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SUMS OF RECIPROCALS OF SOME MULTIPLICATIVE FUNCTIONS

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1. Introduction. Throughout this paper, m denotes a positive integral variable, p denotes a prime and x denotes a real variable ≥ 3. In 1900 Landau [5] established that

$$(1.1)\quad \sum_{m\leq x}\frac{1}{\varphi(m)}=\frac{315\zeta(3)}{2\pi^4}\left(\log x+\gamma-\sum_p\frac{\log p}{p^2-p+1}\right)+O(x^{-1}\log x),$$

where $\varphi(m)$ is the Euler-totient function, $\zeta(s)$ is the Riemann zeta function and γ is the Euler constant. In 1916 Ramanujan [6] established that

$$(1.2)\quad \sum_{m\leq x}\frac{1}{\tau(m)}=x\left\{\frac{A_1}{(\log x)^{\frac{1}{2}}}+\frac{A_2}{(\log x)^{\frac{3}{2}}}+\cdots+\frac{A_r}{(\log x)^{r-\frac{1}{2}}}+O((\log x)^{(-r-\frac{1}{2})})\right\},$$

where $\tau(m)$ is the number of divisors of m, r is any positive integer, $A_1=\pi^{-\frac{1}{2}}\prod_p\left\{(p^2-p)^{\frac{1}{2}}\log\left(\frac{p}{p-1}\right)\right\}$, and $A_2,\cdots,A_r$ are more complicated constants.

In this paper, we establish the asymptotic formulae for the sums $\sum_{m\leq x}\frac{1}{\sigma(m)}$ and $\sum_{m\leq x}\frac{1}{\psi(m)}$, where $\sigma(m)$ is the sum of the divisors of m and $\psi(m)$ is Dedekind's ψ-function (cf. [2], p. 123) which has the following arithmetical form:

$$(1.3)\qquad \psi(m)=\sum_{d\delta=m}\mu^2(d)\delta=m\prod_{p|m}\left(1+\frac{1}{p}\right),$$

μ being the Möbius function. In fact, we prove the following general result and then deduce (see §4) asymptotic expressions for $\sum_{m\leq x}\frac{1}{\sigma(m)}$ and $\sum_{m\leq x}\frac{1}{\psi(m)}$:

Theorem. *Suppose g is a multiplicative function satisfying*

$$(1.4)\qquad g(p)=\frac{1}{p+1},\quad \textit{for all primes } p,$$

and for each $\varepsilon>0$,

(1. 5) $$p^j(g(p^j)-g(p^{j-1}))=O(p^{j\varepsilon}),$$

for all primes p *and positive integers* j. *Then we have*

(1. 6) $$\sum_{m\leq x}\frac{g(m)}{m}=A\log x+B+O(x^{-1}\log^{\frac{2}{3}}x(\log\log x^{\frac{4}{3}}),$$

where

(1. 7) $$A\equiv A(g)=\sum_{m=1}^{\infty}\frac{g^*(m)}{m},$$

and

(1. 8) $$B=A\gamma-\sum_{m=1}^{\infty}\frac{g^*(m)\log m}{m}.$$

In the above, g^* *is the multiplicative function defined by*

(1. 9) $$g^*(m)=\sum_{d|m}\mu(d)g\left(\frac{m}{d}\right).$$

2. **Prerequisites.** In this section, we state some known results and prove some lemmas which are needed in the present discussion. Let $[x]$ denote, as usual, the largest integer $\leq x$. We need the following best known result of its kind which is due to Arnold Walfisz [8] :

Lemma 2.1 (cf. [8], (36), p. 144).

$$\sum_{m\leq x}\frac{\mu(m)}{m}\rho\left(\frac{x}{m}\right)=O(\lambda(x)),$$

where

(2. 1) $$\rho(x)=x-[x]-\frac{1}{2},$$

and

(2. 2) $$\lambda(x)=\begin{cases}\log^{\frac{2}{3}}x(\log\log x)^{\frac{4}{3}}, & if\ x\geq 3,\\ 1, & if\ 0<x<3.\end{cases}$$

Remark 2.1. It is clear that $\lambda(x)$ is increasing for $x\geq 3$. Using this, it can be shown that if $x>0$, then

$$\lambda(x)\leq H\lambda(y),\quad \text{for all } y\leq x,$$

where H is an absolute positive constant.

