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ON A THEOREM OF Y. MIYASHITA

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Throughout the present note, K will represent a ring with the identity 1, and ρ an automorphism of K. Let $R=K[X;\rho]$ be the skew polynomial ring whose multiplication is defined by $aX=X\rho(a)$ for $a\in K$. A monic polynomial f in R is called a separable (resp. Frobenius) polynomial, if $Rf=fR$ and R/Rf is separable (resp. Frobenius) over K.

In his recent paper [1], Y. Miyashita posed the following question : Is any separable polynomial Frobenius ? Some arguments concerning the question have been done in [1, § 3]. The purpose of this paper is to give some sufficient conditions for a separable polynomial to be Frobenius (Theorem 1), and show that a theorem of Miyashita [1, Theorem 3.5] obtained for a quadratic separable polynomial is still valid for any separable polynomial (Theorem 2).

Henceforth, we consider a monic polynomial $f=X^m-X^{m-1}a_1-\dots-Xa_{m-1}-a_m$ in R $(m>1)$ with $Rf=fR$. Since $af=f\rho^m(a)$ for all $a\in K$, we see that $aa_m=a_m\rho^m(a)$ and $aa_{m-1}=a_{m-1}\rho^{m-1}(a)$. In particular, $Ka_m=a_mK$ and $Ka_{m-1}=a_{m-1}K$. If a_m (or a_{m-1}) is one-sided invertible, then it is necessarily invertible. In what follows, these facts will be used freely. We set

$$Y_0=X^{m-1}-X^{m-2}a_1-\dots-Xa_{m-2}-a_{m-1},$$
$$Y_1=X^{m-2}-X^{m-3}a_1-\dots-Xa_{m-3}-a_{m-2},$$
$$\dots\dots\dots\dots$$
$$Y_{m-2}=X-a_1,$$
$$Y_{m-1}=1.$$

Then, there hold the following which have been proved in [1, Theorem 1.8 and Proposition 1.13] and will play essential role in our subsequent study.

(I) *If f is separable, then there exists a polynomial y in R of degree $<m$ such that $\sum_{j=0}^{m-1}Y_jyX^j\equiv 1$ (mod Rf) and $\rho^{m-1}(a)y=ya$ for all $a\in K$, and conversely.*

(II) *If f is Frobenius, then there exists a polynomial r in R of degree $<m$ such that $r+Rf$ is invertible in R/Rf and $\rho^{m-1}(a)r=ra$ (or $r\rho^{m-1}(a)=ar$) for all $a\in K$, and conversely.*

First, as a direct consequence of (I), we state the following key lemma.

Lemma 1. *Let f be separable, and let $y = X^{m-1}c_{m-1} + \dots + Xc_1 + c_0$ be as in* (I). *Then there exists some $d \in K$ such that $a_m d - a_{m-1}c_0 = 1$.*

Proof. By (I), there exists a polynomial g in R such that $\sum_{j=0}^{m-1} Y_j y X^j = 1 - fg$. Comparing the constant terms of the both sides, we readily obtain $-a_{m-1}c_0 = 1 - a_m d$ with some $d \in K$.

Now, we can prove the following

Theorem 1. (a) *If a_m is left or right invertible, then f is Frobenius.*

(b) *If a_{m-1} is left or right invertible, then f is Frobenius.*

(c) *If f is separable and if a_m is in the Jacobson radical* rad(K) *of K, then f is Frobenius.*

(d) *If f is separable and if a_{m-1} is in* rad(K), *then f is Frobenius.*

Proof. (a) Since $a_m \equiv X(X^{m-1} - X^{m-2}a_1 - \dots - a^{m-1})$ (mod Rf), $X+Rf$ is invertible in R/Rf. Then, $X^{m-1} + Rf$ is invertible, and hence f is Frobenius by (II).

(b) Since a_{m-1} is invertible and $aa_{m-1} = a_{m-1}\rho^{m-1}(a)$ for all $a \in K$, f is Frobenius again by (II).

(c) By Lemma 1, we have rad$(K) + a_{m-1}K = K$. Then $a_{m-1}K = K$, which means that a_{m-1} is invertible. Hence, f is Frobenius by (b).

