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COSEMISIMPLE COALGEBRAS AND COSEPARABLE COALGEBRAS OVER COALGEBRAS

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Introduction. Semisimple algebras and separable algebras over commutative rings have been treated from the view-point of module theory (cf. [1]). In this paper we shall deal with cosemisimple coalgebras and coseparable coalgebras over coalgebras from the comodule-theoretical view-point. In our study the notion of cotensor product introduced by J. Milnor and J. C. Moore [3] is an essential tool. We shall use only elementary properties of the cotensor products but not require "co-hom" introduced in [5].

In § 1, by making use of the cotensor product and a coalgebra map ϕ: $C \longrightarrow D$, we shall define a coalgebra C over D. A coalgebra C over D is said to be *left cosemisimple* if any short exact sequence of left C-comodules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

splits as C-comodule whenever it splits as D-comodule. Some basic properties of cosemisimple coalgebras over coalgebras will be given.

In § 2, we define coseparable coalgebras over coalgebra and give some properties of coseparable coalgebras. It seems that the coseparbility can not be defined by the existence of the "coseparability coordinate system".

As for the notaions and terminologies used here, we follow Sweedler [4].

0. **Preliminaries.** Let k be a fixed ground field. All vector spaces and linear maps are k-vector spaces and k-linear maps. Unadorned $\otimes$ means $\otimes_k$, and **Mod** denotes the category of vector spaces.

A *coalgebra* is a triple (C, Δ, ε) where C is a vector space, $\Delta: C \longrightarrow C \otimes C$ and $\varepsilon: C \longrightarrow k$ are linear maps such that $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta: C \longrightarrow C \otimes C \otimes C$ and $(\varepsilon \otimes 1)\Delta = 1 = (1 \otimes \varepsilon)\Delta: C \longrightarrow k \otimes C = C = C \otimes k$. For coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$, a *coalgebra map* $\phi: C \longrightarrow D$ is a linear map such that $\Delta_D \phi = (\phi \otimes \phi)\Delta_C$ and $\varepsilon_D \phi = \varepsilon_C$. Throughout the paper C, D, E and F are coalgebras.

A *right C-comodule* is a pair of a vector space X and a linear map $\rho: X \longrightarrow X \otimes C$ (*right C-comodule structure map*) such that $(1 \otimes \Delta)\rho = (\rho \otimes 1)\rho: X \longrightarrow X \otimes C \otimes C$ and $(1 \otimes \varepsilon)\rho = 1: X \longrightarrow X \otimes k = X$. A *C-colinear map* $f: X \longrightarrow Y$ of right C-comodules is a linear map such that $\rho_Y f = (f\otimes 1)\rho_X$, where ρ_Y and ρ_X are the right C-comodule structure maps of X and Y, respectively. $\mathbf{Com}_{\text{-}C}$ denotes the category of right C-comodules and C-colinear maps. By symmetry left C-comodules and C-colinear maps can be defined, and $\mathbf{Com}_{C\text{-}}$ denotes the category of left C-comodules. When the structure map α of X needs explicit mention, we write $\alpha = \alpha_X$.

If $W \in \mathbf{Mod}$ and $X \in \mathbf{Com}_{\text{-}C}$, then $1\otimes\rho$ is a right C-comodule structure map of $W\otimes X$. A *C-D-bicomodule* is a left C- and right D-comodule M such that the left C-comodule structure map $\rho^{C\text{-}}: M \longrightarrow C\otimes M$ is D-colinear, or equivalently, the right D-comodule structure map $\rho^{\text{-}D}: M \longrightarrow M\otimes D$ is C-colinear, that is, $(\rho^{C\text{-}}\otimes 1)\rho^{\text{-}D} = (1\otimes\rho^{\text{-}D})\rho^{C\text{-}}: M \longrightarrow C\otimes M\otimes D$.

In the following we write X_D, ${}_CY$ and ${}_CZ_D$ to denote that X is a right D-comodule, Y a left C-comodule, and Z a C-D-bicomodule.

For comodules X_D and ${}_DY$, the *cotensor product* $X\square_D Y$ is the kernel of the map

$$\rho_X^{\text{-}D}\otimes 1 - 1\otimes\rho_Y^{D\text{-}}: X\otimes Y \longrightarrow X\otimes D\otimes Y$$

The functors $X\square_D ?$ and $?\square_D Y$ are left exact and preserve direct sums [5, p. 632]. In particular, for $W\in\mathbf{Mod}$,

$$X\square_D(Y\otimes W) = (X\square_D Y)\otimes W$$
$$(W\otimes X)\square_D Y = W\otimes(X\square_D Y)$$

If ${}_CX_D$ and ${}_DY_E$ are bicomodules, the structure maps $\rho_X^{C\text{-}}: X\longrightarrow C\otimes X$ and $\rho_Y^{\text{-}E}: Y\longrightarrow Y\otimes E$ induce the structure maps $\rho_X^{C\text{-}}\square 1: X\square_D Y \longrightarrow (C\otimes X)\square_D Y = C\otimes(X\square_D Y)$ and $1\square\rho_Y^{\text{-}E}: X\square_D Y\longrightarrow X\square_D(Y\otimes E) = (X\square_D Y)\otimes E$ with which $X\square_D Y$ is a C-E-bicomodule [5, p. 632].

The cotensor product is associative: For comodules X_C, ${}_DZ$ and bicomodule ${}_CY_D$, we have

$$(X\square_C Y)\square_D Z = X\square_C(Y\square_D Z)$$

in $X\otimes Y\otimes Z$. This subspace is denoted by $X\square_C Y\square_D Z$. For comodules X_C and ${}_CY$, the structure maps ρ_X and ρ_Y induce C-colinear isomorphisms $X\cong X\square_C C$ and $Y\cong C\square_C Y$. In particular $X\otimes W\cong X\square_C(C\otimes W)$ and $W\otimes Y\cong(W\otimes C)\square_C Y$ for $W\in\mathbf{Mod}$ [5, p. 632].

