Math. J. Okayama Univ. 21 (1979), 179—182

ON HOMOMORPHISMS OF RINGS INTO MATRIX RINGS

KATSUO CHIBA

Throughout the present note, all rings and all ring homomorphisms are assumed to be unital. Let R be a ring, and $M_n(R)$ the ring of all $n \times n$ matrices with entries in R. The (i, j)-entry of an element X of $M_n(R)$ will be denoted by $(X)_{ij}$. If $\eta : R \to S$ is a ring homomorphism, then $M_n(\eta) : M_n(R) \to M_n(S)$ denotes the ring homomorphism defined by $M_n(\eta)(r_{ij}) = (\eta(r_{ij}))$.

Given a ring homomorphism $f : A \to R$, we put $ax = f(a)x$ and $xa = xf(a)$ for $a \in A$ and $x \in R$, and we say that R is an *A-algebra*, if $R = AR^A$ where $R^A = \{r \in R \mid ar = ra$ for all $a \in A\}$. The ring of polynomials in a set of noncommutative variables $X = \{x_i\}$ with coefficients in A is called a *free A-algebra*, and is denoted by $A\langle X \rangle$. An A-algebra R is said to be *central if* $R = AR^R$ where R^R is the center of R. The ring of polynomials in a set of commutative variables X with coefficients in A is called a free *central A-algebra*, and is denoted by $A[X]$. A *finitely generated A-algebra* (resp. *finitely generated central A-algebra*) will mean a homomorphic image of $A\langle X \rangle$ (resp. of $A[X]$) for some finite X (see [3]).

The purpose of this note is to prove the following generalizations of [2, Theorem 2] and [1, Theorem 1].

Theorem 1. *Let R be an A-algebra with an A-homomorphism into the $n \times n$ matrix ring over some central A-algebra. Then there is a central A-algebra S and an A-homomorphism $\rho : R \to M_n(S)$ such that for any A-homomorphism $\sigma : R \to M_n(T)$, T a central A-algebra, there is a unique A-homomorphism $\eta : S \to T$ such that $M_n(\eta)\rho = \sigma$.*

Theorem 2. *Let A be a ring with the ascending chain condition on two-sided ideals, and R a finitely generated A-algebra. Then, for each positive integer n, R satisfies the ascending chain condition on ideals P such that R/P can be embedded as A-algebra into the $n \times n$ matrix ring over some central A-algebra.*

The homomorphism ρ in Theorem 1 is called a *universal A-homomorphism* of R.

In advance of proving our theorems, we establish the following lemmas, whose proofs are heavily due to the technique employed in the proofs of [2, Theorem 1] and [1, Lemma 1].

Lemma 1. *Let* $R = A\langle X\rangle$ *be the free A-algebra in* $X = \{x_\lambda \mid \lambda \in \Lambda\}$, *and* $S = A[X']$ *the free central A-algebra in* $X' = \{x_{ij}^\lambda \mid \lambda \in \Lambda,\ 1 \leqq i, j \leqq n\}$. *Then, the A-homomorphism* $\rho : R \to M_n(S)$ *defined by* $\rho(x_\lambda) = (x_{ij}^\lambda)$ *is a universal A-homomorphism of* R.

Proof. Let T be an arbitrary central A-algebra, and $\sigma : R \to M_n(T)$ an A-homomorphism. We defined an A-homomorphism $\eta : A[X'] \to T$ by $\eta(x_{ij}^\lambda) = (\sigma(x_\lambda))_{ij}$. Then it is easy to see that $M_n(\eta)\rho = \sigma$, and that such an η is uniquely determind.

Lemma 2. *Let* R *be an A-algebra with a universal A-homomorphism* $\rho : R \to M_n(S)$, *and* I *a proper ideal of* R. *Choose an ideal* U *of* S *such that the ideal of* $M_n(S)$ *generated by* $\rho(I)$ *is* $M_n(U)$.

