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ON THE ORBIT SPACES OF LINEAR FREE T^2-ACTIONS ON $S^3 \times S^5$

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In this note we show that the homotopy types of the orbit spaces of linear free T^2-actions on $S^3 \times S^5$ are determined by its integral cohomology rings. The main theorem is theorem 2.5.

1. We recall some general facts concerning the free torus actions. Let E be a simply conneted finite CW complex with finitely generated rational homotopy groups, i. e., $\dim \pi_i(E) \otimes Q < \infty$, where Q is the rational field. If there is a free action of a torus T^r of rank r on E, then $r \leqq -\chi_\pi(E)$ [3], where $\chi_\pi(E) = \sum_{i=1}^{\infty} (-1)^i \dim \pi_i(E) \otimes Q$ is the homotopy Euler number of E. Let $y_1, \ldots, y_m$ and $z_1, \ldots, z_n$ be homogeneous bases of $\sum_{i=1}^{\infty} \pi_{2i-1}(E) \otimes Q$ and $\sum_{i=1}^{\infty} \pi_{2i}(E) \otimes Q$ respectively, $g_i = \deg y_i$ and $k_i = \deg z_i$. Now suppose that E has a maximal free T^r-action (i. e. $r = -\chi_\pi(E)$). Then the orbit space E/T^r has a homotopy type of a simply connected finite CW complex with finitely generated rational homotopy groups and its homotopy Euler number is zero. Now, the following is a direct consequence of [3].

(1) *The Euler number of* $E/T^r = (g_1 + 1) \ldots (g_m + 1)/2^r k_1 \ldots k_n$, *where* $r = m - n$,

(2) $H^*(E/T^r; Q) \cong Q[x_1, \ldots, x_m]/I$, *where* $\deg x_i =$ *even and* I *is the Borel ideal of the polynomial ring* $Q[x_1, \ldots, x_m]$ *generated by* m *elements.*

The fact (2) impies that the cohomology ring of E/T^r is nice, i, e., there are only finitely many homotopy types among the simply connected finite CW complexes whose integral cohomology rings are isomorphic to given $H^*(E/T^r; Z)$ [1].

In the following, we assume $E = S^{2n+1} \times S^{2m+1}$, a product of two spheres of odd dimensions. Then, $\chi_\pi(E) = -2$. Let X be the orbit space of a free T^2-action on E. A routine computation of the cohomology spectral sequence of the bundle $E \longrightarrow X$ gives the following

Proposition 1.1. $H^*(X; Z) \cong Z[x, y] / (f(x, y), g(x, y))$, *where* $\deg x = \deg y = 2$ *and* $(f(x, y), g(x, y))$ *is the ideal generated by homo-*

geneous polynomials f and g of degree $n+1$ and $m+1$ respectively such that the resultant of f and g is ± 1.

More generally, let $A = Z[x, y] / (f(x, y), g(x, y))$ where $\deg x = \deg y = 2$,

$f(x, y) = a_0x^{n+1} + a_1x^ny + \ldots + a_{n+1}y^{n-1}$

$g(x, y) = b_0x^{m+1} + b_1 x^my + \ldots + b_{m+1}y^{m+1}$, a_i and $b_i \in Z$.

We write $A = A_0 + A_2 + \ldots$ as a graded ring, and assume $A_{2(n+m+1)} = 0$, which is equivalent to the fact that the resultant of f and g is ± 1. Then A is a free Z-module of rank $(n+1)(m+1)$. Let $\alpha = a_0 x^n$, $\beta = a_1x^n + \ldots + a_{n+1}y^n$, $\gamma = b_0x^m$ and $\delta = b_1 x^m + \ldots + b_{m+1} y^m$. Since f and g have no common divisor, we have the following

Lemma 1.2. *$\alpha\delta - \beta\gamma \in A_{2(n+m)} \cong Z$ is a generator and the products $A_{2i} \otimes A_{2(n+m-i)} \longrightarrow A_{2(n+m)}$, $i \geqq 0$, are duality pairings.*

