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ENDOMORPHISMS OF MODULES OVER MAXIMAL ORDERS

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1. Introduction. Throughout this paper R *will represent a right order in a right Artinian ring* Q (cf. [4]). Let M be a right R-module, $T=\mathrm{End}(M_R)$, and $M^*=\mathrm{Hom}(M_R, R_R)$. Then there exists a derived Morita context; $(\ ,\ ): M^*\otimes {}_TM \longrightarrow R$ and $[\ ,\ ]: M\otimes_R M^* \longrightarrow T$ such that $f[m, f']=(f, m)f'$ and $m(f, m')=[m, f]m'$ for all $m, m'\in M$, and $f, f'\in M^*$ (see [1] and [8]). The images (M^*, M) and $[M, M^*]$ are denoted by I and K, respectively. Let $\mathfrak{D}_I$ (resp. $\mathfrak{D}_K$) be the smallest Gabriel filter (additive topology [11]) of R (resp. T) which contains I (resp. K). Then, B. Müller [8] showed that there exists an equivalence between quotient categories determined by $\mathfrak{D}_I$ and $\mathfrak{D}_K$. One of the purposes of this paper is to apply the above result of Müller to endomorphism rings of modules over orders in Artinian rings.

Assume that R is a maximal two-sided order in a semi-simple ring Q and M is a finite dimensional torsionless faithful right R-module. Then, J. H. Cozzens has proved in [2, Theorem 2. 8] that (a) $\Longrightarrow$ (b) $\Longleftrightarrow$ (c), where

(a) M_R is reflexive,
(b) T is a maximal two-sided order, and
(c) $T=\mathrm{End}\,({}_RM^*)$.

He has claimed also that (b) $\Longrightarrow$ (a) does not hold in general, but does provided M_R is a generator.

In this paper we shall try to generalize the above result to a maximal one-sided order R in a quasi-Frobenius ring Q, where $M\otimes_R Q$ is Q-projective. In Theorem 2, by making use of a hereditary torsion theory induced by $\mathfrak{D}_I$ [11] and the notion of $\mathscr{C}_R$ (see below), we shall give necessary and sufficient conditions (weaker than reflexivity) for T to be a maximal right order.

A right R-module X is *torsion free in the sense of Levy* [5], if no non-zero element of X is annihilated by a regular element of R. Here there exists a right Q-module MQ ($\simeq M\otimes_R Q$) which contains M as an R-submodule. Let $\mathscr{C}_R$ be the class $\{X_R\ ;\ X$ is embedded in a right R-module W such that W is a direct product of copies of R_R and W/X is torsion free in the sense of Levy$\}$. We shall prove that if Q and S are Morita equivalent right Artinian rings and if R is a maximal right order in Q, then we can pick up a maximal right order T in S such that there exists a category equivalence between $\mathscr{C}_R$ and $\mathscr{C}_T$. Furthermore, when Q is quasi-Frobenius, a necessary and sufficient condition for T to be a maximal right order whose classical right quotient ring is Morita equivalent to Q will be given in Theorem 1.

2. In this paper every ring has an identity, and every homomorphism of modules will be written on the opposite side of scalars. A right R-module M will be said to *satisfy the condition* (A), if M is isomorphic to a direct summand of a right R-module U such that $U \subset \bigoplus_{i=1}^{n} R$ and $UQ = \bigoplus_{i=1}^{n} Q$. If Q is semi-simple, every finite dimensional torsionless right R-module satisfies (A) (see the proof of (ii)$\Longrightarrow$(i) of Theorem 1). In [6], it is proved that if M satisfies (A), then $\mathrm{End}(M_R)$ is a right order in $\mathrm{End}(MQ_Q)$. Furthermore, M^* is canonically embedded in $(MQ_Q)^* = \mathrm{Hom}(MQ_Q, Q_Q)$ as an R-T-bimodule.

