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# ON ABSTRACT MEAN ERGODIC THEOREMS. II

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## 1. Introduction

This is a continuation of [7]. In [7] we proved abstract mean ergodic theorems for weakly right ergodic semigroups $\mathfrak{S}$ of continuous linear operators on a *complete* locally convex topological vector space $E$. Among other things it was proved that the fixed points of $\mathfrak{S}$ separate the fixed points of the adjoint semigroup $\mathfrak{S}^* = \{T^* : T \in \mathfrak{S}\}$ if and only if $E$ is the direct sum of the fixed points of $\mathfrak{S}$ and the closed linear subspace of $E$ determined by the set $\{x - Tx : x \in E \text{ and } T \in \mathfrak{S}\}$. This generalizes Sine's mean ergodic theorem for a single Banach space contraction operator ([5], [6], [9]). In the present paper we shall first derive a criterion, called the finite dimension criterion, for the validity of a mean ergodic theorem for a weakly right ergodic semigroup which has as a special case Sine's and Atalla's criterion for a single Banach space contraction operator ([2], [8], [11]). We shall next study ergodic properties of right ergodic semigroups of Markov operators on $C(X)$, $C(X)$ being the Banach space of all (real or complex) continuous functions on a compact Hausdorff space $X$ with the supremum norm. Sine's results in [10] will be generalized.

## 2. Definitions and the finite dimension criterion

Throughout this section, $E$ will be a *complete* locally convex topological vector space (t. v. s.) and $\mathfrak{S}$ a semigroup of continuous linear operators on $E$. For an $x \in E$ we denote by $A(x)$ the affine subspace of $E$ determined by the set $\{Tx : T \in \mathfrak{S}\}$, i. e.

$$A(x) = \{y : y = \sum_{i=1}^{k} a_i T_i x,\ \sum_{i=1}^{k} a_i = 1,\ T_i \in \mathfrak{S},\ 1 \leq k < \infty\},$$

and by $\bar{A}(x)$ the closure of $A(x)$ in $E$. A net $(T_n, n \in \Delta)$ of linear operators on $E$ is said to be *(weakly) right* [resp. *(weakly) left*] *$\mathfrak{S}$-ergodic* if it satisfies:

(I) For every $x \in E$ and all $n \in \Delta$, $T_n x \in \bar{A}(x)$.

(II) The transformations $T_n$ are equicontinuous.

(III) For every $x \in E$ and all $T \in \mathfrak{S}$,

(weak-) $\lim_n T_n Tx - T_n x = 0$ [resp. (weak-) $\lim_n TT_n x - T_n x = 0$].

The semigroup $\mathfrak{S}$ is said to be *(weakly) right* [resp. *(weakly) left*] *ergodic* if it possesses at least one (weakly) right [resp. (weakly) left] $\mathfrak{S}$-ergodic net $(T_n, n \in \Delta)$. Whenever $(T_n, n \in \Delta)$ is a both (weakly) right and left $\mathfrak{S}$-ergodic net, we call it simply (weakly) $\mathfrak{S}$*-ergodic.* And if $\mathfrak{S}$ possesses at least one (weakly) $\mathfrak{S}$-ergodic net, $\mathfrak{S}$ is said to be (*weakly*) *ergodic.* (See [4] and [7].)

The adjoint semigroup of $\mathfrak{S}$ is the semigroup $\mathfrak{S}^* = \{T^* : T \in \mathfrak{S}\}$, where $T^*$ is the adjoint operator of $T$ defined by $\langle x, T^*x^* \rangle = \langle Tx, x^* \rangle$ for all $x \in E$ and all $x^* \in E^*$, $E^*$ being the topological dual of $E$. We let

$$F = \{ x \in E: Tx = x \text{ for all } T \in \mathfrak{S}\}$$

and

$$F^* = \{x^* \in E^* : T^*x^* = x^* \text{ for all } T^* \in \mathfrak{S}^*\}.$$

**Lemma.** *Let $\mathfrak{S}$ be a weakly right ergodic semigroup. If* $\dim F < \infty$, *then* $\dim F^* \geq \dim F$.

