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NUMERICAL INVESTIGATION OF SUMS OF THE MÖBIUS FUNCTION

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Let $\mu(n)$ denote the Möbius function, i. e. $\mu(1)=1$ and $\mu(n)=0$ if n is divisible by a square >1 and $\mu(n)=(-1)^k$ if n is the product of $k\geqq 1$ distinct prime numbers.

We put

$$M(N)=\sum_{n=1}^{n}\mu(n).$$

The function $M(N)$ is called the Mertens function and it is well known that certain theorems on the Mertens function $M(N)$ have a deep connection with the Prime Number Theorem. Untill now, many authors discussed on the Mertens function $M(N)$ and there are many conjectures about properties of this function [1 — 6].

For examples, it is conjectured that

(1) $\limsup_{N\to+\infty} M(N)=+\infty,$

(2) $\liminf_{N\to-\infty} M(N)=-\infty,$

(3) $|M(N)|\leqq\sqrt{N}.$

About (1) and (2), it is known that $M(N)=\Omega(\sqrt{N})$ holds true for $N\to\infty$ without any hypothesis where the symbol Ω denotes the negation of Landou's o-symbol (Cf. e. g. E. C. Titchmarsh [7] : p. 317, Theorem 14. 26(B)).

And as to (3), by means of Riemann's zeta function, S. Knapowski [2, 3, 4] investigated a problem of estimating $M(N)$ and the oscillatory properties of $M(N)$ in detail.

Taking these conjectures into account, we have numerically pursued the behavior of the function $M(N)$ on a computer, and we have observed some interesting phenomena.

The present note is a report of our experiments concerning the function $M(N)$ within the limit $N\leqq 4\cdot 10^8$.

All of the computation was done with a computer HITAC 20 in the Department of Mathematics, Okayama University. And the programs are written by the assembly language BASY except for the early test computation.

1. Preliminaries. Let Γ be the polygon obtained by joining successively points $(n, M(n))$ with segments in the (n, M)-plane. We consider the following quantities which characterize Γ in various sense.

$$Q(N) = \sum_{n=1}^{N} |\mu(n)|,$$
$$G(N) = \max_{1\leq n\leq N} M(n),$$
$$L(N) = \min_{1\leq n\leq N} M(n).$$

Two functions $G(N)$ and $L(N)$ reveal magnitudes of growth of values of the function $M(N)$.

$Z(N)$ = the number of touching of the polygon Γ to the n-axis in the interval $[1, N]$.

$C(N)$ = the number of changes of signs in the sequence $\{\mu(n);\ n = 1, 2, \ldots, N\}$ ignoring zeros.

Two functions $Z(N)$ and $C(N)$ measure the oscillatory character of the polygon Γ in a certain sense. Especially, $Z(N)$ does in the large, while $C(N)$ does locally.

Moreover, we consider a quantity like a variation,

$$V(N: H) = M(N) - M(N-H) \quad (H \geqq 1).$$

In actual computation, we have counted on the case where $M(n)$ is monotone (nondecreasing or nonincreasing) in the interval $[N-H, N]$. Thus, the function $V(N; H)$ reveals a height (or depth) of slope in the polygon Γ.

$F(N)$ = the maximal number of numbers K such that $\mu(N) = \mu(N-1) = \ldots = \mu(N-K+1) = 0$, whiel $F(N) = 0$, if $\mu(N) \neq 0$.

The function $F(N)$ expresses a length of a flat part (tableland or hollow) in the polygon Γ.

$P_1(N) = \pi(N)$ = the number of prime numbers p such that $p \leqq N$.

For $k \geqq 2$,

$P_k(N)$ = the number of composite numbers n which are product of k distinct prime numbers such that $n \leqq N$. Then, evidently, we have

$$M(N) = 1 - P_1(N) + P_2(N) - \ldots + (-1)^k P_k(N) + \ldots$$

In actual process, we have worked in the following manner.

Step 1° : Read starting values.

Step 2° : Compute the first 3512 prime numbers and store them into the memory. [1)]

Step 3° : Evaluate the value of $\mu(N)$ by means of factoring N.

1) 3512 is the number of prime numbers p less than 2^{15}.