Lemma 2.2. *Let* f *be any multiplicative function satisfying*

(2. 3) $$f(m)=O(m^{\varepsilon}),\quad for\ every\ \varepsilon>0,$$

and

(2.4) $$f(p)+1=O(1/\sqrt{p}) \quad \textit{for all primes } p.$$

Further, let h *be the arithmetic function defined by*

(2.5) $$h(m)=\sum_{d\mid m} f(d).$$

Then the series $\sum_{m=1}^{\infty} h(m)\, m^{-s}$ *converges absolutely for any* $s>\frac{1}{2}$.

Proof. Since f is multiplicative, it follows (cf. [4], Theorem 265, p. 235) that h is also multiplicative. It is known (cf. [3], Theorem 41) that if h is multiplicative and the product $\prod_p \left\{1+\sum_{m=1}^{\infty} \frac{|h(p^m)|}{p^{ms}}\right\}$ converges then the series $\sum_{m=1}^{\infty} h(m)\, m^{-s}$ converges absolutely. Hence, in the present context, it suffices to prove that $\prod_p \left\{1+\sum_{m=1}^{\infty} \frac{|h(p^m)|}{p^{ms}}\right\}$ converges for $s>\frac{1}{2}$. Let $s>\frac{1}{2}$ and $0<\varepsilon<s-\frac{1}{2}$. From (2.3) and (2.5) it follows that $h(m)=O(m^{\varepsilon})$. Since $h(p)=1+f(p)$, by (2.4) we have

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{|h(p^m)|}{p^{ms}} &= \frac{|1+f(p)|}{p^s}+\sum_{m=2}^{\infty} \frac{|h(p^m)|}{p^m} \\ &= O(p^{-s-\frac{1}{2}})+O(\sum_{m=2}^{\infty} p^{-m(s-\varepsilon)}) \\ &= O(p^{-s-\frac{1}{2}})+O(p^{-2(s-\varepsilon)}(1-p^{-(s-\varepsilon)})^{-1}) \\ &= O(p^{-s-\frac{1}{2}})+O(p^{-2(s-\varepsilon)}), \quad \text{for large } p.\end{aligned}$$

Now,

$$\begin{aligned}\sum_p \sum_{m=1}^{\infty} \frac{|h(p^m)|}{p^{ms}} &= O(\sum_p p^{-s-\frac{1}{2}})+O(\sum_p p^{-2(s-\varepsilon)}) \\ &= O(1)+O(1)=O(1),\end{aligned}$$

since $s>\frac{1}{2}$ and $2(s-\varepsilon)>1$. Hence Lemma 2.2 follows.

Lemma 2.3. *Let* f *be as in Lemma* 2.2. *Then we have*

$$\sum_{m\le x} \frac{f(m)}{m}=O(1).$$

Proof. By (2.5) and the Möbius inversion formula ([4], Theorem

266, p. 236) we have

(2. 6)
$$f(m)=\sum_{d\mid m}\mu(d)\,h\left(\frac{m}{d}\right).$$

Since $\sum_{m\leq x}\mu(m)m^{-1}=O(1)$, we have by (2. 6),

$$\sum_{m\leq x}\frac{f(m)}{m}=\sum_{d\delta\leq x}\frac{\mu(d)h(\delta)}{d\delta}=\sum_{\delta\leq x}\frac{h(\delta)}{\delta}\sum_{d\leq x/\delta}\frac{\mu(d)}{d}$$
$$=O(\sum_{\delta\leq x}\frac{|h(\delta)|}{\delta})=O(1),\ \text{by Lemma 2. 2.}$$

Hence Lemma 2. 3 follows.