1. Cosemisimple coalgebras over coalgebras. For a coalgebra D, let $\mathscr{D}$ denote the class of all pairs (C, ϕ) where C is a coalgebra and ϕ is a coalgebra map from C to D. For pairs (C, ϕ), (E, ψ), a *morphism* $f: (C, \phi)\longrightarrow(E, \psi)$ is a coalgebra map $f: C\longrightarrow E$ such that $\phi = \psi f$:

$C \longrightarrow D$. Then $\mathscr{D}$ is a category. A *coalgebra* over D is defined as an object (C, ϕ) in $\mathscr{D}$ together with map $\Delta = \Delta_C : C \longrightarrow C \square_D C$ such that

$$(1 \square \Delta)\Delta = (\Delta \square 1)\Delta : C \longrightarrow C \square_D C \square_D C$$

and

$$(1 \square \phi)\Delta = (\phi \square 1)\Delta = 1 : C \longrightarrow C \square_D D = D \square_D C$$

where C is a D-comodule via ϕ. Given coalgebras (C, ϕ) and (E, ψ) over D, we define a morphism $\theta : (C, \phi) \longrightarrow (E, \psi)$ which is a k-linear map $\theta : C \longrightarrow E$ such that the following diagrams are commutative

$$\begin{array}{ccc} C & \xrightarrow{\Delta_C} & C \square_D C \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta \square \theta} \\ E & \xrightarrow[\Delta_E]{} & E \square_D E \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\theta} & E \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & D & \end{array}$$

We denote the category of coalgebras over D by $\mathbf{Coalg}_D$. If (C, ϕ) is in $\mathbf{Coalg}_D$, then every C-comodule X is a D-comodule with the structure map $\rho_X^D = (\phi \otimes 1)\rho_X^C$.

Let $\phi : C \longrightarrow D$ be a coalgebra map. An exact sequence of (left) C-comodules is said to be *(C, D)-exact* if it splits as a sequence of D-comodules. A left C-comodule X is defined to be *(C, D)-injective* if for every diagram of left C-comodules and C-colinear maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & & & & & \\ & & X & & & & & & \end{array}$$

with (C, D)-exact row, there exists a C-colinear map $g : M \longrightarrow X$ such that $g\beta = f : L \longrightarrow X$. We can define dually *$(C, D)$-projective* comodules. The following proposition is easy by definition.

Proposition 1.1. *For $(C, \phi) \in \mathbf{Coalg}_D$, the following conditions are equivalent.*

(1) *Every left C-comodule is (C, D)-injective.*

(2) *Every (C, D)-exact sequence of C-comodules splits as C-comodule.*

(3) *Every subcomodule of a left C-comodule which is a D-direct*

summand is a C-direct summand as comodule.

(4) *Every left C-comodule is (C, D)-projective.*

Definition 1.2 (cf. [4, p. 290, Def.]). A coalgebra (C, ϕ) over D is said to be *left cosemisimple* if it satisfies the equivalent conditions in Prop. 1.1. Similarly we can define a right cosemisimple coalgebra over D.

The following two lemmas are useful in our study.

Lemma 1.3. *Let* $(C, \phi) \in \mathbf{Coalg}_D$, *and* $X \in \mathbf{Com}_{D^-}$. *Then* $C\square_D X$ *is* (C, D)*-injective, where* $C\square_D X$ *is a left C-comodule via* $\Delta\square 1:\ C\square_D X \longrightarrow (C\otimes C)\square_D X = C\otimes(C\square_D X)$.

Proof. Consider a diagram of left C-comodules

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \underset{\longleftarrow}{\overset{\alpha}{\longrightarrow}} & M \\ & & f\big\downarrow & \gamma & \\ & & C\square_D X & & \end{array}$$

where γ is a D-colinear map such that $\gamma\alpha = 1$. We have to show that there exists a C-colinear map $g:\ M \longrightarrow C\square_D X$ with $g\alpha = f$. Recalling that $(C, \phi) \in \mathbf{Coalg}_D$, we have the k-linear map $\phi\square 1:\ C\square_D X \longrightarrow D\square_D X$. Now we consider the following diagram

$$\begin{array}{ccc} 0 \longrightarrow L & \overset{\alpha}{\longrightarrow} & M \\ \rho_L\big\downarrow & & \big\downarrow\rho_M \\ C\square_D L & \underset{1\square\alpha}{\longrightarrow} & C\square_D M \\ 1\square f\big\downarrow & & \\ C\square_D C\square_D X & & \big\downarrow 1\square(\phi\square 1)f\gamma \\ 1\square\phi\square 1\big\downarrow & & \\ C\square_D D\square_D X & = & C\square_D X \end{array}$$

Since α is a C-colinear map, the upper square is commutative and the lower square is also commutative by $\gamma\alpha = 1$. Therefore we have

$$\begin{aligned}(1\square(\phi\square 1)f\gamma)\rho_M\alpha &= (1\square(\phi\square 1)f\gamma)(1\square\alpha)\rho_L \\ &= (1\square\phi\square 1)(1\square f)\rho_L\end{aligned}$$

$= (1 \square \phi \square 1)(\Delta \square 1) f$ (since f is a C-colinear)
$= ((1 \square \phi) \Delta \square 1) f$
$= f$ (since $(C, \phi) \in \mathbf{Coalg}_D$).

Moreover an easy computation shows that $(1 \square (\phi \square 1) f \gamma) \rho_M$ is a C-colinear map. Thus $C \square_D X$ is (C, D)-injective.