*Proof.* Let $f$ be any linear functional on $F$. Then $f$ is continuous on $F$, as $F$ is finite dimensional. Therefore by the Hahn-Banach theorem there exists an $f^* \in E^*$ such that

$$f^* = f \quad \text{on} \quad F.$$

Write

$$U = \{x \in E: |\langle x, f^* \rangle| < 1\}.$$

Now if $(T_n, n \in \Delta)$ is a weakly right $\mathfrak{S}$-ergodic net, then by the equicontinuity of the operators $T_n$ we can choose a neighborhood $W$ of the origin of $E$ such that $W \subset U$ and also such that

$$T_n W \subset U \quad \text{for all } n \in \Delta.$$

Let

$$A^* = \{x^* \in E^* : x^* = f \text{ on } F \text{ and } |\langle x, x^* \rangle| \leq 1 \text{ for all } x \in U\}$$

and

$$B^* = \{x^* \in E^* : x^* = f \text{ on } F \text{ and } |\langle x, x^* \rangle| \leq 1 \text{ for all } x \in W\}.$$

It is easily seen that $f^* \in A^* \subset B^*$ and that

$$x^* \in A^* \text{ implies } T_n^* x^* \in B^* \text{ for all } n \in \Delta.$$

The Banach-Alaoglu theorem shows that $B^*$ is weak*-compact, thus there exists a subnet $(T_{n'}^* f^*, n' \in \Delta')$ of the net $(T_n^* f^*, n \in \Delta)$ which converges in the weak*-topology to a point $g^*$ in $B^*$. Hence for every $T^* \in \mathfrak{S}^*$ and all $x \in E$

$$\langle x, T^* g^* \rangle = \lim_{n'} \langle x, T^* T_{n'}^* f^* \rangle = \lim_{n'} \langle T_{n'} T x, f^* \rangle$$

$$=\lim_{n'}\langle T_{n'}x,\ f^*\rangle=\lim_{n'}\langle x,\ T_{n'}{}^*f^*\rangle$$
$$=\langle x,\ g^*\rangle.$$

It follows that $g^*\in F^*$, and since $g^*=f$ on $F$, we immediately conclude that $\dim F^*\geq\dim F$. The proof is complete.

**Remark 1.** The above-given argument can easily be modified to show that if $\mathfrak{S}$ is a weakly right ergodic semigroup, then $\dim F^*<\infty$ implies $\dim F\leq\dim F^*$. Any continuous linear functional on $F$ can be extended to a continuous linear functional on $E$ belonging to $F^*$.

**Theorem 1.** *Let $\mathfrak{S}$ be a weakly right ergodic semigroup of continuous linear operators on a complete locally convex t. v. s. $E$. If either* $\dim F<\infty$ *or* $\dim F^*<\infty$ *then the following conditions are equivalent*:

(a) $\dim F=\dim F^*$.

(b) *$E$ is the direct sum of $F$ and $N$, where $N$ is the closed linear subspace of $E$ determined by the set $\{x-Tx:\ x\in E$ and $T\in\mathfrak{S}\}$.*

*Proof.* By the previous lemma and remark, we see that (a) is equivalent to the following: For any nonzero $x^*\in F^*$ there exists an $x\in F$ satisfying $\langle x, x^*\rangle\neq0$, i. e. $F$ separates $F^*$. And this condition is equivalent to (b), as is stated in Introduction. The proof is complete.

**Remark 2.** In the above theorem, the hypothesis that $F$ is finite dimensional is not omitted. In fact there are many spaces $E$ such that $\dim E<\dim E^*$. If we let $S=\{I\}$, where $I$ denotes the identity operator on such a space $E$, then clearly (b) holds but (a) does not.

