \in L_{p_1}(Y, \mu)$ and $|f| \leq \alpha 1_Y$ for some $\alpha \geq 0$, then $1_Z \cdot A_n f$ converges in $L_p(Z, \mu)$.*

To prove this, let

$$U = 1_Y \cdot T \text{ and } W = 1_Z \cdot T.$$

Then it follows from (ii) that $1_Y \cdot T^n = U^n$, $UW = 0$ and

$$T^n = U^n + WU^{n-1} + \cdots + W^n;$$

hence

$$1_Z \cdot T^n = WU^{n-1} + \cdots + W^n.$$

Since the assumption $T1 = 1$ implies $U1_Y = 1_Y$, the above idetity shows that the positive function $h = W1_Y$ satisfies

$$h + \cdots + W^{n-1}h = 1_Z \cdot T^n 1_Y,$$

and hence

$$\sum_{i=0}^{\infty} \| W^i h \|_1 = \| \sum_{i=0}^{\infty} W^i h \|_1 \leq \| \sum_{i=0}^{\infty} W^i h \|_p$$
$$\leq \sup_n \| T^n \|_p < \infty.$$

Therefore the sequence $\sum_{i=m+1}^{\infty} W^i h$ converges to 0 in measure as $m \longrightarrow \infty$. Since for each i,

$$| \sum_{k=0}^{\infty} a_{n,i+k} U^k f | \leq \alpha (\sum_{k=0}^{\infty} | a_{n,i+k} U^k 1_Y |)$$
$$\leq \alpha\beta 1_Y$$

where we put

$$\beta = \sup_n \sum_i | a_{ni} |,$$

it follows that

$$0 \leq \sum_{i=m+1}^{\infty} W^i | \sum_{k=0}^{\infty} a_{n,i+k} U^k f |$$
$$\leq \alpha\beta \sum_{i=m}^{\infty} W^i h \longrightarrow 0 \text{ in measure as } m \longrightarrow \infty.$$

Thus we have proved that $\sum_{i=m+1}^{\infty} W^i [\sum_{k=0}^{\infty} a_{n,i+k} U^k f]$ converges to 0 in measure (uniformly in n) as $m \longrightarrow \infty$. On the other hand, for each fixed i, the matrix $(a_{n,i+k})_{n,k}$ is uniformly regular, so that by Step II the sequence

$$\sum_{k=0}^{\infty} a_{n,i+k} U^k f = 1_Y \cdot (\sum_{k=0}^{\infty} a_{n,i+k} T^k f)$$

converges in $L_p(\mu)$ as $n \longrightarrow \infty$, and so does

$$W^i [\sum_{k=0}^{\infty} a_{n,i+k} U^k f] \text{ as } n \longrightarrow \infty.$$

Now, the convergence of $1_Z \cdot A_n f$ in $L_p(Z, \mu)$ results from Lemma 1

with $S_n = 1_Z \cdot A_n$, because the relation

$$\begin{aligned} 1_Z \cdot A_n f &= \sum_{i=1}^{\infty} a_{ni} 1_Z \cdot T^i f \\ &= \sum_{i=1}^{\infty} a_{ni} [WU^{i-1} f + \cdots + W^i f] \\ &= \sum_{i=1}^{m} W^i [\sum_{k=0}^{\infty} a_{n,i+k} U^k f] \\ &+ \sum_{i=m+1}^{\infty} W^i [\sum_{k=0}^{\infty} a_{n,i+k} U^k f] \end{aligned}$$

implies that the sequence $1_Z \cdot A_n f$ is a Cauchy sequence for convergence in measure and hence converges in measure.

Step IV. *Suppose that* $T1 = 1$ *and that* T^n *converges weakly in* $L_p(\mu)$. *Then* A_n *converges strongly in* $L_p(\mu)$.

In fact, by Steps I, II and III, $A_n f$ converges in $L_p(\mu)$ whenever $f \in L_{p_1}(\mu)$ and $1_Y \cdot |f| \leq \alpha 1_Y$ for some $\alpha \geq 0$. But since the set of those f is dense in $L_p(\mu)$, and since $\sup_n \|A_n\|_p < \infty$, the sequence $A_n g$ converges for every $g \in L_p(\mu)$.

This completes the proof.

Proof of the Theorem. (a) $\Longrightarrow$ (b): We may choose an $h \in L_p(\mu)$, with $0 \leq h = Th$, such that if $g \in L_p(\mu)$ and $Tg = g$ then $\{x : g(x) \neq 0\} \subset \{x : h(x) > 0\}$. Put $X^+ = \{x : h(x) > 0\}$ and $X^0 = \{x : h(x) = 0\}$. It can be readily seen that $L_p(X^+, \mu)$ is invariant for T, and that if we set $Sg = 1_{X^0} \cdot Tg$ for $g \in L_p(X^0, \mu)$, then S^n converges to 0 in $L_p(X^0, \mu)$. Hence, as in Step I of the proof of Lemma 2, we have

$$\lim_n \|S^n g\|_p = \lim_n \|1_{X^0} \cdot T^n g\|_p = 0$$

for every $g \in L_p(X^0, \mu)$. On the other hand, since $L_p(X^+, \mu)$ is isometrically isomorphic to $L_p(X^+, h^p d\mu)$ by the mapping $g \longrightarrow g/h$, and since $(Th)/h = 1$, Lemma 2 shows that if $g \in L_p(X^+, \mu)$ and (a_{ni}) is a uniformly regular matrix then the sequence $A_n g = \sum_i a_{ni} T^i g$ converges in $L_p(X^+, \mu)$.

Now, fix any $f \in L_p(\mu)$. Given an $\varepsilon > 0$, take an N with $\|1_{X^0} \cdot T^i f\|_p \leq \varepsilon / \sup_n \|T^n\|_p$ for all $i \geq N$. Then we get

$$\|T^i f - T^{i-N}(1_{X^+} \cdot T^N f)\|_p \leq \varepsilon \qquad (i \geq N).$$

Since the sequence

$$C_n f := \sum_{i=N}^{\infty} a_{ni} T^{i-N}(1_{X^+} \cdot T^N f)$$

converges in $L_p(X^+, \mu)$, the above inequality implies that the sequence $A_n f = \sum_i a_{ni} T^i f$ is a Cauchy sequence in $L_p(\mu)$. Hence $A^n f$ converges in $L_p(\mu)$.

(b) $\Longrightarrow$ (a) : This is straightforward and is omitted.

Proof of the Corollary. Considering S on $L_p(h^p d\mu)$ representing T on $L_p(\mu)$, the Corollary follows immediately from the Theorem, because the assumption $T^n h \leq Mh$ for all $n \geq 1$ implies $\|S^n\|_\infty \leq M$ for all $n \geq 1$.

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