Lemma 1.4. *Let* $(C, \phi) \in \mathbf{Coalg}_D$, *and* $X \in \mathbf{Com}_{C^-}$. *Then* X *is* (C, D)*-injective if and only if the comodule structure map* $\rho^C : X \longrightarrow C \square_D X$ *splits is* $\mathbf{Com}_{C^-}$.

Proof. By Lemma 1. 3, "if" part is clear. Suppose X is (C, D)-injective. Since the left D-comodule structure map of X is given by $(\phi \otimes 1) \rho^C = \rho^D$, we have the following commutative diagram of D-comodules

$$\begin{array}{ccc} X & \xrightarrow{\rho^C} & C \square_D X \\ \| & & \downarrow \phi \square 1 \\ X & \xrightarrow[\rho^D]{} & D \square_D X \end{array}$$

But ρ^D is an isomorphism as a D-comodule map. Since X is (C, D)-injective, ρ^C splits as a C-colinear map, that is, X is a direct summand of $C \square_D X$ as C-comodule.

Proposition 1.5. *Let* (C, ϕ), $(E, \psi) \in \mathbf{Coalg}_D$, *and let* $\theta : (C, \phi) \longrightarrow (E, \psi)$ *be in* $\mathbf{Coalg}_D$. *If* θ *is a monomorphism and if* (E, ψ) *is left cosemisimple, then* (C, ϕ) *is left cosemisimple.*

Proof. If X is a left C-comodule, then X is a left E-comodule via $(\theta \otimes 1) \rho^C$. Consider the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\rho^C} & C \square_D X \\ | & & \downarrow \theta \square 1 \\ X & \xrightarrow[\rho^E]{} & E \square_D X \end{array}$$

where $\rho^E = (\theta \otimes 1) \rho^C$. Since ρ^E is a monomorphism and X is (E, D)-injective, by Lemma 1. 4 there exists an E-colinear map $g : E \square_D X \longrightarrow X$ such that $g \rho^E = g(\theta \square 1) \rho^C = 1$. It remains therefore to show that the map $g(\theta\,\square\,1): C\,\square_D X \longrightarrow X$ is a C-colinear map. Since

$$\begin{aligned}(\theta\otimes 1)\rho^C g(\theta\,\square\,1) &= (1\otimes g)(\Delta_E\,\square\,1)(\theta\,\square\,1) \quad \text{(since } g \text{ is an } E\text{-colinear)}\\ &= (1\otimes g)((\theta\,\square\,\theta)\Delta_C\,\square\,1) \quad \text{(since } \theta\in \mathbf{Coalg}_D)\\ &= (\theta\otimes 1)(1\otimes g(\theta\,\square\,1))(\Delta_C\,\square\,1),\end{aligned}$$

and $\theta\otimes 1$ is a monomorphism, we have

$$\rho^C g(\theta\,\square\,1) = (1\otimes g(\theta\,\square\,1))(\Delta_C\,\square\,1).$$

This shows that $g(\theta\,\square\,1)$ is a C-colinear map.

Theorem 1.6. *If* $(C, \phi)\in \mathbf{Coalg}_D$ *and* $(D, \psi)\in \mathbf{Coalg}_E$ *are left cosemisimple, then* $(C, \psi\phi)$ *is left cosemisimple.*

Proof. Let X be a left C-comodule. Then X is a left D-comodule via $\rho^D = (\phi\otimes 1)\rho^C$. Since (D, ψ) is left cosemisimple and X is a left E-comodule via $\rho^E = (\psi\otimes 1)\rho^D$, the sequence of D-comodules

$$0 \longrightarrow X \xrightarrow{\rho^D} D\,\square_E X$$

splits by Lemma 1. 4, and so

$$0 \longrightarrow C\,\square_D X \xrightarrow{1\,\square\,\varepsilon_C} C\,\square_D(D\,\square_E X) \cong C\,\square_E X$$

splits as C-comodule. By the left cosemisimplicity of (C, ϕ), the left C-comodule X is a direct summand of $C\square_D X$, and so of $C\square_E X$. Hence X is (C, E)-injective.

Proposition 1.7. *Let* $\theta: (C, \phi)\longrightarrow (E, \psi)$ *be in* $\mathbf{Coalg}_D$. *If* (C, ϕ) *is left cosemisimple, then* (C, θ) *is left cosemisimple.*

Proof. If X is a left C-comodule, then the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\rho^C} & C\,\square_E X & \longrightarrow & C\otimes X & \xrightarrow{g} & C\otimes E\otimes X \\ \Big\| & & \Big\downarrow i & & \Big\downarrow 1 & & \Big\downarrow 1\otimes\psi\otimes 1 \\ X & \xrightarrow[\rho^C]{} & C\,\square_D X & \longrightarrow & C\otimes X & \xrightarrow[f]{} & C\otimes D\otimes X \end{array}$$

is commutative, where $f = (1\otimes\phi\otimes 1)((\Delta_C\otimes 1) - (1\otimes\rho^C))$, $g = (1\otimes\theta\otimes 1)((\Delta_C\otimes 1) - (1\otimes\rho^C))$, i is the restriction of 1 and the unlabeled arrows are the canonical injections. If (C, ϕ) is left cosemisimple, then by Lemma 1.4 $\rho^C : X \longrightarrow C\square_D X$ splits in $\mathbf{Com}_{C^-}$, and therefore the restriction $\rho^C : X \longrightarrow C\square_E X$ so does. Thus (C, θ) is left cosemisimple.

Definition 1.8. A coalgebra C is called an *augmented coalgebra* if there exists a coalgebra map $\eta : k \longrightarrow C$ such that $\varepsilon\eta = 1$.