Step 4° : Renew the values of $M(N)$, $G(N)$, $L(N)$, $Z(N)$, $C(N)$, $V(N; H)$, $F(N)$ and $P_k(N)$ $(k = 1, 2, \ldots, 10)$ according as the values of $\mu(N)$.

Step 5° : If some of these values make the new records in the previously determined rules, then print out the record and related data.

Step 6° : Replace $N+1$ to N and skip to the Step 3°.

2. **Observations.** In our experiments, we have obtained enormous data more than 300 sheets of computer output and it will be inconvenient to reproduce here all of them.

So, we extract some distinctive data from various view-points.

Table I gives the values of $M(N)$ and $|M(N)|/\sqrt{N}$ for some specified values of N. Here, in order to reveal the aspect of growth of $|M(N)|$, we select the point N_i such that $M(N)$ attains to the extremal value at N_i in the interval $[1, N_i]$, i. e. they correspond to hills and dales of Γ. From the Table I, we can see the state that the values of $\max_{1 \leqq n \leqq N} |M(N)|$ grow up gradually as N increases, nevertheless, Mertens' conjecture (3) is valid in the interval we examined.

The Table II is a table of values of $Q(N)$ and $Q(N)/N$. We can observe the fact that when N increases, in the first the ratio $Q(N)/N$ approaches rapidly to the neighborhood of the theoretical limit $6/\pi^2 = 0.6079271018\ldots$ up to the order 10^{-5}. But, thereafter, the ratio $Q(N)/N$ wanders about the theoretical limit. We may consider that this phenomenon implies an influence of the second term in the asymptotic expansion of $Q(N)$. (cf. [8 ; V. § 6]). But, as is seen in the Table III, the data we obtained are not so many to discuss the second term.

The Table IV is a table of the function $Z(N)$ and $C(N)$ which are concerned with the oscillatory character of Γ. Evidently, the ratio $C(N)/N$ shows a tendancy to converge to a certain limit in the neighborhood of 0.3 as N increases. On the other hand, it seems likely that the logarithmic ratio $\log Z(N)/\log N$ shows some indication to converge, but we cannot say yet that it certainly attains to a stationary state.

The Table V is a list of values of $V(N; H)$ which renovate the records at N. It is observed that the ratio $|V|/N$ gradually diminishes.

The Table VI is a list of the records of $F(N)$. We wonder if the values of $F(N)$ maynot be bounded.

The Table VII is a table of $P_k(N)$ $(k=1, 2, \ldots, 8)$. On the functions $P_k(N)$, it is well known that for $k \geqq 2$,

$$P_k(N) \sim \frac{N(\log\log N)^{k-1}}{(k-1)!\,\log N} \qquad \text{as } N \to \infty .$$

But it seems likely that the limit we examined is yet so small to apply these aproximations.

Table I. Growth of Absolute Values of $M(N)$.

N	$M(N)$	$\|M(N)\|/\sqrt{N}$	N	$M(N)$	$\|M(N)\|/\sqrt{N}$
1	1	1.000000	463139	244	0.358537
31	−4	0.718421	603151	−278	0.357958
95	2	0.205196	693255	264	0.317072
199	−8	0.567196	962265	−368	0.382367
221	5	0.336336	1066854	432	0.418245
443	−9	0.427603	1497305	−388	0.317086
586	7	0.289167	1793918	550	0.410640
1109	−15	0.450428	3239797	−683	0.379456
1418	13	0.345227	4549130	633	0.296687
2803	−25	0.472203	5343761	−847	0.366404
3295	23	0.400682	6841601	1060	0.416356
7021	−29	0.346098	7109100	−1078	0.404307
8511	35	0.379383	10194458	1240	0.388365
9861	−43	0.433020	12874814	−1447	0.403272
19291	51	0.367192	25734597	1419	0.279720
24185	−72	0.462977	30919091	−2573	0.462729
31990	73	0.408146	61913863	2845	0.361567
42961	−88	0.424566	76015339	−3448	0.395473
48433	96	0.436215	92418127	3290	0.342229
96014	−132	0.425997	110103729	−4610	0.439340
119545	132	0.381776	152353222	4279	0.346670
230399	−154	0.320834	179919749	−6226	0.414162
300551	240	0.437776	360718458	6695	0.352506
355733	−258	0.432571			