Lemma 2.4. *Under the hypothesis of Lemma* 2. 2, *we have*

(2. 7)
$$F(x)\equiv\sum_{m\leq x}\frac{f(m)}{m}\rho\left(\frac{x}{m}\right)=O(\lambda(x)),$$

where $\rho(x)$ *and* $\lambda(x)$ *are given by* (2. 1) *and* (2. 2) *respectively.*

Proof. By (2. 6), Lemma 2. 1, Remark 2. 1 and Lemma 2. 2, we have

$$F(x)=\sum_{d\delta\leq x}\frac{\mu(d)h(\delta)}{d\delta}\rho\left(\frac{x}{d\delta}\right)=\sum_{\delta\leq x}\frac{h(\delta)}{\delta}\sum_{d\leq x/\delta}\frac{\mu(d)}{d}\rho\left(\frac{x}{d\delta}\right)$$
$$=O\left(\sum_{\delta\leq x}\frac{|h(\delta)|\lambda\left(\frac{x}{\delta}\right)}{\delta}\right)=O(\lambda)(x)\sum_{\delta\leq x}\frac{|h(\delta)|}{\delta})=O(\lambda(x)).$$

Hence Lemma 2. 4 follows.

Lemma 2. 5. *Let* g^* *be given by* (1. 9). *Then we have*

(2. 8)
$$G^*(x)=\sum_{m\leq x}g^*(m)=O(1).$$

Proof. From (1. 9) it follows that for $j\geq 1$,

(2. 9)
$$g^*(p^j)=g(p^j)-g(p^{j-1}).$$

Since g^* is multiplicative, it follows from (1. 5) and (2. 9) that

(2. 10)
$$mg^*(m)=O(m^{\varepsilon}),\quad\text{for every }\varepsilon>0.$$

Further, by (2. 9) and (1. 4), we have

$$pg^*(p)=p\,(g(p)-1)=p\left(\frac{p}{p+1}-p\right)=-\frac{p}{p+1},$$

so that

$$(2.11)\qquad 1+pg^*(p)=1-\frac{p}{p+1}=\frac{1}{p+1}=O\left(\frac{1}{p}\right)=O\left(\frac{1}{p^{\frac{1}{2}}}\right).$$

Hence if we take $f(m)=mg^*(m)$, from (2. 10) and (2. 11), it is clear that the conditions (2. 3) and (2. 4) are satisfied. Now, Lemma 2. 5 follows from Lemma 2. 3.

Lemma 2. 6. *We have*

$$\sum_{m\leq x} g^*(m)\,\rho\left(\frac{x}{m}\right)=O(\lambda(x)),$$

where g^* *is given by* (1. 9).

Proof. Taking $f(m)=mg^*(m)$ in Lemma 2. 4, we obtain Lemma 2. 6, in virtue of (2. 10) and (2. 11).

Lemma 2.7. *We have*

$$\sum_{m\leq x}\frac{g^*(m)}{m}=A+O(x^{-1}),$$

where A *is given by* (1. 7).

Proof. The series $\sum_{m=1}^{\infty}\frac{g^*(m)}{m}$ converges absolutely by (2. 10). If $G^*(x)$ is given by (2. 8), then by Lemma 2. 5 and partial summation we have

$$\sum_{m>x}\frac{g^*(m)}{m}=-\frac{G^*(x)}{([x]+1)}+\sum_{m>x}G^*(m)\left(\frac{1}{m}-\frac{1}{m+1}\right)$$
$$=O(x^{-1})+O\left(\sum_{m>x}\frac{1}{m^2}\right)=O(x^{-1})+O(x^{-1})=O(x^{-1}).$$

Since by (1. 7), $\sum_{m\leq x}\frac{g^*(m)}{m}=A-\sum_{m\leq x}\frac{g^*(m)}{m}$, we obtain Lemma 2. 7.