Lemma 1. *Let* $T = \mathrm{End}(M_R)$, *and* $S = \mathrm{End}(MQ_Q)$, *where* M *satisfies* (A). *Then there hold the following* :

(1) $M^*S = (MQ_Q)^*$, *where* $M^*S = \{\sum f_i s_i\,;\ f_i \in M^*,\ s_i \in S\}$.

(2) *The trace ideal* I *of* M_R *contains a regular element if and only if* MQ_Q *is a generator.*

Proof. Set $T_0 = \mathrm{End}(U_R)$, and $S_0 = \mathrm{End}(UQ_Q)$, where U is as above. Let $u_1, u_2, \dots, u_n$ be the canonical free Q-basis of UQ. There exists a right R-module N such that $U = M \oplus N$ and hence $UQ = MQ \oplus NQ$. Therefore, $(MQ_Q)^*$ is canonically embedded in $(UQ_Q)^*$. Let $\phi \in (MQ_Q)^*$, and put $\phi(u_i) = q_i$. Then we can choose a regular element $d \in R$ such that $u_i d \in U$ and $q_i d \in R$, $i = 1, 2 \dots, n$. Define $t \in S_0$ by $t(\sum u_i p_i) = \sum u_i d p_i$, where $p_i \in Q$. Since $\sum u_i p_i \in U$ implies $p_i \in R$, t is contained in T_0. Cleary, t is a regular element of T_0. On the other hand, $\phi \cdot t \in U^* = \mathrm{Hom}(U_R, R_R)$, since $\phi \cdot t(\sum u_i p_i) = \sum q_i d p_i$. It follows that $T = \{s \in T_0\,; \phi \cdot s \in U^*\}$ is a right ideal of T_0 containing a regular element. Now, let $e \in T_0$ be the projection $M \oplus N \longrightarrow M$. Since T_0 is a right order in a right Artinian ring S_0 and e is an idempotent element, the proof of [9, Lemma 6] enables us to see that eLe contains a regular element of $eT_0e = T$. On the other hand, it is evident that $\phi \cdot eLe \subset M^*$ and then we have $M^*S \supset (MQ_Q)^*$, proving (1).

We consider here the Morita context $(\ ,\ ) : (MQ_Q)^* \otimes_R MQ \longrightarrow Q$ and $[\ ,\ ] : MQ \otimes_R (MQ_Q)^* \longrightarrow S$. Then, $IQ = (M^*, M)Q = (M^*, SMQ) =$

$(M^*S, MQ) = ((MQ_Q)^*, MQ)$. Hence I contains a regular element if and only if MQ_Q is a generator.

Lemma 2. *Let R be a maximal right order in Q. If $M \in \mathscr{C}_R$ satisfies (A) and I contains a regulat element, then $T = \mathrm{End}\,(M_R)$ is a maximal right order in $S = \mathrm{End}(MQ_Q)$.*

Proof. Suppose there exists a ring T_1 such that $T \subset T_1 \subset S$ and $t_1 T_1 t_2 \subset T$ for some units $t_1, t_2 \in S$. Then it follows that $t_1[M, M^*]T_1 [M, M^*]t_2 \subset T$. Hence, $(M^*t_1, M)(M^* T_1, M)(M^*t_2, M) = (M^*t_1 [M, M^*] T_1[M, M^*]t_2, M) \subset R$. Since t_i is invertible, there holds $t_i MQ = MQ$, so that $(M^*t_i, M)Q = (M^*S, MQ) = Q$ by Lemma 1. Thus, (M^*t_i, M) contains a regular element of Q. Put $R_0 = \{q \in Q\,;\ q(M^*T_1, M) \subset (M^*T_1, M)\}$. Then, $(M^*t_1, M)R_0 (M^*T_1, M)(M^*t_2, M) \subset R$. Now, R is a maximal right order contained in R_0, and hence $R = R_0$. It follows then $(M^*T_1, M) \subset R$. On the other hand, $M \in \mathscr{C}_R$ implies that there exists an R-monomorphism $f: M \longrightarrow \prod_{\lambda \in \Lambda} R^{(\lambda)}$, where $R^{(\lambda)}$ is a copy of R_R, such that $(\mathrm{Im}\, f)Q \cap \prod R^{(\lambda)} = \mathrm{Im}\, f$. Then f can be extended to a Q-monomorphism $\bar{f}: MQ \longrightarrow \prod_{\lambda \in \Lambda} Q^{(\lambda)}$, since M_R is torsion free in the sense of Levy. Let $s \in T_1$, and $m \in M$. Each projection $\bar{f}_\lambda: MQ \longrightarrow Q^{(\lambda)}$ can be regarded as an element of M^*. Then, $\bar{f}_\lambda(s(m)) \in (M^*T_1, M) \subset R$ and $\bar{f}(s(m)) \in f(M)$. Thus, $s(m) \in M$ and $T_1 \subset T$. Hence T is a maximal right order.