(i) *If* $U \neq S$, *then the A-homomorphism* $\bar\rho : R/I \to M_n(S/U)$ *induced by* ρ *is a universal A-homomorphism of* R/I.

(ii) R/I *can be embbed into the* $n \times n$ *matrix ring over some central A-algebra if and only if* $\rho^{-1}(M_n(U)) = I$.

Proof. (i) Let $\pi : R \to R/I$ and $\tau : S \to S/U$ be the canonical maps, and $\sigma : R/I \to M_n(T)$ (T a central A-algebra) an A-homomorphism. Then we have an A-homomorphism $\eta : S \to T$ such that $M_n(\eta)\rho = \sigma\pi$. Since $M_n(\eta)\rho(I) = \sigma\pi(I) = 0$, we have $\eta(U) = 0$, and hence there is an A-homomorphism $\bar\eta : S/U \to T$ with $\bar\eta\tau = \eta$. It follows then $M_n(\bar\eta)\bar\rho\pi = M_n(\bar\eta)M_n(\tau)\rho = M_n(\bar\eta)\tau\rho = M_n(\eta)\rho = \sigma\pi$. Hence $M_n(\bar\eta)\bar\rho = \sigma$, since π is surjective. Now, let $\eta' : S/U \to T$ an arbitrary A-homomorphism with $M_n(\eta')\bar\rho = \sigma$. Then we have $M_n(\eta'\tau)\rho = M_n(\eta')M_n(\tau)\rho = M_n(\eta')\bar\rho\pi = \sigma\pi$. By the uniqueness of η, we obtain $\eta = \eta'\tau = \bar\eta\tau$. Hence $\eta' = \bar\eta$, sincc τ is surjective.

(ii) By (i), it is easy to see that R/I can be embedded into the $n \times n$ matrix ring over some central A-algebra if and only if $\bar\rho$ is injective, whence it follows our assertion.

Proof of Theorem 1. We may assume that $R = A\langle X\rangle/I$ with some free A-algebra $A\langle X\rangle$ and its ideal I. By Lemma 1, $A\langle X\rangle$ has a universal A-homomorphism $\rho : A\langle X\rangle \to M_n(S)$. Choose an ideal U of S such that the ideal of $M_n(S)$ generated by $\rho(I)$ is $M_n(U)$. Since $A\langle X\rangle/I$ has an A-homomorphism into the $n \times n$ matrix ring over some central A-algebra, the proof of Lemma 2 (i) enables us to see that $U \neq S$

and the A-homomorphism $\bar{\rho}: A\langle X\rangle/I \to M_n(S/U)$ induced by ρ is a universal A-homomorphism of $A\langle X\rangle/I$.

Remark. If $\rho: R \to M_n(S)$ and $\rho': R \to M_n(S')$ are universal A-homomorphisms of R, then there is an A-isomorphism $\eta: S \to S'$ such that $\rho' = M_n(\eta)\rho$. Therefore, under the notations of Theorem 1, $\{(\rho(r))_{ij} | r \in R^A,\ 1 \leqq i,\ j \leqq n\}$ generates S as A-algebra. As a consequence, if R is a finitely generated A-algebra, then S is a finitely generated central A-algebra.

Proof of Theorem 2 (cf. also [5, p. 106, Theorem 2. 1]). Let $\rho: R \to M_n(S)$ be a universal A-homomorphism of R, and let $I_1 \subset I_2 \subset \cdots \subset I_k \subset \cdots$ be an ascending chain of ideals of R such that R/I_k can be embedded into the $n \times n$ matrix ring over some central A-algebra. Then we have the following ascending chain of ideals in $M_n(S)$: $\{\rho(I_1)\} \subset \{\rho(I_2)\} \subset \cdots \subset \{\rho(I_k)\} \subset \cdots$, where $\{\rho(I_k)\}$ is the ideal of $M_n(S)$ generated by $\rho(I_k)$. As was noted in the above remark, S is a finitely generated central A-algebra. Hence, $M_n(S)$ satisfies the ascending chain condition on two-sided ideals. Since there exists then a positive integer k such that $\{\rho(I_k)\} = \{\rho(I_{k+1})\} = \cdots$, by Lemma 2 (ii) we obtain $I_k = I_{k+1} = \cdots$.