### 3. Frgodic properties of Markov operator semigroups

Let $X$ be a compact Hausdorff space and $C(X)$ the Banach space of all (real or complex) continuous functions on $X$ with the supremum norm. A linear operator $T$ on $C(X)$ is said to be a *Markov operator* if $T1=1$ and if $f\geq0$ implies $Tf\geq0$. Let $\mathfrak{S}$ be a fixed semigroup of Markov operators on $C(X)$, and put

$$C_i(X)=\{f\in C(X):\ Tf=f\ \text{for all}\ T\in\mathfrak{S}\}.$$

It is well-known ([3], p. 265) that the topological dual space $C^*(X)$ of $C(X)$ is identified with the space of all regular finite (countably additive) measures on the $\sigma$-field $\Sigma$ of Borel subsets of $X$. Denote by $\mathscr{P}(X)$ the regular probability measures on $\Sigma$, and put

$$\mathscr{P}_i(X)=\{\mu\in\mathscr{P}(X):\ T^*\mu=\mu \text{ for all } T^*\in\mathfrak{S}^*\}.$$

We define, as in Sine [10], the center $M$ of $\mathfrak{S}$ by

$$M=\text{closure}\bigcup\{\text{supp}\ \mu:\ \mu\in\mathscr{P}_i(X)\}.$$

A closed subset $K$ of $X$ is said to be *$\mathfrak{S}$-invariant* if supp $T^*e_x\subset K$ for every $x\in K$, where $e_x$ denotes the unit mass concentrated at $x$. It is easily seen from Sine [8] that $M$ is $\mathfrak{S}$-invariant.

**Proposition.** *Let $\mathfrak{S}$ be a weakly right ergodic semigroup of Markov operators on $C(X)$. Then any $g\in C(X)$ with $g=0$ on $M$ is in the closed linear subspace $N$ of $C(X)$ determined by the set $\{f-Tf:\ f\in C(X)$ and $T\in\mathfrak{S}\}$.*

*Proof.* Let $(T_n,\ n\in\Delta)$ be a weakly right $\mathfrak{S}$-ergodic net. If $\mu\in\mathscr{P}(X)$, then as in the proof of the lemma there exists a subnet $(T^*_{n'}\mu,\ n'\in\Delta')$ of the net $(T^*_n\mu,\ n\in\Delta)$ and an element $\tilde{\mu}\in C^*(X)$ such that

$$\text{weak*-}\lim_{n'} T^*_{n'}\mu=\tilde{\mu}.$$

It follows that $T^*\tilde{\mu}=\tilde{\mu}$ for all $T^*\in\mathfrak{S}^*$. Since $\|T^*\|=1$ for all $T^*\in\mathfrak{S}^*$, it follows that $\tilde{\mu}$ is a finite linear combination of elements of $\mathscr{P}_i(X)$. Hence supp $\tilde{\mu}\subset M$, and so we have

$$\lim_{n'}\langle T_{n'}g,\ \mu\rangle=\lim_{n'}\langle g,\ T^*_{n'}\mu\rangle=\langle g,\tilde{\mu}\rangle=0.$$

By this and an easy induction argument, the zero function 0 is a weak cluster element of the net $(T_n g,\ n\in\Delta)$, and thus we have $0\in\overline{A}(g)$. Therefore given an $\varepsilon>0$ there exists an $h=\sum_{i=1}^k a_iT_ig$ with $\|h\|<\varepsilon$, where $\sum_{i=1}^k a_i=1$ and $T_i\in\mathfrak{S}$ for each $i$. Consequently

$$g=h+\sum_{i=1}^k a_i(g-T_ig),$$

and this proves the proposition.

In Theorem 2 below we study ergodic properties of $\mathfrak{S}$ restricted to the center $M$. A semigroup $\mathfrak{S}$ is said to be *continuously scattered* if there exists a family of functions in $C(X)$ so that each function in the family is constant on the support of each extreme measure of $\mathscr{P}_i(X)$ and the family separates the extreme measures of $\mathscr{P}_i(X)$.