Lemma 2. 8. *We have*

$$\sum_{m\leq x}\frac{g^*(m)\log m}{m}=\sum_{m=1}^{\infty}\frac{g^*(m)\log m}{m}+O(x^{-1+\varepsilon}),\quad \textit{for every}\ \varepsilon>0.$$

Proof. By (2. 10) we have

$$\sum_{m\leq x}\frac{g^*(m)\log m}{m}=O\left(\sum_{m>x}\frac{1}{m^{2-\varepsilon}}\right)=O(x^{-1+\varepsilon}).$$

Hence Lemma 2. 8 follows.

Now, we are in a position to prove the following important

Lemma 2. 9. *Let g be as given in the statement of Theorem. Then we have*

(2. 12) $$\sum_{m\le x} g(m) = Ax + O(\lambda(x)),$$

where A is given by (1. 7) *and* $\lambda(x)$ *as defined in* (2. 2).

Proof. From (1. 9) and the converse of the Möbius inversion formula (cf. [4], Theorem 267, p. 236)

(2. 13) $$g(m) = \sum_{d\mid m} g^*(d).$$

Now, by (2. 13), (2. 1), Lemmas 2. 6 and 2. 7, and (2. 8), we have

$$\sum_{m\le x} g(m) = \sum_{d\delta\le x} g^*(d) = \sum_{d\le x} g^*(d)\left[\frac{x}{d}\right]$$
$$= x\sum_{x\le d}\frac{g^*(d)}{d} - \sum_{d\le x} g^*(d)\rho\left(\frac{x}{d}\right) - \frac{1}{2}\sum_{d\le x} g^*(d)$$
$$= Ax + O(1) + O(\lambda(x)) + O(1) = Ax + O(\lambda(x)).$$

Hence Lemma 2. 9 follows.

Lemma 2.10 (cf. [4], Theorem 422, p. 347). *For* $x\ge 2$,

$$\sum_{m\le x}\frac{1}{m} = \log x + \gamma + O(x^{-1}).$$

3. Proof of Theorem. Let

$$G(x) = \sum_{m\le x} g(m) \text{ and } \Delta(x) = G(x) - Ax.$$

Then by (2. 12) we have $\Delta(x) = O(\lambda(x))$, Now, by partial summation, we have

$$\sum_{m\le x}\frac{g(m)}{m} = \frac{G(x)}{x} + \int_1^x \frac{G(t)}{t^2}\,dt$$
$$= A + \frac{\Delta(x)}{x} + \int_1^x \frac{1}{t^2}\{At + \Delta(t)\}\,dt$$
$$= A + \frac{\Delta(x)}{x} + A\int_1^x \frac{dt}{t} + \int_1^x \frac{\Delta(t)}{t^2}\,dt$$
$$= A + \frac{\Delta(x)}{x} + A\log x + \int_1^\infty \frac{\Delta(t)}{t^2}\,dt - \int_x^\infty \frac{\Delta(t)}{t^2}\,dt$$
$$= A(\log x + C) - \int_x^\infty \frac{\Delta(t)}{t^2}\,dt + O(x^{-1}\lambda(x)),$$

where $C = 1 + \frac{1}{A}\int_1^\infty \frac{\Delta(t)}{t^2}\,dt$, is a constant. (Here, of course, we assume that $A \neq 0$). Also, we have

$$\int_x^\infty \frac{\Delta(t)}{t^2}\,dt = O\left(\int_x^\infty \frac{\lambda(t)}{t^2}\,dt\right) = O\left(x^{-\varepsilon}\lambda(x)\int_x^\infty \frac{dt}{t^{2-\varepsilon}}\right)$$
$$= O(x^{-\varepsilon}\lambda(x)x^{-1+\varepsilon}) = O(x^{-1}\lambda(x)),$$