Proposition 1.9. *Let* $\theta : (C, \phi) \longrightarrow (E, \psi)$ *be in* $\mathbf{Coalg}_D$. *If* (C, ϕ) *is left cosemisimple and* C *is an augmented coalgebra, then* (E, ψ) *is left cosemisimple.*

Proof. Let X be a left E-comodule, and consider a diagram of E-comodules

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & M \\ & & {\scriptstyle f}\downarrow & & \\ & & X & & \end{array}$$

where the row is (E, D)-exact. Since the functor $C\square_D ?$ is left exact, we obtain the following diagram of left C-comodules with (C, D)-exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & C\square_D L & \xrightarrow{1\square\alpha} & C\square_D M \\ & & {\scriptstyle 1\square f}\downarrow & & \\ & & C\square_D X & & \end{array}$$

Since (C, ϕ) is left cosemisimple, $C\square_D X$ is (C, D)-injective by Lemma 1.3, and so there exists a C-colinear map $g : C\square_D M \longrightarrow C\square_D X$ with $g(1\square\alpha) = 1\square f$. Since C is an augmented coalgebra, every $W \in \mathbf{Mod}$ is trivially a C-comodule. Now, consider the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & M \\ & & {\scriptstyle \rho_L}\downarrow & & \downarrow{\scriptstyle \rho_M} \\ & & C\square_D L & \xrightarrow{1\square\alpha} & C\square_D M \\ & & {\scriptstyle 1\square f}\downarrow & \swarrow{\scriptstyle g} & \\ & & C\square_D X & & \\ & & {\scriptstyle \varepsilon\square 1}\downarrow & & \\ & & X & & \end{array}$$

where ρ_L, ρ_M are the trivial C-comodule structure maps of L, M, respectively. Then it is easy to see that $(\varepsilon \square 1) g \rho_M$ is an E-colinear map, and for any $x \in L$ we have

$$((\varepsilon \square 1) g \rho_M) \alpha(x) = (\varepsilon \square 1)(1 \square f) \rho_L(x) = f(x).$$

Therefore (E, ψ) is left cosemisimple.

Remark 1.10. Following M. Takeuchi [4], a comodule ${}_CX$ is said to be *finitely cogenerated* if it is isomorphic to a subcomodule of $C \otimes W$ for some finite dimensional vector space W. A coalgebra (C, ϕ) over D is called *weakly left cosemisimple,* if every finitely cogenerated left C-comodule is (C, D)-injective. In Props. 1. 5, 1. 7 and 1. 9, we can replace "left cosemisimple" by "weakly left cosemisimple", and Th. 1. 6 is also true for weakly left cosemisimple coalgebras provided ${}_CD$ and ${}_DE$ are finitely cogenerated.

Definition 1.11. Let C be a coalgebra. In [5], a subspace J in C is called a *left coideal* if $\Delta(J) \subset C \otimes J$, or equivalently, J is a left C-subcomodule in C. A left coideal J is called *cofinite* if the quotient comodule C/J is finite dimensional over k.

Proposition 1.12. *Let* $(C, \phi) \in$ **Coalg**$_D$ *be weakly left cosemisimple. If a cofinite left coideal* J *is a direct summand of* C *as* D*-comodule, then* J *is a direct summand of* C *as* C*-comodule.*

Proof. By assumption, the sequence of C-comodules

$$0 \longrightarrow J \longrightarrow C \longrightarrow C/J \longrightarrow 0$$

is (C, D)-exact and C/J is finitely cogenerated. Since (C, ϕ) is weakly left cosemisimple, the above sequence splits in **Com**$_{C}$.

Theorem 1.13. *Let* $(C, \phi) \in$ **Coalg**$_D$ *be left cosemisimple. If* ${}_CX$ *is an injective* D*-comodule, then* ${}_CC \square_D X$ *is injective, and so* ${}_CX$ *is injective.*

Proof. Consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\beta} & M \\ & & {\scriptstyle h}\downarrow & & \\ & & C \square_D X & & \\ & & {\scriptstyle \phi \square 1}\downarrow & & \\ & & D \square_D X \cong X & & \end{array}$$

where β, h are C-colinear maps. Since ${}_DX$ is injective, there exists a D-colinear map $g: M \longrightarrow X$ such that $g\beta = (\phi \square 1)h$. Next we consider the following commutative diagram

$$\begin{array}{ccccc}
0 \longrightarrow & L & \xrightarrow{\beta} & M & \\
 & \rho_L \big\downarrow & & \big\downarrow \rho_M & \\
 & C \square_D L & \xrightarrow{1 \square \beta} & C \square_D M & \\
 & 1 \square h \big\downarrow & & \Big| & \\
 & C \square_D C \square_D X & & \Big| \; 1 \square g & \\
 & 1 \square \phi \square 1 \big\downarrow & & \big\downarrow & \\
 & C \square_D D \square_D X & \cong & C \square_D X &
\end{array}$$

Then

$$\begin{aligned}
(1 \square g)\rho_M\beta &= (1 \square \phi \square 1)(1 \square h)\rho_L \\
&= (1 \square \phi \square 1)(\Delta \square 1)h \quad \text{(since } \rho_L \text{ is } C\text{-colinear)} \\
&= h \quad \text{(since } (C, \phi) \in \mathbf{Coalg}_D\text{)},
\end{aligned}$$

and $(1 \square g)\rho_M$ is a C-colinear map. Therefore ${}_C C \square_D X$ is injective. Since X is a direct summand of $C \square_D X$ as C-comodule (Lemma 1. 4), ${}_CX$ is injective.