In conclusion, as application of Theorem 2 together with the following proposition, we shall present several results concerning *PI*-rings. Recall that a prime *PI*-ring R has a central simple quotient ring $Q = RK$, where K is the quotient field of the center of R [6, Corollary 1], and that p. i. deg R is the square root of $\dim_K Q$ (see e. g. [4]).

Proposition 1. *If an A-algebra R is a semiprime PI-ring, then R can be embedded into the* $n \times n$ *matrix ring over some central A-algebra, where n is the least common multiple of* p. i. deg R/P *for all prime ideals P of R.*

Proof. First, we consider the case that R is prime. Let $Q = RK$ be the central simple quotient ring of R, where K is the quotient field of the center of R. According to $R = AR^A$, we see that $Q = RK = (AR^A)K = (AK)R^A$ is (Artinian) simple. Hence, AK is a prime ring whose center is K, and so by [6, Theorem 2], is a central simple K-algebra. Now, there holds $Q = AK \otimes_K V_Q(AK)$, where the centralizer $V_Q(AK)$ of AK in Q is a central simple algebra. If $L \supset K$ is a splitting field of $V_Q(AK)$, then $V_Q(AK) \otimes_K L \simeq M_m(L)$ and $m \mid$ p. i. deg R. Obviously, $Q \otimes_K L \simeq (AK \otimes_K V_Q(AK)) \otimes_K L \simeq AK \otimes_K M_m(L) \simeq$

$M_m(AK\otimes_K L)$ and $AK\otimes_K L$ is a central A-algebra. Now, we come back to the genereal case. For every prime ideal P of R, we have seen that R/P can be embedded in the $n\times n$ matrix ring over some central A-algebra S. Hence, we have an embedding $R\to\Pi(R/P)\to\Pi M_n(S)\to M_n(\Pi S)$.

In the following corollaries, we assume that A is a ring with the ascending chain conditionon on two-sided ideals and that R is a finitely generated A-algebra satisfying a polynomial identity. The former generalizes [7, Lemma 2] and [5, p. 106, Corollary 2. 2], and the latter is a generalization of [5, p. 108, Theorem 2. 5].

Corollary 1. *R satisfies the ascending chain condition on semiprime ideals.*

Proof. This is immediate by Theorem 2 and Proposition 1.

Corollary 2. *If R is semiprime and S is the set of regular elements of the center of R, then the natural localization R_S is the finite direct sum of finite dimensional central simple algebras.*

Proof. Since R satisfies the ascending chain condition on semiprime ideals (Corollary 1), by [5, p. 108, Corollary 2. 4] we have $P_1\cap\cdots\cap P_t=0$, where $P_1,\cdots,P_t$ are all the minimal prime ideals of R. Now, we can proceed in the same way as that of [5, p. 108, Theorem 2. 5] did.

REFERENCES

[1] S. A. AMITSUR: A noncommutative Hilbert basis theorem and subrings of matrices, Trans. Amer. Math. Soc. **149** (1970), 133—142.
[2] S. A. AMITSUR: Embeddings in matrix rings, Pacific J. Math. **36** (1971), 21—29.
[3] M. ARTIN: On Azumaya algebras and finite dimensional representations of rings, J. Algebra **11** (1969), 532—563.
[4] G. M. BERGMAN and L. W. SMALL: P. I. degrees and prime ideals, J. Algebra **33** (1975), 435—462.
[5] C. PEOCESI: Rings with Polynomial Identity, Marcer Dekker, New York, 1973.
[6] L. ROWEN: Some results on the center of a ring with polynomial identity, Bull. Amer. Math. Soc. **79** (1973), 219—223.
[7] W. SCHELTER: Non-commutative affine P. I. rings are catenary. J. Algebra **51** (1978), 12—18.

DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

(Received April 3, 1979)