Now we consider the converse of proposition 1.1. Let X be a simply connected finite CW complex such that its integral cohomology ring is the above ring A, ant led $E_{T^2} \longrightarrow B_{T^2}$ be the universal principal T^2 bundle. Let $h: X \longrightarrow B_{T^2}$ be a continuous map such that h^*; $H^2(B_{T^2}; Z) \longrightarrow H^2(X; Z)$ is an isomorphism, and let $E \longrightarrow X$ be the bundle induced by h. Clearly E is a simply connected finite CW complex. From the cohomology spectral sequence of the bundle $E \longrightarrow X$ and lemma 1.2, we have the following

Proposition 1.3. $E \underset{Z}{\sim} S^{2n+1} \times S^{2m-1}$.[1]

Remark. Let A be a graded ring in lemma 1.2. Then by [1] we see that there is a simply connected CW complex such that the rational cohomology ring is $A \otimes Q$. However, A is not necessarily the integral cohomology ring of a topological space, $A = Z[x, y]/(x^2 + xy + y^2, xy^2)$ is such an example.

2. It is well known that $SU(3) \underset{Z}{\sim} S^3 \times S^5$. But the difference between $SU(3)$ and $S^3 \times S^5$ reflects on the cohomology rings of the orbit spaces of free T^2-actions on $SU(3)$ and $S^3 \times S^5$.

Proposition 2.1. *Let X be the orbit space of a continuous free T^2-action on $S^3 \times S^5$. Then $H^*(X; Z_2) \cong Z_2[x, y] / (x^2, y^3)$ or $Z_2[x, y]/$*

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1) $X \underset{Z}{\sim} Y$ means that $H^*(X; Z)$ and $H^*(Y; Z)$ are isomorphic as rings.

$(x^2 + xy,\ y^3)$, *where* $\deg x = \deg y = 2$.

Proof. Since X is a mod 2 Poincaré space, the following are only possible types of the mod 2 cohomology rings of X:

1) $Z_2[x, y] / (x^2, y^3)$,
2) $Z_2[x, y] / (x^2 + xy, y^3)$ and
3) $Z_2[x, y] / (x^2 + xy + y^2, y^3)$.

Now, we show that the case 3) does not occur. If this were not so, we arrive at a contradiction as follows. Put $E = S^3 \times S^5$, and consider the mod 2 cohomology spectral sequence of the bundle $q: E \times_{T^2} E_{T^2} \longrightarrow B_{T^2}$. Since $E \times_{T^2} E_{T^2}$ is homotopy equivalent to X, we have the following commutative diagram:

$$\begin{array}{ccccc}
H^5(E; Z_2) & \longrightarrow & H^6(X, E; Z_2) & \xleftarrow{q^*} & H^6(B_{T^2}; Z_2) \\
\uparrow Sq^2 & & \uparrow Sq^2 & & \uparrow Sq^2 \\
H^3(E; Z_2) & \longrightarrow & H^4(X, E; Z_2) & \xleftarrow{q^*} & H^4(B_{T^2}; Z_2)
\end{array}$$

Let z be the generator of $E_4^{0,3} \cong H^3(E; Z_2)$. Then $d_4(z) = x^2 + xy + y^2 \in E_4^{4,0} \cong H^4(B_{T^2}; Z_2)$; and $Sq^2(x^2+xy+y^2) = x^2y + xy^2$. It is sufficient to show that ker q^* does not contain x^2y+xy^2, since $Sq^2 = 0$ on $H^3(E; Z_2)$. Let $\hat{E}_r^{p,q}$ be the spectral sequence of $(X, E) \longrightarrow (B_{T^2}; *)$. Then $q^*: H^6(B_{T^2}; Z_2) \cong \hat{E}_4^{6,0} \longrightarrow \hat{E}_5^{6,0} \subset H^6(X, E; Z_2)$. Hence $\ker q^* = \operatorname{Im} d_4$, where $d_4: \hat{E}_4^{2,3} \longrightarrow \hat{E}_4^{6,0}$. Since xz and yz generate $\hat{E}_4^{2,3}$, ker q^* is generated by $x(x^2+xy+y^2)$ and $y(x^2+xy+y^2)$, and hence $\ker q^* \not\ni x^2y + xy^2$. q. e. d.