In what follows, let us denote by $L_I(\ \ )$ the *quotient functor* (*localization functor*) with respect to $\mathfrak{D}_I$.

Lemma 3. *Assume that R is a maximal right order in Q, M is a torsionless right R-module with trace ideal I containing a regular element, and that $\mathrm{End}\,(MQ_Q)$ is a classical right quotient ring of $\mathrm{End}\,(M_R)$ as a canonical extension. If $Y \in \mathscr{C}_R$, then $L_I(Y) = Y$ and $\mathrm{Hom}\,(M_R, Y_R) \in \mathscr{C}_T$.*

Proof. Let Y be a submodule of $\prod R^{(\lambda)}$ such that $YQ \cap \prod R^{(\lambda)} = Y$. Suppose $L_I(Y) \neq Y$, and let $E(Y)$ be the injective hull of Y. Then $E(Y)/Y$ is not $\mathfrak{D}_I$-torsion free (cf. [11]). Hence, there exists $x \in E(Y)$ such that $x \notin Y$ and $xI \subset Y$. Since I contains a regular element, we obtain $x \in YQ \subset \prod Q^{(\lambda)}$. Let x_λ be the projection of x into $Q^{(\lambda)}$. Since $x_\lambda I \subset R$, there holds $x_\lambda M^* \subset M^*$ and then $x_\lambda I \subset I$. Since R is a maximal right order, it follows $x_\lambda \in R$. This implies that $x \in YQ \cap \prod R^{(\lambda)} = Y$, a contradiction. This proves that $L_I(Y) = Y$.

Put $A=\mathrm{Hom}(MQ_Q, YQ_Q)$. Since $Mq\neq 0$ for every non-zero $q\in Q$, the right S-module A is embedded in $\prod_{(m,\lambda)\in M\times\Lambda} S^{(m,\lambda)}$ by the map $\partial: A\longrightarrow \prod S^{(m,\lambda)}$ such that $[\pi_{(m,\kappa)}\partial(a)](x)=m(\pi_\kappa a(x))$, where $x\in MQ$, $a\in A$, $(m,\kappa)\in M\times\Lambda$, and $\pi_{(m,\kappa)}$ and π_κ are projections $\prod S^{(m,\lambda)}\longrightarrow S^{(m,\kappa)}$ and $\prod Q^{(\lambda)}\longrightarrow Q^{(\kappa)}$, respectively. Put $B=\mathrm{Hom}(M_R, Y_R)$. Then B is a T-submodule of A and $\partial(B)\subset \prod T^{(m,\lambda)}$. Now, let $b_0\in A$ such that $\partial(b_0)\in\partial(B)S\cap\prod T^{(m,\lambda)}$. Since $\pi_{(m,\lambda)}\partial(b_0)\in T$ for every $(m,\lambda)\in M\times\Lambda$, it follows $m\pi_\lambda b_0(M)\subset M$. Therefore, $I\pi_\lambda b_0(M)\subset R$ and $\pi_\lambda b_0(M)\subset R$. Consequently, $b_0(M)\subset YQ\cap\prod R^{(\lambda)}=Y$ and $\partial(b_0)\in\partial(B)$. This implies $B\in\mathscr{C}_T$.

