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ON THE RATIONAL HOMOTOPY OF FIXED POINT SETS OF CIRCLE ACTIONS

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1. Introduction. Let S^1 be the circle group consisting of the complex numbers with absolute value 1. Let X be a simply connected finite CW complex on which S^1 acts continuously. Let F be a connected component of the fixed point set of X which is a simply connected subcomplex of X. Choose and fix a point x_0 in F, and let ΩX (resp. ΩF) be the loop space of X (resp. F) with x_0 as its base point. For $g \in S^1$ and $x \in X$, $g \cdot x$ denotes the point of X obtained by transforming x by g. Then S^1 acts on ΩX by $(gh)(t) = g \cdot h(t)$ for $g \in S^1$, $h \in \Omega X$ and $t \in [0, 1]$. The fixed point set of this S^1 action on ΩX is ΩF.

Let $ES^1 \longrightarrow BS^1$ be the universal S^1 bundle. For an S^1 space Y, let Y_G be the quotient space of $ES^1 \times Y$ by the diagonal S^1 action. Y_G is a fibre bundle over BS^1 with fibre Y. The equivariant cohomology ring of Y with rational coefficient is defined by $H_G^*(Y, \boldsymbol{Q}) \equiv H^*(Y_G, \boldsymbol{Q})$, the singular cohomology ring of Y_G with rational coefficient. Throughout this paper the coefficients of the cohomology rings are always $\boldsymbol{Q}$, and we shall ommit it. $H_G^*(Y)$ is a $H^*(BS^1)$ module via the bundle projection $Y_G \longrightarrow BS^1$. Now choose and fix a generator t of $H^2(BS^1)$. Then $H^*(BS^1)$ is a polynomial ring $\boldsymbol{Q}[t]$. Let $H_G^*(Y)[t^{-1}]$ be the localization of $H_G^*(Y)$ with respect to the multiplicative system of powers of t ([3], [4]). Our first theorem is the following localization theorem.

Theorem 1. *Let X, F, ΩX and ΩF be as above. Then the inclusion $\Omega F \longrightarrow \Omega X$ induces an isomorphism,*

$$H_G^*(\Omega X)[t^{-1}] \longrightarrow H_G^*(\Omega F)[t^{-1}].$$

We note that the usual localization theorem of the equivariant cohomology ([3], [4]) cannot be applied to ΩX, for ΩX is neither compact nor of fiinite cohomological dimension over $\boldsymbol{Q}$ in general (the cohomological dimension over $\boldsymbol{Q}$ of a space Y means the supremum, finite or infinte, of the integer m such that there exists a sheaf $\mathscr{F}$ of $\boldsymbol{Q}$ modules with $H^m(Y, \mathscr{F}) \neq 0$). In fact there are many S^1 spaces for which the usual localization theorem does not hold (the simplest example is $S^r * S^\infty$ with trivial S^1 action on S^r and free S^1 action on S^∞).

As an application of Theorem 1, we shall prove the following theorem.

Theorem 2. *Let X and F be as above. Assume that X has a finite rational homotopy type, that is, $\sum_{k=0}^{\infty} \dim_Q \pi_k(X) \otimes \boldsymbol{Q} < \infty$. Then F has a finite rational homotopy type and the following inequality holds for each $i = 0, 1, \cdots\cdots,$*

$\sum_{n=0}^{\infty} \dim_Q (\pi_{i+2n}(F) \otimes \boldsymbol{Q}) \leqq \sum_{n=0}^{\infty} \dim_Q (\pi_{i+2n}(X) \otimes \boldsymbol{Q})$, where $\dim_Q$ denotes the dimension over $\boldsymbol{Q}$.

The inequality of the above theorem is a substantial improvement of the inequalities given in G. E. Bredon [2] and C. Allday [1] (Theorem 3. 3, 3. 7).