Theorem 1. 14. *For* $(C, \phi) \in \mathbf{Coalg}_D$, *the following conditions are equivalent.*

(1) *Every* (C, D)*-injective* C*-comodule is an injective* C*-comodule.*

(2) *Every exact sequence of left* C*-comodule is* (C, D)*-exact.*

Proof. (1) $\Longrightarrow$ (2). It suffices to show that any short exact sequence of C-comodules splits as D-comodule. Consider a diagram of C-comodules

$$\begin{array}{ccccccc}
0 \longrightarrow & L & \xrightarrow{\alpha} & M & \longrightarrow & N & \longrightarrow 0 \\
 & \rho \big\downarrow & & & & & \\
 & C \square_D L & & & & & \\
 & \phi \square 1 \big\downarrow & & & & & \\
 & D \square_D L \cong L & & & & &
\end{array}$$

where the row is exact. Since $C\square_D L$ is (C, D)-injective by Lemma 1. 3, ${}_C C\square_D L$ is injective. Therefore there exists a C-colinear map $h: M \longrightarrow C\square_D L$ such that $h\alpha = \rho$. Since $(\phi\square 1)\rho$ is a D-comodule structure map of L, we have $(\phi\square 1) h\alpha = (\phi\square 1)\rho = 1$.

(2) $\Longrightarrow$ (1). Let M be a (C, D)-injective comodule. Then there exists an exact sequence of C-comodules

$$0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0$$

such that ${}_C X$ is injective. By assumption, this sequence is (C, D)-exact. Since M is (C, D)-injective, M is a direct summand of X as C-comodule.

2. Coseparable coalgebras over coalgebras. Let (C, Δ, ε) be a coalgebra. Then $C^{\circ} = (C, t\Delta, \varepsilon)$ is a coalgebra, where $t(x\otimes y) = y\otimes x$ $(x, y\in C)$. As is easily seen, the category $\mathbf{Com}_{C}$ (resp. $\mathbf{Com}_{\text{-}C}$) is isomorphic to $\mathbf{Com}_{C^{\circ}}$ (resp. $\mathbf{Com}_{C^{\circ}\text{-}}$). Now let D be another coalgebra. If M is a (C, D)-bicomodule, then M is a left $C\otimes D^{\circ}$-comodule with the structure map

$$\rho^{C\otimes D^{\circ}\text{-}} = (1\otimes t)(1\otimes \rho^{\text{-}D})\rho^{C\text{-}}: M \longrightarrow C\otimes D^{\circ}\otimes M.$$

Conversely, if M is a left $C\otimes D^{\circ}$-comodule, then M has a left C-comodule structure

$$\rho^{C\text{-}} = (1\otimes\varepsilon\otimes 1)\rho^{C\otimes D^{\circ}\text{-}}: M \longrightarrow C\otimes M$$

and has a right D-comodule structure

$$\rho^{\text{-}D} = t(\varepsilon\otimes 1\otimes 1)\rho^{C\otimes D^{\circ}\text{-}}: M \longrightarrow M\otimes D.$$

By the above structure, M is a (C, D)-bicomodule.

Definition 2. 1 (cf. [2, pp. 262-263]). A coalgebra (C, ϕ) over D is said to be *coseparable* if the exact sequence

$$0 \longrightarrow C \xrightarrow{\Delta_C} C\square_D C$$

splits as (C, C)-bicomodule, or equivalently, the exact sequence splits in $\mathbf{Com}_{C\otimes D^{\circ}\text{-}}$.

Let $\phi: C \longrightarrow D$ be a coalgebra map. Let X be a right C-comodule, and Y a left C-comodule. Then X is a right D-comodule via $\rho_X^{\text{-}D} = (1\otimes\phi)\rho_X^{\text{-}C}$ and Y is a left D-comodule via $\rho^{D\text{-}} = (\phi\otimes 1)\rho^{C\text{-}}$. If $f: X \longrightarrow U$ is a right D-comodule map and $g: Y \longrightarrow V$ is a left D-comodule map, then we have the following diagram

$$\begin{array}{ccccc}
X\square_C Y & \xrightarrow{i} & X\otimes Y & \xrightarrow{\rho_X^{\cdot C}\otimes 1-1\otimes\rho_Y^{C\cdot}} & X\otimes C\otimes Y \\
h\downarrow & & \downarrow f\otimes g & & \downarrow f\otimes\phi\otimes g \\
U\square_D V & \xrightarrow[j]{} & U\otimes V & \xrightarrow[\rho_U^{\cdot D}\otimes 1-1\otimes\rho_V^{D\cdot}]{} & U\otimes D\otimes V
\end{array}$$

where i and j are the canonical injections. As is easily seen, the right square of the above diagram is commutative, and so there exists a k-linear map $h: X\square_C Y\longrightarrow U\square_D V$ such that $jh=(f\otimes g)i$. Therefore h is the restriction of $f\otimes g$ on $X\square_C Y$.

Proposition 2.2 (1) *Let $\theta: (C, \phi)\longrightarrow(E, \psi)$ be in* **Coalg**$_D$. *If $(C, \phi)\in$* **Coalg**$_D$ *is coseparable, then $(C, \theta)\in$* **Coalg**$_E$ *is coseparable.*

(2) *Let $\theta: C\longrightarrow E$ be a coalgebra map which is a monomorphism. If $(E, \psi)\in$* **Coalg**$_D$ *is coseparable, then $(C, \psi\theta)$ is coseparable.*

Proof. (1) Consider the following commutative diagram

$$\begin{array}{ccccccc}
C & \xrightarrow{\Delta_C} & C\square_E C & \longrightarrow & C\otimes C & \xrightarrow{\rho^{\cdot E}\otimes 1-1\otimes\rho^{E\cdot}} & C\otimes E\otimes C \\
\| & & \downarrow i & & \downarrow 1\otimes 1 & & \downarrow 1\otimes\psi\otimes 1 \\
C & \xrightarrow[\Delta_C]{} & C\square_D C & \longrightarrow & C\otimes C & \xrightarrow[\rho^{\cdot D}\otimes 1-1\otimes\rho^{D\cdot}]{} & C\otimes D\otimes C
\end{array}$$

where i is the restriction of $1\otimes 1$ and the unlabeled arrows are the canonical inclusions. Since $\Delta_C: C\longrightarrow C\square_D C$ splits as (C, C)-bicomodule, there exists a (C, C)-bicomodule map $\gamma: C\square_D C\longrightarrow C$ such that $\gamma\Delta_C=1$. Then $\Delta_C: C\longrightarrow C\square_E C$ splits by i. Thus (C, θ) is coseparable.