Proposition 1. *If Q and S are Morita equivalent right Artinian rings and R is a maximal right order in Q, then there exists a maximal right order T in S such that there exists a category equivalence between $\mathscr{C}_R$ and $\mathscr{C}_T$.*

Proof. There exists a right Q-module P and P' such that $P\oplus P'=\bigoplus_{i=1}^{n} Q$ and $S=\mathrm{End}(P_Q)$, where P_Q is a generator (cf. [1], [7]). Put $M=P\cap\bigoplus_{i=1}^{n} R$, and $M'=P'\cap\bigoplus_{i=1}^{n} R$. Evidently $M\in\mathscr{C}_R$. Since $(M\oplus M')Q=\bigoplus_{i=1}^{n} Q$, M_R satisfies (A) and I contains a regular element, Hence, $T=\mathrm{End}(M_R)$ is a maximal right order in $S=\mathrm{End}(MQ_Q)$ (Lemma 2), and M_R and M_T^* are faithful and torsionless. Then, M_R is canonically embedded in $\mathrm{Hom}(M_T^*, T_T)$, and $\mathrm{Hom}(M_T^*, T_T)I\subset M$. Since $L_I(M)=M$ by Lemma 3, we have $M=\mathrm{Hom}(M_T^*, T_T)$, whence it follows that $K=[M, M^*]$ is the trace ideal of M_T^*. From $KS=[M, M^*S]=[M, Q(M^*S)]=S$, we see that K contains a regular element of T. Obviously, $\mathrm{End}(M_T^*)$ $(\supset R)$ is a right order in $\mathrm{End}(M^*S_S)=\mathrm{End}((MQ_Q)^*S)=Q$. If we put $R_1=\{q\in Q; qM^*\subset M^*\}$, we can say $\mathrm{End}(M_T^*)=R_1$. Since $R_1I\subset R$, we have $R=\mathrm{End}(M_T^*)$. As was mentioned in the introduction there exists a category equivalence between quotient categories with respect to $\mathfrak{D}_I$ and $\mathfrak{D}_K$ via the functor $\mathrm{Hom}(M_R, \ )$: Mod-$R\longrightarrow$ Mdo-T and $\mathrm{Hom}(M_T^*, \ )$: Mod-$T\longrightarrow$ Mod-R. The consequence is immediate by Lemma 3, since $\mathscr{C}_R$ and $\mathscr{C}_T$ are full subcategories of these quotient categories, respectively.

Theorem 1. *If Q is a quasi-Frobenius ring, then the follwing conditions on a ring T are equivalent :*

(i) *T is a maximal right order whose classical right quotient ring is Morita equivalent to Q.*

(ii) *There exists a maximal right order* R *in* Q *and a finite dimensional faithful right R-module* M *in* $\mathscr{C}_R$ *such that* $M\otimes_R Q$ *is Q-projective and* $T=\mathrm{End}(M_R)$.

Proof. (i)$\Longrightarrow$(ii). There exists a ring S such that $Q=\mathrm{End}(V_S)$ and T is a maximal right order in S, where V_S is a direct summand of $\bigoplus_{i=1}^{n} S$ and is a generator. As in the proof of Proposition 1, putting $N_T=V\cap\bigoplus_{i=1}^{n} T$ and $R=\mathrm{End}(N_T)$, we have $T=\mathrm{End}\,(N_R^*)$, where $N^*=\mathrm{Hom}(N_T, T_T)$. Since T is in $\mathscr{C}_T$, the consequence is immediate by Lemma 3.