where we used that $x^{-\varepsilon}\lambda(x)$ is decreasing for every $\varepsilon > 0$. Hence we obtain

$$\sum_{m\le x}\frac{g(m)}{m} = A(\log x + C) + O(x^{-1}\lambda(x)). \tag{3.1}$$

On the other hand, we have by (2.13), Lemmas 2.10, 2.7, 2.8 and (1.8),

$$\sum_{m\le x}\frac{g(m)}{m} = \sum_{d\delta<x}\frac{g^*(d)}{d\delta} = \sum_{d\le x}\frac{g^*(d)}{d}\sum_{\delta\le x/d}\frac{1}{\delta}$$
$$= \sum_{d\le x}\frac{g^*(d)}{d}\left\{\log x - \log d + \gamma + O\left(\frac{d}{x}\right)\right\}$$
$$= (\log x + \gamma)\sum_{d\le x}\frac{g^*(d)}{d} - \sum_{d\le x}\frac{g^*(d)\log d}{d} + O(x^{-1}\sum_{d\le x}|g^*(d)|)$$
$$= (\log x + \gamma)(A + O(x^{-1})) - \sum_{d=1}^{\infty}\frac{g^*(d)\log d}{d}$$
$$+ O(x^{-1+\varepsilon}) + O(x^{-1}\sum_{d\le x}|g^*(d)|)$$
$$= A\log x + B + O(x^{-1+\varepsilon}) + O(x^{-1}\sum_{d\le x}|g^*(d)|).$$

Now, by (2.10), we have

$$\sum_{d\le x}|g^*(d)| = O\left(x^{\varepsilon}\sum_{d\le x}\frac{1}{d}\right) = O(x^{\varepsilon}\log x) = O(x^{2\varepsilon}),$$

for every $\varepsilon > 0$. Hence we have

$$\sum_{m\le x}\frac{g(m)}{m} = A\log x + B + O(x^{-1+2\varepsilon}), \tag{3.2}$$

for every $\varepsilon > 0$. Now, comparing (3.1) and (3.2), we find that $AC = B$, from which the theorem follows.

4. Applications. First we have

Corollary 4.1. *For* $x \ge 3$

$$\sum_{m\le x}\frac{1}{\sigma(m)} = a\left(\log x + \gamma + \sum\frac{(p-1)^2\beta(p)\log p}{p\alpha(p)}\right)$$
$$+ O\left(x^{-1}\log^{\frac{2}{3}} x\,(\log\log x)^{\frac{4}{3}}\right), \tag{4.1}$$

where

(4.2) $$a = \prod_p \alpha(p),$$

(4.3) $$\alpha(p) = 1 - \frac{(p-1)^2}{p} \sum_{j=1}^{\infty} \frac{1}{(p^j-1)(p^{j+1}-1)},$$

and

(4.4) $$\beta(p) = \sum_{j=1}^{\infty} \frac{j}{(p^j-1)(p^{j+1}-1)}.$$

Proof. Taking $g(m) = \frac{m}{\sigma(m)}$ in Theorem, we see that g satisfies (1.4). Since $\sigma(p^j) = \frac{p^{j+1}-1}{p-1}$, we have for $j \geq 1$,

(4.5) $$\begin{aligned} g^*(p^j) &= g(p^j) - g(p^{j-1}) \\ &= \frac{p^j(p-1)}{p^{j+1}-1} - \frac{p^{j-1}(p-1)}{p^j-1} = -\frac{p^{j-1}(p-1)^2}{(p^{j+1}-1)(p^j-1)}, \end{aligned}$$

so that

$$\begin{aligned} p^j \mid g(p^j) - g(p^{j-1}) \mid &= \frac{p^j}{p^j-1} \cdot \left(\frac{p-1}{p}\right)^2 \cdot \frac{p^{j+1}}{p^{j+1}-1} \\ &\leq \frac{p^j}{p^j-1} \cdot \frac{p^{j+1}}{p^{j+1}-1} \leq 2 \cdot 2 = 4. \end{aligned}$$