Table II. $Q(N)$ **and** $Q(N)/N$

N	$Q(N)$	$Q(N)/N$
10	7	0.70000000
$2\cdot 10$	13	0.65000000
$5\cdot 10$	31	0.62000000
10^2	61	0.61000000
$2\cdot 10^2$	122	0.61000000
$5\cdot 10^2$	306	0.61200000
10^3	608	0.60800000
$2\cdot 10^3$	1215	0.60750000
$5\cdot 10^3$	3042	0.60840000
10^4	6083	0.60830000
$2\cdot 10^4$	12160	0.60800000
$5\cdot 10^4$	30401	0.60802000
10^5	60794	0.60794000
$2\cdot 10^5$	121581	0.60790500
$5\cdot 10^5$	303958	0.60791600
10^6	607926	0.60792600
$2\cdot 10^6$	1215877	0.60793850
$5\cdot 10^6$	3039633	0.60792660
10^7	6079291	0.60792910
$2\cdot 10^7$	12158575	0.60792875
$5\cdot 10^7$	30396344	0.60792688
10^8	60792694	0.60792694
$2\cdot 10^8$	121585426	0.60792713
∞	—	0.60792710

Table III. Table of $R(N)= \left| Q(N)-\frac{6}{\pi^2}N \right| / \sqrt{N}$

N	$Q(N)-\frac{6}{\pi^2}R$	$R(N)$
10^5	1.2898	0.004079
10^6	−1.1018	0.001102
10^7	19.9815	0.006318
10^8	−16.1854	0.001619
$2\cdot 10^8$	5.6219	0.000398
$3\cdot 10^8$	−4.5562	0.000263
$4\cdot 10^8$	64.2584	0.003212

Table IV. $Z(N)$ **and** $C(N)$.

N	$C(N)$	$Z(N)$	$\log Z/\log N$
10	4	1	0.000000
10^2	33	5	0.009897
10^3	326	59	0.590284
10^4	3038	256	0.602060
10^5	30394	949	0.595453
10^6	303632	3269	0.585735
10^7	3037372	7646	0.554776
10^8	30397746	25525	0.550871

Table V. Local Variation of $M(N)$.

N	$M(N)$	$\|V(N;H)\|$	H	$\|V\|/H$
3	−1	2	2	1.000000
5	−2	3	4	0.750000
31	−4	3	3	1.000000
95	2	4	5	0.800000
105	−3	5	9	0.555556
146	1	5	6	0.833333
210	−1	7	10	0.700000
221	5	7	9	0.777777
610	−2	8	13	0.615385
1945	5	8	12	0.666667
2699	−6	11	17	0.647058
3107	8	13	18	0.722222
3842	−25	15	22	0.681818
41251	−12	16	26	0.571429
294602	134	19	29	0.655172
2156879	−451	20	31	0.645161
4819868	−140	21	33	0.636364
8812189	−811	22	36	0.611111
10309555	1052	25	42	0.595238
39479430	−507	26	45	0.577778

Table VI. Table of $F(N)$.

N	$M(N)$	$F(N)$
9	−2	2
50	−3	3
245	−2	4
848	−1	5
22025	−1	6
217076	−20	7
1092754	219	8
8870032	−466	9
221167432	5363	11

Table VII. $P_k(N)$ $(k = 1, 2, \dots, 8)$

N	$M(N)$	$P_1(N)$ $P_5(N)$	$P_2(N)$ $P_6(N)$	$P_3(N)$ $P_7(N)$	$P_4(N)$ $P_8(N)$
10	−1	4 0	2 0	0 0	0 0
10^2	1	25 0	30 0	5 0	0 0
10^3	2	168 0	288 0	135 0	16 0
10^4	−23	1229 24	2600 0	1800 0	429 0
10^5	−48	9592 910	23313 20	19919 0	7039 0
10^6	212	78498 18387	209867 1235	206964 8	92966 0
10^7	1037	664579 286758	1903878 32396	2086746 1044	1103888 1
10^8	1928	5761455 3884936	17426029 605939	20710806 38186	12364826 516

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