(ii)$\Longrightarrow$(i). M_R is a submodule of $\prod_{\lambda\in\Lambda} R^{(\lambda)}$ $(\subset \prod Q^{(\lambda)})$ such that $MQ\cap\prod R^{(\lambda)}=M$. Since M_R is faithful and Q is quasi-Frobenius, MQ_Q is a generator. Clearly, MQ_Q is a finite direct sum of injective indecomposable Q-modules, since MQ_Q is finite dimensional. Thus, MQ_Q is finitely generated and hence Artinian. Therefore, we can choose a finite subset F of Λ so that $\bigcap_{f\in F}\mathrm{Ker}\ \pi_f=0$, where π_f is the projection $MQ\longrightarrow Q^{(f)}$. Now, M_R is embedded in $\bigoplus_{\lambda\in F} R$ and MQ_Q is a direct summand of $\bigoplus_{\lambda\in F} Q$, it is easy to see that M_R satisfies (A) and T is a maximal right order in $S=\mathrm{End}(MQ_Q)$ by Lemma 2.

Every prime Goldie maximal right order need not be Morita equivalent to a right Ore domain (see [10]). However, from Theorem 1 we have the following

Corollary. *A ring* T *is a prime Goldie maximal right order, if and only if there exists a maximal right order* R *in a division ring and a finite dimensional right R-module* M *in* $\mathscr{C}_R$ *such that* $T\simeq\mathrm{End}(M_R)$.

Proposition 2. *If* M *is a reflexive right R-module, then* $M\in\mathscr{C}_R$. *The converse is true, provided* R *is a maximal two-sided order in a quasi-Frobenius ring* Q *and* M *is a finite dimensional faithful module such that* $M\otimes_R Q$ *is Q-projective.*

Proof. Since M is R-reflexive, we can say that there exists a left R-module N such that $M=\mathrm{Hom}({}_RN, {}_RR)$. Define $\theta: M_R\longrightarrow\prod_{n\in N} R^{(n)}$ by $\theta(v)=\{(n)v\}_{n\in N}\in\prod R^{(n)}$, $v\in M$. This is a monomorphism. Now, let $y=\{r_n\}_{n\in N}\in\prod R^{(n)}$ be such that $yd\in\mathrm{Im}\ \theta$ for some regular $d\in R$. Let $v_0\in M$ be such that $(n)v_0=r_nd$, $n\in N$, and define $v_0\cdot d^{-1}\in M$ by $(n)\,v_0\cdot d^{-1}=r_n$, $n\in N$. Then, $\theta(v_0\cdot d^{-1})=y$. Hence $\prod R^{(n)}/\mathrm{Im}\ \theta$ is R-torsion free in the sense of Levy.

Next, we shall prove the latter assertion. Assume that $M \in \mathscr{C}_R$. By [6] and Theorem 1, $T = \mathrm{End}(M_R)$ is a maximal two-sided order. Now, ${}_TM$ is embedded in a direct product of ${}_TT$ by the map $\alpha : {}_TM \longrightarrow \prod_{f \in M^*} T^{(f)}$ such that each projection of $(m)\alpha$ is $[m, f]$, $m \in M$. On the other hand, since M_R is torsionless, there is a canonical monomorphism $\sigma : {}_TM_R \longrightarrow {}_TC_R = \mathrm{Hom}({}_RM^*, {}_RR)$. Evidently, $KC \subset \mathrm{Im}\ \sigma$ and K contains a regular element (see the proof of Proposition 1). Then, since ${}_TM$ is torsionless and T is a left order in $\mathrm{End}(MQ_Q)$, ${}_TC$ is embedded in $SM = [MQ, (MQ_Q)^*]\,M = MQ((MQ_Q)^*, M) = MQ$. Now, M_R is a submodule of $\Pi R^{(\lambda)}$ such that $MQ \cap \Pi R^{(\lambda)} = M$. If $z \in C \subsetneq MQ$, then $M \supsetneq Kz = M(M^*, z)$, and hence $(M^*, z) \subset R$. It follows $z \in \Pi R^{(\lambda)}$, i. e., $z \in M$. M is therefore reflexive.

Remark. In virture of Theorem 1 and Proposition 2, Proposition 2.9 of [2] remains true even if R is semi-prime, provided M_R is faithful.