2. **Proof of Theorem 1.** Let X and F be as in §1, and x_0 be a point of F. Let PX and PF be the path spaces of X and F respectively with x_0 as the common base point. S^1 acts on PX by $(gp)(u) = g \cdot (p(u))$ for $g \in S^1$, $p \in PX$ and $u \in [0, 1]$. The fixed point set of this action on PX is PF. We obtain the following two fibre squares,

$$\begin{array}{ccccccc} \Omega X & \longrightarrow & PX & & \Omega F & \longrightarrow & PF \\ \downarrow & & \downarrow & \longleftarrow & \downarrow & & \downarrow \\ x_0 & \longrightarrow & X & & x_0 & \longrightarrow & F \end{array}$$

where the vertical maps are the maps obtained by taking the endpoint of each path, and the horizontal maps are the inclusions. All the maps in the above fibre squares are S^1 equivariant and the fixed point set of the left fibre square is the right one. Hence we obtain the following two fibre squares,

$$\begin{array}{ccccccc} (\Omega X)_G & \longrightarrow & (PX)_G & & (\Omega F)_G & \longrightarrow & (PF)_G \\ \downarrow & & \downarrow & \longleftarrow & \downarrow & & \downarrow \\ (x_0)_G & \longrightarrow & X_G & & (x_0)_G & \longrightarrow & F_G \end{array}$$

where Y_G denotes the space defined in §1 for $Y = \Omega X$, PX, $\cdots$. Now as X and F are finite CW complexes, $H^*(X)$ and $H^*(F)$ are finite dimensional, so that $H_G^*(X)$ and $H_G^*(F)$ are finite dimensional in each degree.

PX and PF are equivariantly contractible to x_0, and $H_G^*(PX) = H_G^*(PF) = H_G^*(x_0)$. Since X and F are simply connected, X_G and F_G are of homotopy type of simply connected CW complexes. Therefore we may apply the Eilenberg-Moore spectral sequence to the cohomologies of the above fibre squares ([5]),

$$\begin{array}{ccc} E_2^{p,q} = \mathrm{Tor}^{p,q}{}_{H_G^*(X)}(H_G^*(x_0),\ H_G^*(x_0)) & \Longrightarrow & H_G^*(\Omega X) \\ \downarrow & & \downarrow \\ E_2^{p,q} = \mathrm{Tor}^{p,q}{}_{H_G^*(F)}(H_G^*(x_0),\ H_G^*(x_0)) & \Longrightarrow & H_G^*(\Omega F). \end{array}$$

where the vertical maps are the maps induced by the inclusion and the naturality of the spectral sequences. Now the localization functor is an exact functor, hence we obtain the localized versions of the above spectral sequences,

$$\begin{array}{ccc} \mathrm{Tor}^{p,q}{}_{H_G^*(X)[t^{-1}]}(H_G^*(x_0)[t^{-1}],\ H_G^*(x_0)[t^{-1}]) & \Longrightarrow & H_G^*(\Omega X)[t^{-1}] \\ \downarrow & & \downarrow \\ \mathrm{Tor}^{p,q}{}_{H_G^*(F)[t^{-1}]}(H_G^*(x_0)[t^{-1}],\ H_G^*(x_0)[t^{-1}]) & \Longrightarrow & H_G^*(\Omega F)[t^{-1}], \end{array}$$

here we regard all the cohomology rings as Z-graded ($\deg t^{-1} = -2$). Let F' be the disjoint union of all the components of the fixed point set in X other than F. Then the usual localization theorem gives the isomorphism induced by the inclusion,

$H_G^*(X)[t^{-1}] \longrightarrow H_G^*(F)[t^{-1}] \oplus H_G^*(F')t^{-1}]$,

where $H_G^*(F)[t^{-1}]$ and $H_G^*(F')[t^{-1}]$ annihilate each other. As x_0 is a point of F, $H_G^*(F')[t^{-1}]$ annihilates $H_G^*(x_0)[t^{-1}]$. Therefore the inclusion induces an isomorphism of the E_2 terms of the above localized spectral sequences, and it induces an isomorphism $H_G^*(\Omega X)[t^{-1}] \longrightarrow H_G^*(\Omega F)[t^{-1}]$. This proves Theorem 1.

3. Proof of Theorem 2. Let X be as in Theorem 2 in §1. Since X is simply connected and of finite rational homotopy type, the rational homotopical property of ΩX is the same as that of a finite product of Eilenberg Maclane complexes $K(\boldsymbol{Q}, n_1) \times \cdots \times K(\boldsymbol{Q}, n_r)$ ($n_1 \leqq \cdots \leqq n_r$). Now BS^1 is $K(\boldsymbol{Z}, 2)$ and $(\Omega X)_G$ is a fiber space over BS^1 with fiber ΩX. From the rational Postnikov decompsition of $(\Omega X)_G$, we may construct a minimal model of $(\Omega X)_G$ such that,

(1) $\mathscr{M}((\Omega X)_G) = S\langle x_1, \cdots, x_r\rangle \otimes \boldsymbol{Q}[t]$ as a graded algebra, where *deg* $x_j = n_j$ ($j = 1, \cdots, r$), *deg* $t = 2$ and $S\langle x_1, \cdots, x_r\rangle$ denotes the free graded algebra generated by $x_1, \cdots, x_r$, that is, the tensor product of the polynomial algebra generated by the even graded elements and the exterior algebra generated by the odd graded elements,

the differential is given by $dt=0$ and $dx_j=$ a polynomial of $x_1, \cdots,$
(2) x_{j-1} and t which is divisible by t.