(2) Consider the following commutative diagram

$$\begin{array}{ccccccc}
C & \xrightarrow{\Delta_C} & C\square_D C & \longrightarrow & C\otimes C & \xrightarrow{\rho^{\cdot D}\otimes 1-\otimes\rho^{D\cdot}} & C\otimes D\otimes C \\
\theta\downarrow & & \downarrow j & & \downarrow\theta\otimes\theta & & \downarrow\theta\otimes 1\otimes\theta \\
E & \xrightarrow[\Delta_E]{} & E\square_D E & \longrightarrow & E\otimes E & \xrightarrow[\rho^{\cdot D}\otimes 1-1\otimes\rho^{D\cdot}]{} & E\otimes D\otimes E
\end{array}$$

where j is the restriction of $\theta\otimes\theta$ and the unlabeled arrows are the canonical inclusions. Then by the coseparability of (E, ψ), we have an (E, E)-bicomodule map $\pi: E\square_D E\longrightarrow E$ such that $\pi\Delta_E=1$, and so $\pi(\theta\otimes\theta)\Delta_C=\theta$. Moreover, since θ is an (E, E)-bicomodule monomorphism, π and θ are (C, C)-bicomodule maps. Therefore Δ_C splits as (C, C)-comodule map, that is, $(C, \psi\theta)$ is coseparable.

Theorem 2.3. *If* $(C, \phi) \in \mathbf{Coalg}_D$ *is coseparable, then* (C, ϕ) *is left and right cosemisimple.*

Proof. Let X be a left C-comodule. Then, by Lemma 1.3, $C\square_D X$ is a (C, D)-injective comodule. By the coseparability of (C, ϕ), the sequence

$$0 \longrightarrow C \xrightarrow{\Delta_C} C\square_D C$$

splits as (C, C)-bicomodule. Cotensoring each term of the sequence with X over C, we have then the following commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow & C\square_C X & \xrightarrow{\Delta_C\square 1} & (C\square_D C)\square_C X \\ & \| & & \| \\ & X & \xrightarrow[\rho_X^C]{\hspace{4em}} & C\square_D X \end{array}$$

whose row splits as left C-comodule. Therefore X is left (C, D)-injective, and dually, right (C, D)-injective.

Proposition 2.4. *If* $(C, \phi) \in \mathbf{Coalg}_D$ *and* $(E, \psi) \in \mathbf{Coalg}_F$ *are coseparable, then* $(C\otimes E, \phi\otimes\psi)$ *is coseparable.*

Proof. Let $f\colon C\otimes C\otimes E\otimes E \longrightarrow C\otimes E\otimes C\otimes E$ be the map defined by $f(x\otimes y\otimes z\otimes w) = x\otimes z\otimes y\otimes w$ ($x, z \in C$, $y, w \in E$). Then we have the following commutative diagram

$$\begin{array}{ccc} C\otimes E & \xrightarrow{\Delta_{C\otimes E}} & (C\otimes E)\square_{D\otimes F}(C\otimes E) \\ \| & & \downarrow \bar{f} \\ C\otimes E & \xrightarrow[\Delta_C\otimes\Delta_E]{} & (C\square_D C)\otimes(E\square_F E) \end{array}$$

where $\bar{f}$ is the isomorphism obtained by the restriction of f. Since $\pi_C\Delta_E = 1$ and $\pi_E\Delta_E = 1$, $\Delta_{C\otimes E}$ splits as $(C\otimes E, C\otimes E)$-bicomodule.

Corollary 2.5. *If* $(C, \phi) \in \mathbf{Coalg}_D$ *is cosemisimple and if* $(E, \psi) \in$

$\mathbf{Coalg}_k$ *is coseparable, then* $(C\otimes E,\ \phi\otimes\psi)$ *is cosemisimple.*

Proof. By Prop. 2. 4 and Th. 2. 3, $(C\otimes E,\ 1_C\otimes\psi)\in\mathbf{Coalg}_C$ is left cosemisimple. Hence $(C\otimes E,\ \phi\otimes\psi)$ is left cosemisimple by Th. 1. 6.

Lemma 2. 6. *Let* $L\in\mathbf{Com}_{\text{-}C}$, $M\in\mathbf{Com}_{D\text{-}}$, *and* $N\in\mathbf{Com}_{C\text{-}}$. *If* N *is a left D-comodule as well and* $(1\otimes\rho_N^{D\text{-}})\,\rho_N^{C\text{-}}=(t\otimes 1)(1\otimes\rho_N^{C\text{-}})\,\rho_N^{D\text{-}}$, *then* $(L\otimes M)\,\square_{C\otimes D}N=L\square_C(M\square N)$ *in* **Mod**, *where the left* $C\otimes D$*-comodule structure of* N, *the right* $C\otimes D$*-comodule structure of* $L\otimes M$, *and the left C-comodule structure of* $M\square_D N$ *are given by* $(1\otimes\rho_N^{D\text{-}})\rho_N^{C\text{-}}$, $(1\otimes t\otimes 1)(\rho_L^{\text{-}C}\otimes\rho_M^{D\text{-}})$, *and* $(t\otimes 1)(1\otimes\rho_N^{C\text{-}})$, *respectively.*

Proof. By definition of the contensor product, $L\square_C(M\square_D N)$ and $(L\otimes M)\square_{C\otimes D}N$ are k-subspaces of $L\otimes M\otimes N$. Then $l\otimes m\otimes n$ is in $L\square_C(M\square_D N)$ if and only if the following equalities hold:

(1) $\sum_{(l),(m)} l_{(L)}\otimes l_{(C)}\otimes m_{(M)}\otimes m_{(D)}\otimes n=\sum_{(n)} l\otimes n_{(C)}\otimes m\otimes n_{(D)}\otimes n_{(N)}$

(2) $\sum_{(l),(m)} l_{(L)}\otimes l_{(C)}\otimes m_{(M)}\otimes m_{(D)}\otimes n=\sum_{(m),(n)} l\otimes n_{(C)}\otimes m_{(M)}\otimes m_{(D)}\otimes n_{(N)}$

(3) $\sum_{(l),(n)} l_{(L)}\otimes l_{(C)}\otimes m\otimes n_{(D)}\otimes n_{(N)}=\sum_{(n)} l\otimes n_{(C)}\otimes m\otimes n_{(D)}\otimes n_{(N)}$

in $L\otimes C\otimes M\otimes D\otimes N$, where $\rho_L^{-C}(l)=\sum_{(l)} l_{(L)}\otimes l_{(C)}\in L\otimes C$, $\rho_M^{-D}(m)=\sum_{(m)} m_{(M)}\otimes m_{(D)}\in M\otimes D$, etc. On the other hand, $l\otimes m\otimes n$ is in $(L\otimes M)\square_{C\otimes D}N$ if and only if

(4) $\sum_{(l),(m)} l_{(L)}\otimes m_{(M)}\otimes l_{(C)}\otimes m_{(D)}\otimes n=\sum_{(n)} l\otimes m\otimes n_{(C)}\otimes n_{(D)}\otimes n_{(N)}$.

It is easy to see that this implies (1) (and conversely). Moreover, applying $(1\otimes 1\otimes\rho_M^{D\text{-}}\otimes\varepsilon_D\otimes 1)(1\otimes t\otimes 1\otimes 1)$ to both sides of (4), we have (2). Finally, applying $(1\otimes 1\otimes\varepsilon_D\otimes\rho_N^{D\text{-}})(1\otimes t\otimes 1\otimes 1)$ to both sides of (4), we obtain (3). Thus $(L\otimes M)\square_{C\otimes D}N\cong L\square_C(M\square_D N)$.

Theorem 2. 7. *If* $(C,\ \phi)\in\mathbf{Coalg}_D$ *and* $(D,\psi)\in\mathbf{Coalg}_E$ *are coseparable, then* $(C,\ \psi\phi)$ *is coseparable.*

Proof. By assumption, the sequence

$$0\longrightarrow D\xrightarrow{\Delta_D} D\square_E D$$

splits as left $D\otimes D^\circ$-comodule. Cotensoring each term of the sequence with $C\otimes C^\circ$ over $D\otimes D^\circ$, we get the split exact sequence of left $C\otimes C^\circ$-comodules

$$0\longrightarrow(C\otimes C^\circ)\,\square_{D\otimes D^\circ}D\xrightarrow{(1\otimes 1)\,\square\,\Delta_C}(C\otimes C^\circ)\,\square_{D\otimes D^\circ}(D\,\square_E D)$$

Now by Lemma 2. 6, we obtain the canonical isomorphisms

$$C\square_D C \overset{f_1}{\cong} C\square_D (D\square_D C) \overset{f_2}{\cong} C\square_D(C^{\circ}\square_{D^{\circ}}D) \overset{f_3}{\cong} (C\square C^{\circ})\square_{D\otimes D^{\circ}}D$$

For any $x\otimes y \in C\otimes C$, $f_2 f_1 (x\otimes y) = \sum_{(y)} x\otimes y_{(2)} \otimes \phi(y_{(1)})$ in $C\square_D(C^{\circ}\square_{D^{\circ}} D)$. By the proof of Lemma 2.6, $\sum_{(y)} x\otimes y_{(2)}\otimes\phi(y_{(1)})$ is in $C\square_D(C^{\circ}\square_{D^{\circ}}D)$ if and only if

(5) $\sum_{(x),(y)} x_{(1)}\otimes\phi(x_{(2)})\otimes y_{(2)}\otimes\phi(y_{(1)}) = \sum_{(y)} x\otimes y_{(1)}\otimes\phi(y_{(2)})\otimes\phi(y_{(3)})$

in $C\otimes D\otimes C^{\circ}\otimes D$. Again by Lemma 2.6, we have the canonical isomorphisms

$$C\square_E C \overset{g_1}{\cong} C\square_D (D\square_E D)\square_D C \overset{g_2}{\cong} C\square_D (C^{\circ}\square_{D^{\circ}}(D\square_E D))$$
$$\overset{g_3}{\cong} (C\square C^{\circ})\square_{D\otimes D^{\circ}}(D\square_E D)$$

and so

(6) $g_3 g_2 g_1 (x\otimes y) = \sum_{(x),(y)} x_{(1)}\otimes y_{(2)}\otimes\phi(x_{(2)})\otimes\phi(y_{(1)})$

in $C\otimes C^{\circ}\otimes D\otimes D$. Therefore by (5) and (6), we obtain the following commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow & C\square_D C & \xrightarrow{\quad i \quad} & C\square_D C \\ & \downarrow f_1 f_2 f_3 & & \downarrow g_1 g_2 g_3 \\ & (C\otimes C^{\circ})\square_{D\otimes D^{\circ}} D & \xrightarrow[(1\otimes 1)\square\Delta_D]{} & (C\otimes C^{\circ})_{D\otimes D^{\circ}}(D\square_E D) \end{array}$$

where i is the canonical inclusion. Since $(1\otimes 1)\square\Delta_D$ splits in $\mathbf{Com}_{C\otimes C^{\circ}-}$, so does i in $\mathbf{Com}_{C\otimes C^{\circ}}$. This enables us to see that the sequence

$$0 \longrightarrow C \xrightarrow{\Delta_C} C\square_E C$$

splits in $\mathbf{Com}_{C\otimes C^{\circ}}$, completing the proof.