(d) By Lemma 1, we have rad$(K) + a_m K = K$. Then $a_m K = K$, which means that a_m is invertible. Hence, f is Frobenius by (a).

Corllary 1. ([1, Theorem 3.4 (1)]). *If* rad(K) *is a maximal ideal of K, then every separable polynomial f is Frobenius.*

Proof. By Lemma 1, we have $a_m K + a_{m-1}K = K$. Since rad(K) is the unique maximal ideal of K, it follows then either $a_m K = K$ or $a_{m-1}K = K$. Now, the conclusion is immediate by Theorem 1 (a) and (b).

Corollary 2. *If the center C of K is a local ring, then every separable polynomial f is Frobenius.*

Proof. Let $y = X^{m-1}c_{m-1} + \dots + Xc_1 + c_0$ be as in (I). Since $\rho^{m-1}(a)y = ya$ for all $a \in K$, it is easy to see that $a_{m-1}c_0$ is in C. By Lemma 1, $a_m d - a_{m-1}c_0 = 1$ with some $d \in K$. Then, $a_m d$ is in C, as well. Since $a_m dC + a_{m-1}c_0 C = C$ and C is a local ring, it follows that either $a_m dC = C$ or $a^{m-1}c_0 C = C$, that is, either $a_m d$ or $a_{m-1}c_0$ is invertible in C. Hence, f is Frobenius by Theorem 1 (a) and (b).

In the rest of this note, we assume that K is the direct sum of (directly) indecomposable rings $K_i (i = 1, 2, \dots, r)$. Obviously, the center C of K is the direct sum of the centers C_i of K_i. Let e_i be the identity of K_i. Then ρ induces a permutation τ of $\{1, 2, \dots, r\}$ such that $\rho(e_i) = e_{\tau(i)}$. Let $\gamma_1, \gamma_2, \dots, \gamma_k$ be the orbits of τ, and set $A_j = \bigoplus_{i \in \gamma_j} K_j (j = 1, 2, \dots, k)$. Then, there holds $R = \bigoplus_{j=1}^{k} A_j[X; \rho_j]$, where ρ_j is the restriction of ρ onto A_j.

Under the above hypothesis and notations, there holds the following lemma.

Lemma 2. *If τ is a cycle of length $r > 1$, then every separable polynomial f is Frobenius.*

Proof. Without loss of generality, we may assume that $\tau = (1, 2, \dots, r)$. If r does not divide $m - 1$, then $\rho^{m-1}(e_1) \neq e_1$. Hence, $e_1 a_{m-1} = a_{m-1} \rho^{m-1}(e_1) = \rho^{m-1}(e_1) a_{m-1}$, whence it follows $e_1 a_{m-1} = 0$. Similarly, we can prove that $e_i a_{m-1} = 0$, and therefore $a_{m-1} = 0$. Hence, f is Frobenius by Theorem 1 (d). On the other hand, if r divides $m-1$, then $\rho^m(e_i) = \rho(e_i) = e_{i+1}$ with the convention $e_{r+1} = e_1$. Hence, $e_i a_m = a_m e_{i+1} = e_{i+1} a_m$, whence it follows $a_m = 0$. Then, f is Frobenius by Theorem 1 (c).

Now, let e_j^* be the identity of A_j. According to (I) (resp. (II)), one can easily see that h is a separable (resp. Frobenius) polynomial in R if and only if each he_j^* is a separable (resp. Frobenius) polynomial in $A_j[X; \rho_j]$. Hence, as a combination of Lemma 2 and Corollary 1, we readily obtain the following which includes [1, Theorem 3.5].

Theorem 2. *Assume that K is the direct sum of indecomposable rings K_i, and that each* rad(K_i) *is a maximal ideal of K_i. Then every separable polynomial in R is Frobenius.*

As is well known, every commutative artinian ring is a direct sum of local rings. This together with Lemma 2 and Corollary 2 yields the following

Theorem 3. *If the center of K is an artinian ring, then every separable polynomial in R is Frobenius.*

Corollary 3. *If K is a commutative artinian ring, then every separable polynomial in R is Frobenius.*

REFERENCE

[1] Y. MIYASHITA: On a skew polynomial ring, J. Math. Soc. Japan **31** (1979), 317—330.

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