Now let $\mathcal{M}(\Omega X)$ and $\mathcal{M}(BS^1)$ be the minimal model of ΩX and BS^1 respectively. The differential of them are trivial. The map induced by the fibre inclusion, $\mathcal{M}((\Omega X)_G) \longrightarrow \mathcal{M}(\Omega X)$, is given by putting $t=0$, and the map induced by the projection, $\mathcal{M}(BS^1) \longrightarrow \mathcal{M}((\Omega X)_G)$ is given by the inclusion $\boldsymbol{Q}[t] \longrightarrow S < x_1, \cdots, x_r > \otimes \boldsymbol{Q}[t]$. Since F is simply connected, the rational homotopical property of ΩF is the same as that of a product of Eilenberg Maclane complexes. Hence the minimal model of $(\Omega F)_G=(\Omega F) \times BS^1$ is given by $\mathcal{M}((\Omega F)_G)=S < y_1, y_2 \cdots >$ $\otimes \boldsymbol{Q}[t]$ with trivial differential, where each y_s corresponds to an element of a base of $\pi_{m_s}(\Omega F) \otimes \boldsymbol{Q}=\pi_{m_s+1}(F) \otimes \boldsymbol{Q}$ and $deg\ y_s=m_s (s=1, 2, \cdots)$. The inclusion $(\Omega F)_G \longrightarrow (\Omega X)_G$ induces a differential graded algebra map $i: \mathcal{M}((\Omega X)_G) \longrightarrow \mathcal{M}((\Omega F)_G)$, and the inclusion $(x_0)_G \longrightarrow (\Omega F)^G$ (x_0 is regarded as the constant loop) induces a differential graded algebra map $\varepsilon: \mathcal{M}((\Omega F)_G \longrightarrow \mathcal{M}((x_0)_G)=\boldsymbol{Q}[t]$. Replacing x_j by $(x_j-\varepsilon \circ i(x_j))$ if $\varepsilon \circ i(x_j) \neq 0$, we may assume that $\varepsilon \circ i(x_j)=0$ for each $j=1, \cdots, r$. Now put $\overline{\mathcal{M}}_X=\ker\ \varepsilon \circ i$, $\overline{\mathcal{M}}_F=\ker\ \varepsilon$ and put $\overline{\mathcal{M}}_X \overline{\mathcal{M}}_X=\{ab \in \overline{\mathcal{M}}_X \mid a, b \in \overline{\mathcal{M}}_K\}$ and $\overline{\mathcal{M}}_F \overline{\mathcal{M}}_F=\{cd \in \overline{\mathcal{M}}_F \mid c, d \in \overline{\mathcal{M}}_F\}$. Define $Q_X=\overline{\mathcal{M}}_X / \overline{\mathcal{M}}_X \overline{\mathcal{M}}_X$ and $Q_F=\overline{\mathcal{M}}_F / \overline{\mathcal{M}}_F \overline{\mathcal{M}}_F$. Then Q_X and Q_F are $\boldsymbol{Q}[t]$ module, and i induces a $\boldsymbol{Q}[t]$ homomorphsim $i_Q: Q_X \longrightarrow Q_F$.