Lemma 2.8. *Let $\phi: C \longrightarrow D$ be a coalgebra map, and E a coalgebra. Let M be a left $C\otimes E$-comodule. If M is $(C\otimes E, D\otimes E)$-injective, then M is (C, D)-injective, where the left C-comodule structure of M is given by $\rho^C = (1\otimes\varepsilon_E\otimes 1)\ \rho^{C\otimes E}$.*

Proof. By assumption, the sequence $0\longrightarrow M\longrightarrow(C\otimes E)\square_{D\otimes E} M$ splits in $\mathbf{Com}_{C\otimes E^-}$, and by Lemma 2.6 $(C\otimes E)\square_{D\otimes E} M\cong C\square_D M$ in $\mathbf{Com}_{C^-}$. Hence $0\longrightarrow M\longrightarrow C\square_D M$ splits in $\mathbf{Com}_{C^-}$.

Theorem 2. 9. *Let* $(C, \phi) \in \mathbf{Coalg}_D$, *and* E *an augmented coalgebra. If* $(C \otimes E,\ \phi \otimes 1_E) \in \mathbf{Coalg}_{D\otimes E}$ *is coseparable, then* (C, ϕ) *is coseparable.*

Proof. Since E is an augmented coalgebra, there exists a coalgebra map $\eta : k \longrightarrow E$ such that $\varepsilon\eta = 1$. Consider the following commutative diagram

$$\begin{array}{ccc}
C & \xrightarrow{\quad \Delta_C \quad} & C \square_D C \\
{\scriptstyle 1\otimes\eta}\downarrow\ \uparrow{\scriptstyle 1\otimes\varepsilon} & & \downarrow{\scriptstyle 1\square 1\square\eta} \\
C \otimes E & \xrightarrow[\Delta_C \otimes 1]{} & (C\square_D C) \otimes E \\
{\scriptstyle \Delta_{C\otimes E}}\downarrow & & \downarrow{\scriptstyle \beta} \\
(C\otimes E)\square_{D\otimes E}(C\otimes E) & \xleftarrow[\alpha]{} & C\square_D(C\otimes E)
\end{array}$$

where β is the canonical map, and $\alpha(x \otimes y \otimes e) = \sum_{(e)} x \otimes e_{(1)} \otimes y \otimes e_{(2)}$. By assumption, there exists a $(C \otimes E,\ C \otimes E)$-bicomodule map $\pi : (C \otimes E) \square_{D\otimes E} (C \otimes E) \longrightarrow C \otimes E$ such that $\pi\Delta_{C\otimes E} = 1$. Then we have

$$(1 \otimes \varepsilon)\, \pi\alpha\beta\, (1 \square 1 \square \eta)\, \Delta_C = (1 \otimes \varepsilon)\, \pi\alpha\beta\, (\Delta_C \otimes 1)\, (1 \otimes \eta) = 1$$

and $(1 \otimes \varepsilon)\ \pi\alpha\beta\ (1 \square 1 \square \eta)$ is a (C, C)-bicomodule map. Hence (C, ϕ) is coseparable.

We shall conclude our study with the following

Proposition 2. 10. *Let* $(C, \phi) \in \mathbf{Coalg}_D$. *If* C *is an augmented coalgebra, then the following conditions are equivalent.*

(1) $(C, \phi) \in \mathbf{Coalg}_D$ *is coseparable.*

(2) $(C \otimes C^{\circ},\ \phi \otimes \phi) \in \mathbf{Coalg}_{D\otimes D^{\circ}}$ *is coseparable.*

(3) $(C \otimes C^{\circ},\ \phi \otimes \phi) \in \mathbf{Coalg}_{D\otimes D^{\circ}}$ *is left and right cosemisimple.*

(4) $(C \otimes C^{\circ},\ \phi \otimes \phi) \in \mathbf{Coalg}_{D\otimes D^{\circ}}$ *is weakly left and right cosemisimple.*

(5) $(C \otimes C^{\circ},\ \phi \otimes \phi) \in \mathbf{Coalg}_{D\otimes C^{\circ}}$ *is left and right cosemisimple.*

(6) $(C \otimes C^{\circ},\ \phi \otimes \phi) \in \mathbf{Coalg}_{D\otimes C^{\circ}}$ *is weakly left and right cosemisimple.*

Proof. (1)$\Longrightarrow$(2). Clearly $(C^{\circ}, \phi) \in \mathbf{Coalg}_{D^{\circ}}$ is cosepararable. Then by Prop. 2. 4, (1) implies (2).

(2)$\Longrightarrow$(3). By Th. 2. 3.

(3)$\Longrightarrow$(4). Trivial.

(3) $\Longrightarrow$ (5), (4) $\Longrightarrow$ (6). By Prop. 1.7.

(6) $\Longrightarrow$ (1). Since C is an augmented coalgebra, ${}_{C\otimes C^{\circ}}C$ is finitely cogenerated and $(C\otimes C^{\circ}, D\otimes C^{\circ})$-injective by assumption. Therefore the sequence $0 \longrightarrow C \longrightarrow C\otimes C^{\circ}\,\Box_{D\otimes D^{\circ}}C$ splits in $\mathbf{Com}_{C\otimes C^{\circ}}$ and $(C\otimes C^{\circ})\,\Box\,D_{D\otimes D^{\circ}} \cong C\,\Box_D C$ by Lemma 2.6. Similarly we can prove (5) $\Longrightarrow$ (1).

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Added in proof. After submitting the paper, the author has found that the results of Lemmas 1.3 and 1.4, Th. 2.3, and Cor. 2.5 was obtained essentially by Y. Doi in his paper: Homological coalgebra, to appear.