**Theorem 2.** *Let $\mathfrak{S}$ be a semigroup of Markov operators on $C(X)$ and*

*$(T_n, n \in \Delta)$ a right $\mathfrak{S}$-ergodic net of linear operators on $C(X)$. Then the following conditions are equivalent:*

(a) *$\mathfrak{S}$ is continuously scattered.*

(b) *For any $f \in C(X)$ the net $(T_n f,\ n \in \Delta)$ converges uniformly on the center $M$, and further $\lim_n TT_n f - T_n f = 0$ uniformly on $M$ for all $T \in \mathfrak{S}$.*

*Proof.* We proceed partly as in Sine [10]. Since $M$ is $\mathfrak{S}$-invariant, we may and will assume without loss of generality that $M$ equals the whole space $X$.

(a) $\Longrightarrow$ (b). Let $\mathscr{A}$ be the family of all $f \in C(X)$ that are constant on the support of each extreme measure of $\mathscr{P}_i(X)$. Then we see that $\mathscr{A}$ is a norm closed algebra and that if $f \in \mathscr{A}$ and if $\mu$ is an extreme measure of $\mathscr{P}_i(X)$ then $Tf = f$ on supp $\mu$ for all $T \in \mathfrak{S}$, because supp $\mu$ is $\mathfrak{S}$-invarinat. By the Krein-Milman theorem, the union

$$\cup \{\text{supp } \mu : \mu \text{ is an extreme measure of } \mathscr{P}_i(X)\}$$

is dense in $X(=M)$, and thus the continuity of $f$ implies that $Tf = f$ on $X$ for all $T \in \mathfrak{S}$. Let $Y$ be the quotient topological space $X/\mathscr{A}$. The quotient map $q$ is defined by

$$q(x) = \{z \in X : f(z) = f(x) \text{ for all } f \in \mathscr{A}\} \quad (\in Y)$$

for all $x \in X$. $Y$ is then a compact Hausdorff space and $q$ is continuous. The Stone-Weierstrass theorem implies that $\mathscr{A}$ can be identified with the Banach space $C(Y)$, and from this it may be readily seen that for any $y \in Y$ the set $q^{-1}(y)$ is $\mathfrak{S}$-invariant. Since by assumption $\mathfrak{S}$ is continuously scattered, there exists a unique measure $\mu$ in $\mathscr{P}_i(X)$ such that supp $\mu \subset q^{-1}(y)$. It follows from Corollary 1 of [7] and the results of the preceding section that $(T_n f,\ n \in \Delta)$ converges uniformly on $q^{-1}(y)$ to a constant function for each $f \in C(X)$.

Let

$$F(x) = \lim_n T_n f(x) \quad (x \in X).$$

To prove the uniform convergence of $(T_n f,\ n \in \Delta)$ to $F$, let $x \in q^{-1}(y)$. Since $(T_n f,\ n \in \Delta)$ converges uniformly on $q^{-1}(y)$, given an $\varepsilon > 0$ there exists an $N \in \Delta$ such that

$$q^{-1}(y) \subset \{z \in X : |T_N f(z) - F(x)| < \varepsilon\}.$$

The latter set is open, and hence there exists an open set $U$ in $Y$ so that

$$q^{-1}(y) \subset q^{-1}(U) \subset \{z \in X : |T_N f(z) - F(x)| < \varepsilon\}.$$

Since $T_N f \in \overline{A}(f)$, we can choose $\sum_{i=1}^{k} a_i T_i f \in A(f)$ so that

$$\| T_N f - \sum_{i=1}^{k} a_i T_i f \| < \varepsilon .$$

It then follows that

$$| \sum_{i=1}^{k} a_i T_i f - F(x) | < 2\varepsilon \text{ on } q^{-1}(U)$$

and that

$$T_n f = T_n( \sum_{i=1}^{k} a_i (f - T_i f)) + T_n( \sum_{i=1}^{k} a_i T_i f - F(x)) + T_n F(x).$$