Lemma 1. *For each element $y \in Q_F$, there is an element $x \in Q_X$ such that $t^u y=i_Q(x)$ for some integer $u \geqq 0$.*

Proof. We may constuct $\boldsymbol{Z}$-graded differential algebras $\mathcal{M}((\Omega X)_G)$ $[t^{-1}]$, $\mathcal{M}((\Omega F)_G)[t^{-1}]$, $\overline{\mathcal{M}}_X[t^{-1}]$ and $\overline{\mathcal{M}}_F[t^{-1}]$, and $\boldsymbol{Q}[t, t^{-1}]$ modules $Q_X[t^{-1}]$ and $Q_F[t^{-1}]$. i induces $\boldsymbol{Q}[t, t^{-1}]$-homomorphism $\mathcal{M}((\Omega X_G))$ $[t^{-1}] \longrightarrow \mathcal{M}((\Omega F)_G)[t^{-1}]$, and $Q_X[t^{-1}] \longrightarrow Q_F[t^{-1}]$. Now it suffices to prove that the latter map is onto. Since the localization functor is an exact functor, the homology of $\mathcal{M}((\Omega X)_G)[t^{-1}]$ with respect to its differential is $H_G^*(\Omega X)[t^{-1}]$. By Theorem 1, the map induced by the inclusion $H_G^*(\Omega X)[t^{-1}] \longrightarrow H_G^*(\Omega F)[t^{-1}]=\mathcal{M}((\Omega F)_G)[t^{-1}]$ is an isomorphism. Hence the above map $\mathcal{M}((\Omega X_G)[t^{-1}] \longrightarrow \mathcal{M}((\Omega F)_G)[t^{-1}]$ is onto. Clearly $i(\overline{\mathcal{M}}_X \overline{\mathcal{M}}) \subset \overline{\mathcal{M}}_F \overline{\mathcal{M}}_F$, so that the above map $Q_X[t^{-1}] \longrightarrow$ $Q_F[t^{-1}]$ is onto. q. e. d.

Now let $\bar{x}_j$ be the class of x_j in $Q_X (j=1, \cdots, r)$ and let $\bar{y}_s$ be the class of y_s in Q_F $(s=1,2, \cdots)$. Then the set $\{\bar{x}_1, \cdots, \bar{x}_r\}$ is linearly independent over $\boldsymbol{Q}[t]$ and generates Q_X, and the set $\{\bar{y}_1, \bar{y}_2, \cdots\}$ is linearly independent over $\boldsymbol{Q}[t]$ and generates Q_F. Therefore from

Lemma 1, it follows that the number of the set $\{y_1, y_2, \cdots\}$ is finite, say k, and $k \leqq r$. This proves that F has a finite rational homotopy type.

By Lemma 1, we can choose a subset $\{z_1, \cdots, z_k\}$ of $\{x_1, \cdots, x_r\}$ such that $\{i_Q(z_1), \cdots, i_Q(z_k)\}$ is linearly independent over $\boldsymbol{Q}[t]$. Put $i_Q(z_j) = a_{j1}\bar{y}_1 + a_{j2}\bar{y}_2 + \cdots + a_{jk}\bar{y}_k$, where $a_{j1}, \cdots, a_{jk} \in \boldsymbol{Q}[t]\,(j = 1, \cdots, k)$. We construct a fundtion ψ from the set $\{\bar{y}_1, \cdots, \bar{y}_k\}$ to $\{z_1, \cdots, z_k\}$ as follows. Consider the coefficient matrix of the above equations,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{21} & \cdots & a_{kk} \end{pmatrix}$$

As det $A \neq 0$, there is such an index (j, k) that both a_{jk} and its cofactor $\tilde{a}_{jk}$ are not zero. Choose such an index arbitrarily and put $\psi(\bar{y}_k) = z_j$. Let A' be the matrix obtained from A by deleting the k-th column and the j-th row. Then det A' is not zero. Choose such an index $(j', k\text{-}1)$ $(j' \neq j)$ that both $a_{j'(k-1)}$ and its cofactor in A' are not zero, and define $\psi(\bar{y}_{k-1}) = z_{j'}$, and so on. Repeating this process, we obtain a function $\psi\,\{y_1, \cdots, y_k\} \to \{z_1, \cdots, z_k\}$ which is one to one. If $\psi(y_s) = z_j$, the $i_Q(z_j)$ contain a term of the form $a_{js}y_s$, $a_{js} \neq 0$. Hence $deg\, z_j = deg\ y_s + 2n$ for some $n \geqq 0$ (z_j and y_s are both homogeneous). Now each $\bar{y}_s$ corresponds to an element of a base of $\pi_{m_s}(\Omega F) \otimes \boldsymbol{Q} = \pi_{m_s+1}(F) \otimes \boldsymbol{Q}$, and each z_j corresponds to an element of a base of $\pi_{m_s+2n}(\Omega X) \otimes \boldsymbol{Q} = \pi_{m_s+1+2n}(X) \otimes \boldsymbol{Q}$. This proves the inequality in Theorem 2.

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