A right R-module M is said to be *$\mathfrak{D}_l$-quasi-injective* if every $f \in \mathrm{Hom}(J_R, M_R)$ can be extended to an element of $\mathrm{End}(M_R)$, provided $J_R \subset M_R$ and M/J is a $\mathfrak{D}_l$-torsion module. If $L_l(M) = M$, then we can see that M is $\mathfrak{D}_l$-quasi-injective. Now, we go back to the result of [2, Theorem 2.8] mentioned in the introduction. When M_R is reflexive in this case, $L_l(M) = M \in \mathscr{C}_R$ by Lemma 3 and Proposition 2. Therefore, the following includes this theorem.

Theorem 2. *Let R be a maximal right order in a quasi-Frobenius ring Q, and M a finite dimensinal torsionless faithful right R-module such that $M \otimes_R Q$ is Q-projective. Then the following conditions are equivalent, where $T = \mathrm{End}(M_R)$ and $S = \mathrm{End}(MQ_Q)$.*

(i) *T is a maximal right order.*

(ii) *M_R is $\mathfrak{D}_l$-quasi-injective and $L_l(M) \in \mathscr{C}_R$.*

(iii) $T = \{s \in S\,;\ M^*s \subset M^*\}$.

Proof. Put $T_0 = \{s \in S\,;\ M^*s \subset M^*\}$. Assume that T_1 is a subring of S such that $T_0 \subset T_1$ and $t_1 T_1 t_2 \subset T_0$ with some units $t_1, t_2 \in S$. Then, as in the proof of Lemma 2, we obtain $(M^* T_1, M) \subset R$, and hence $T_1 = T_0$ is a maximal right order in S. Furthermore, T_0 and T are equivalent orders and $T_0 \supset T$. Therefore, the equivalence of (i) and (iii) is evident.

(i)$\Longrightarrow$(ii). We can define a monomorphism $\theta : M_R \longrightarrow D_R = \mathrm{Hom}(M_T^*, T_T)$ by $\theta(m)(f) = [m, f]$, $m \in M$, $f \in M^*$. Then, $DI \subset M$, and so $D \subset L_l(M)$. Since $T \in \mathscr{C}_T$, we have $L_l(M) = L_l(D) = D$ by Lemma 3. Assume that $J_R \subset M_R$ and M/J is $\mathfrak{D}_l$-torsion. Then, every

$f \in \operatorname{Hom}(J_R, M_R)$ can be extended to an $\bar{f} \in \operatorname{Hom}(L_l(J)_R, L_l(M)_R) = \operatorname{End}(L_l(M_R))$. Since $L_l(M)$ is contained in the injective hull of M_R, i. e., $M \subset L_l(M) \subset MQ$, it follows that $T_1 = \operatorname{End}(L_l(M)_R)$ coincides with $\{s \in S;\ sL_l(M) \subset L_l(M)\}$. Since $R \in \mathscr{C}_R$, we can easily see that $M^* = \operatorname{Hom}(L_l(M)_R, R_R) = L_l(M)^*$. Then $[M, M^*]\,T_1 = [M, L_l(M)^* T_1] \subset T$. Since T is a maximal right order, we have $T_1 = T$. Thus, $\bar{f}(M) \subset M$, which proves (ii).

(ii) $\Longrightarrow$ (i). Let $h \in T_1 = \operatorname{End}(L_l(M)_R)$, and set $J = \{x \in M;\ h(x) \in M\}$. Since M/J is $\mathfrak{D}_l$-torsion, the restriction $h|J \in \operatorname{Hom}(J_R, M_R)$ is extended uniquely to an $h' \in T$. Hence, $T_1 \simeq T$ by $h \longrightarrow h'$. Since $L_l(M) \in \mathscr{C}_R$ and $L_l(M)Q$ coincides with the Q-projective module MQ, $T\ (\simeq T_1)$ is a maximal right order by Theorem 2. This completes the proof.