Since for every $z \in q^{-1}(U)$ and all $n \in \Delta$, supp $T_n^* e_z \subset q^{-1}(U)$, we have that

$$| T_n f - F(x) | = | T_n f - T_n F(x) |$$
$$\leq \sum_{i=1}^{k} | a_i | \| T_n f - T_n T_i f \| + 2\Lambda\varepsilon \text{ on } q^{-1}(U)$$

where $\Lambda = \sup_n \| T_n \|$, and that

$$\lim_n \| T_n f - T_n T_i f \| = 0 \ (i = 1, \cdots, k).$$

Hence we see that $| F(z) - F(x) | \leq 2\Lambda\varepsilon$ for all $z \in q^{-1}(U)$, and furthermore that there exists an $N(x) \in \Delta$ such that if $n \geq N(x)$ then

$$| T_n f - F | < 5\Lambda\varepsilon \text{ on } q^{-1}(U).$$

Since $X$ is compact, the uniform convergence of $(T_n f,\ n \in \Delta)$ to $F$ on $X$ follows. Since $F \in \mathscr{A} \subset C_i(X)$, we also have

$$\lim_n \| T T_n f - T_n f \| = \| TF - F \| = 0$$

for all $T \in \mathfrak{S}$.

(b) $\Longrightarrow$ (a). If (b) holds then by Corollary 1 of [7] $C_i(X)$ separates the extreme measures of $\mathscr{P}_i(X)$. On the other hand, every $f \in C_i(X)$ is constant on the support of each extreme measure of $\mathscr{P}_i(X)$ ([8]). Therefore $\mathfrak{S}$ is continuously scattered.

The following theorem may be regarded as a generalization of Theorem 3.2 of Atalla [1].

**Theorem 3.** *Let $\mathfrak{S}$ be a continuously scattered semigroup of Markov operators on $C(X)$ and $(T_n,\ n \in \Delta)$ a right $\mathfrak{S}$-ergodic net of linear operators on $C(X)$. Then the following conditions are equivalent:*

(a) *For any $f \in C(X)$ the net $(T_n f,\ n \in \Delta)$ converges uniformly on $X$.*

(b) *There exists a continuous linear operator $S$ on $C(X)$ such that for every $f \in C(X)$ $\lim_n \| S T_n f - T_n f \| = 0$ and such that, for each $f \in C(X)$ with $f = 0$ on $M$, $Sf = 0$ on $X$.*

*Proof.* (a)$\Longrightarrow$(b). Let $Sf=\lim_n T_n f$ for all $f\in C(X)$. Since $(T_n, n\in \Delta)$ is right $\mathfrak{S}$-ergodic, if $f\in C(X)$ satisfies $f=0$ on $M$ then by the Proposition $Sf=0$ on $X$. Furthermore for every $T\in\mathfrak{S}$ and all $f\in C(X)$,

$$STf-Sf=\lim_n T_n Tf-T_n f=0.$$

Hence, immediately, $ST_n=S$ for all $n\in\Delta$, and so (b) follows.

(b)$\Longrightarrow$(a). Since $\mathfrak{S}$ is continuously scattered by hypothesis, Theorem 2 shows that the net $(T_n f,\ n\in\Delta)$ converges uniformly on the center $M$ for every $f\in C(X)$. Choose an $F\in C(X)$ so that

$$F(x)=\lim_n T_n f(x) \quad (x\in M).$$

(b) implies that supp $S^* e_x\subset M$ for all $x\in X$, and hence we have

$$\lim_n \|SF-ST_n f\| \leq \|S\| \lim_n (\sup \{|F(z)-T_n f(z)| : z\in M\})=0.$$

Therefore, by (b) again, we have

$$\lim_n \|SF-T_n f\| \leq \lim_n \|SF-ST_n f\| + \lim_n \|ST_n f-T_n f\| =0,$$

completing the proof.

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