Math. J. Okayama Univ. 21 (1979), 165—166

STRUCTURE AND COMMUTATIVITY OF RINGS WITH CONSTRAINTS ON NILPOTENT ELEMENTS. II

HAZAR ABU-KHUZAM and ADIL YAQUB

The purpose of this note is to generalize the principal theorem of the previous paper [1] as follows:

Theorem. *Let R be an associative ring and let N be the set of all nilpotent elements of R. Suppose n is a fixed positive integer. Suppose, further, that* (i) *N is commutative,* (ii) *for every x in R, there exists an element x' in the subring $\langle x \rangle$ generated by x such that $x^m = x^{m+1}x'$ with some positive integer $m = m(x)$,* (iii) *$x - y \in N$ implies that $x^n - y^n$ is in the center Z of R.*

(a) *If $na = 0$, $a \in N$ imply $a = 0$, then R is a subdirect sum of nil commutative rings and local commutative rings.*

(b) *If n is a prime, then R is a subdirect sum of nil commutative rings and local commutative rings.*

In preparation for the proof, we establish the following lemmas.

Lemma 1. *Hypothesis* (iii) *implies that $ab^n = b^n a$ for all $a \in N$ and all $b \in R$, and necessarily all idempotents of R are in Z.*

Proof. Since $(a+b) - b \in N$, by (iii) we have $c = (a+b)^n - b^n \in Z$. Hence $b^n(a+b) = \{(a+b)^n - c\}(a+b) = (a+b)\{(a+b)^n - c\} = (a+b)b^n$, which simplifies to $b^n a = ab^n$. As is well known, every idempotent commuting with all nilpotents is central.

Lemma 2. *Hypotheses* (i), (ii), (iii) *imply the following*:

(a) *N is a commutative nil ideal.*

(b) *If e is an idempotent and a is in N, then $nea \in Z$.*

(c) *If φ is a homomorphism of R onto R^*, then $\varphi(N)$ coincides with the set of all nilpotent elements of R^*.*

Proof. (a) and (c) have been proved in Lemma 2 [1]. We shall prove (b). Since N is a commutative nil ideal, it can be easily seen that $a^k \in Z$ for all $k > 1$. By (iii), $(e+a)^n - e^n$ is in Z. Hence, $a^n + na^{n-1}e + \cdots + nae \in Z$, since e is central by Lemma 1. This implies that $nae \in Z$.

Corollary 1. *If R satisfies the hypotheses* (i), (ii), (iii), *then any subring of R and any homomorphic image of R satisfy* (i), (ii), (iii).

Now, we are ready to prove our theorem.

Proof of Theorem. Careful scrutiny of the proof of Theorem 2 [1] shows that it suffices to prove that if φ is a homomorphism of R onto a local ring R^* with (nil) radical N^* such that $R^*/N^*=GF(r)$, where $r=p^\alpha$, p prime, $\alpha\geq 1$, then every element a^* in N^* is central.

(a) By (ii) and Lemma 1, we can easily see that there exists a central idempotent e of R such that $\varphi(e)=1$. Let b^* be an arbitrary element of R^*. Then, by Lemma 2, $a^*=\varphi(a)$ with some $a\in N$, and $b^*=\varphi(b)$ with some $b\in R$. Since $nea\in Z$ (Lemma 2 (b)), therefore $ne[a, b]=0$. By hypothesis, it follows then $e[a, b]=0$, and therefore $[a^*, b^*]=0$.

(b) Obviously, R^* is of characteristic p^β for some positive integer β. By Lemma 2 (b) and Corollary 1, na^* is central. If $n\neq p$, then it is easy to see that a^* is central. On the other hand, if $n=p$ then Lemma 1 enables us to proceed as in the latter part of the proof of Theorem 2 [1].

The following example was pointed out to us by Prof. H. G. Moore. Let $R=\left\{\begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c\in \mathrm{GF}(4)\right\}$. It is readily verified that R is not commutative and satisfies all the hypotheses of Theorem (a) except the hypothesis that $na=0$, $a\in N$ imply $a=0$ $(n=6)$. Next, we consider the ring R constructed in Remark [1]. Then R is not commutative, and satisifies all the hypotheses of Theorem (b) except the hypothesis that n is prime $(n=6)$.

In conclusion, we would like to express our gratitude and indebtedness to Prof. H. Tominaga for his helpful suggestions and valuable comments.

REFERENCE

[1] D. L. OUTCALT and Adil YAQUB: Structure and commutativity of rings with constraints on nilpotent elements, Math. J. Okayama Univ. **21** (1979), 15—19.

DEPARTMENT OE MATHEMATICS
UNIVERSITY OF CALIFORNIA, SANTA BARBARA
SANTA BARBARA, CALIFNRNIA 93106, U. S. A.

(Received March 9, 1979)