Thus g satisfies (1.5) also. Further by (4.5) and the Euler infinite product theorem (cf. [4], Theorem 286) for $s > 0$ we have

(4.6) $$\begin{aligned} \sum_{m=1}^{\infty} \frac{g^*(m)}{m^s} &= \prod_p \left(1 + \sum_{j=2}^{\infty} \frac{g^*(p^j)}{p^{js}}\right) \\ &= \prod_p \left(1 - (p-1)^2 \sum_{j=1}^{\infty} \left(\frac{p^{j-1}}{(p^{j+1}-1)(p^j-1)p^{js}}\right)\right). \end{aligned}$$

From (4.6) ($s = 1$), (1.7), (4.2) and (4.3), we have

(4.7) $$A = a.$$

Now, differentiating the series in (4.6) with respect to s termwise, and then putting $s = 1$, we obtain

(4.8) $$\sum_{m=1}^{\infty} \frac{g^*(m)\log m}{m} = -a \sum_p \frac{(p-1)^2 \beta(p) \log p}{p\alpha(p)},$$

where a, $\alpha(p)$ and $\beta(p)$ are respectively given by (4.2), (4.3) and (4.4).

Now, (4. 1) follows from (4. 7) and (4. 8), by taking $g(m)=\frac{m}{\sigma(m)}$ in (1. 6).

Remark 4.1. Asymptototic formula for the sum $\sum_{2\leq m\leq x}\frac{1}{\log\sigma(m)}$ has been established by J. -M. De Koninck and J. Galambos (cf. [1], § 3, Theorem).

Corollary 4. 2. *For* $x\geq 3$, *we have*

$$(4.9)\quad \sum_{m\leq x}\frac{1}{\psi(m)}=\alpha\left(\log x+\gamma+\sum_{p}\frac{\log p}{p^2+p-1}\right)+O\left(x^{-1}\log^{\frac{2}{3}}x(\log\log x)^{\frac{3}{4}}\right),$$

where

$$(4.10)\quad \alpha=\prod_{p}\left(1-\frac{1}{p(p-1)}\right).$$

Proof. Taking $g(m)=\frac{m}{\psi(m)}$ in Theorem, we see that g satisfies (1. 4). By (1. 3) we hav e for $j\geq 1$,

$$(4.11)\quad g^*(p^j)=g(p^j)-g(p^{j-1})=\begin{cases}-\frac{1}{p+1}, & \text{if } j=1,\\ 0, & \text{if} j\geq 2.\end{cases}$$

Thus g satisfies (1. 5) also. Now, by the Euler infinite product theorem and (4. 11), we have for $s>0$,

$$(4.12)\quad \sum_{m=1}^{\infty}\frac{g^*(m)}{m^s}=\prod\left(1-\frac{1}{p^s(p+1)}\right).$$

From (4. 12) ($s=1$), (1. 7) and (4. 10), we see that

$$(4.13)\quad A=\alpha.$$

Now, differentiating the series in (4. 12) with respect to s termwise, and then putting $s=1$, we obtain

$$(4.14)\quad \sum_{m=1}^{\infty}\frac{g^*(m)\log m}{m}=-\alpha\sum_{p}\frac{\log p}{p^2+p-1},$$

where α is given by (4. 10). Now, (4. 9) follows from (4. 13) and (4. 14), by taking $g(m)=\frac{m}{\psi(m)}$ in (1. 6).

Remark 4. 2. Formula (4. 9) has been established by Suryanarayana (cf. [7], Lemma 2. 10, p. 12) with a weaker O-estimate for the error term, namely $O(x^{-1} \log x)$.

Remark 4. 3. It may be interesting to improve the O-estimate of the error term in (1. 1), which we could not do.

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