Under the hypothesis of Theorem 2, we notice that the proof of Proposition 1 enables us to see thar $R = \operatorname{End}(M_T^*)$. On the other hand, there exists a canonical ring monomorphism from T to $T_2 = \operatorname{End}({}_RM^*)$. Let $\alpha \in T_2$, and $m \in M$. Define $\overline{\alpha \cdot m} : {}_RM^* \longrightarrow {}_RR$ by $(f)\overline{\alpha \cdot m} = ((f)\alpha)m$, $f \in M^*$. If M_R is reflexive, regarding $\overline{\alpha \cdot m}$ as an element of M, we obtain $\alpha \cdot [m, M^*] = [\overline{\alpha \cdot m}, M^*]$. Thus, $T_2[M, M^*] \subset T$. Since $[M, M^*]$ contains a regular element of T, T_2 is contained in the injective hull S of T_T, and hence $T = T_2$. Conversely, assume that T is canonically isomorphic to $\operatorname{End}({}_RM^*)$. Let $\beta \in \operatorname{Hom}({}_RM^*, {}_RR)$, and $g \in M^*$. Define $\overline{\beta \cdot g} \in T_2$ by $(f)\overline{\beta \cdot g} = ((f)\beta)\,g$, $f \in M^*$. Regarding this as an element of T, we obtain $\beta \cdot (g, M) = \operatorname{Im} \overline{\beta \cdot g} \subset M$, i. e., $\operatorname{Hom}({}_RM^*, {}_RR) \cdot I \subset M$. We have therefore seen that *if R is a maximal right order in a quasi-Frobenius ring Q and N is a finite dimensional torsionless faithful right R-module such that $M \otimes_R Q$ is Q-projective, then the following conditions are equivalent* :

(i) *M_R is reflexive.*

(ii) *M_T^* satisfies the double cenralizer property and* $L_l(M) = M$.

(iii) *T is canonically isomorphic to* $\operatorname{End}({}_RM^*)$ *and* $L_l(M) = M$.

Example. There exists an R-module M satisfying the condition (ii) in Theorem 2 such that $L_l(M) \neq M$; Let $\boldsymbol{Z}$ and $\boldsymbol{Q}$ be integers and rationals, respectively. Obviously, $R = \boldsymbol{Z}[X]$ is a maximal right order in $\boldsymbol{Q}(X)$, the rational function field. Now, let $m \neq 1$ be a positive integer, and put $M = xR + mR$. Then, $M^* = R$, $I = M$, and $L_l(M) = R$.

REFERENCES

[1] H. BASS: The Morita theorem, Lecture Notes University of Oregon, 1962.
[2] J. H. COZZENS: Maximal orders and reflexive modules, Trans. Amer. Math. Soc. **219** (1976), 323—336.
[3] J. H. COZZENS and C. FAITH: Simple Noetherian Rings, Cambridge Tracts in Math. **69**, Cambridge Univ. Press, Cambridge, 1976.
[4] N. Jacobson: The Theory of Rings, Amer. Math. Soc. Math. Surveys **2**, Providence, 1943.
[5] L. LEVY: Torsion free and divisible modules over non-integral domains, Canad. J. Math. **15** (1963), 132—151.
[6] K. MASAIKE: Endomorphism rings of modules over orders in Artinian rings, Proc. Japan Acad. **46** (1970), 89—93.
[7] K. MORITA: Duality for modules and its applications to the theory of rings with minimum condition, Sci. Rep. Tokyo Kyoiku Daigaku, Sec. A, **6** (1958), 83—142.
[8] B. J. MÜLLER: The quotient category of Morita context, J. Algebra **28** (1974). 389—407.
[9] L. W. SMALL: Orders in Artinian rings II, J. Algebra **9** (1968), 266—273.
[10] T. STAFFORD: A simple Noetherian ring not Morita equivalent to a domain, Proc. Amer. Math. Soc. **68** (1978), 159—160.
[11] B. STENSTRÖM: Rings of Quotients, Grundl. Math. Wiss. **217**, Springer-Verlag, Berlin, 1975.

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