Since $Sq^2 \neq 0$ in $H^*(SU(3); Z_2)$, the above proof gives also the following

Corollary 2.2. *Let* X *be the orbit space of a free* T^2*-action on* $SU(3)$. *Then* $H^*(X; Z_2) \cong Z_2[x, y]/(x^2 + xy + y^2, y^3)$.

Let T be a maximal torous of $SU(3)$, then it is well known that $H^*(SU(3)/T; Z) \cong Z[x, y]/(x^2 + xy + y^2, y^3)$. However we have the following

Corollary 2.3. *There is no free* T^2-action on $S^3 \times S^5$ *such that the cohomology ring of the orbit space is isomorphic to* $Z[x, y]/(x^2+xy+y^2, y^3)$.

In order to define linear free T^2-actions, let

$T^2 = \{(s, t) \mid s, t \in C, |s| = |t| = 1\}$,

$S^{2n+1} = \{(z_0, z_1, \ldots, z_n) \mid \sum_i |z_i|^2 = 1, \ z_i \in C\}$ and

$S^{2m+1} = \{(w_0, w_1, \ldots, w_m) \mid \sum_i |w_i|^2 = 1, \ w_i \in C\}$,

where C is the field of complex numbers. We define the linear free T^2-action $S^{2n+1} \times S^{2n+1}$ as follows:

$$\phi(s, t; z_0, z_1, \ldots, z_n; w_0, w_1, \ldots, w_m)$$
$$= (s^{k_0}t^{l_0}z_0, \ldots, s^{k_n}t^{l_n}z_n; s^{p_0}t^{q_0}w_0, \ldots, s^{p_m}t^{q_m}w_m)$$

where k_i, l_i, p_j and q_j are integers such that $k_iq_j - l_ip_j = \pm 1$. Let X_ϕ be the orbit space of this action.

Proposition 2.4. $H^*(X_\phi; Z) \cong Z[x, y]/(\prod_{i=0}^{n}(k_ix + l_iy)), \ \prod_{j=0}^{m}(p_jx + q_jy)$.

Proof. Let $\xi_i : (S^{2n+1} \times S^{2m+1}) \times T^2 C \longrightarrow X_\phi \ (i = 1, 2)$ be the complex line bundles where T^2-acts on C as usual multiplication by s and t respectively. Then the first Chern classes $C_1(\xi_1) = x$ and $C_1(\xi_2) = y$ generate $H^2(X; Z)$. Let η : $(S^{2n+1} \times S^{2m-1}) \times T^2 C^{n+1} \longrightarrow X_\phi$ be the complex vector bundle where T^2 acts on C^{n+1} as follows: $(s, t; v_0, \ldots, v_n) \longrightarrow (s^{k_0}t^{l_0}v_0, \ldots, s^{k_n}t^{l_n}v_n)$, $v_i \in C$. We then have $\eta = \xi_1^{k_0} \otimes \xi_2^{l_0} \oplus \ldots \oplus \xi_1^{k_n} \otimes \xi_2^{l_n}$. Since η has a non zero cross section, $C_{n+1}(\eta) = \prod_{i=0}^{n}(k_ix + l_iy) = 0$. Analogously we have $\prod_{i=0}^{m}(p_ix + q_iy) = 0$. These relations have no common divisor and hence the resultant is ± 1. This completes the proof by the proposition 1. 1. q. e. d.

In particular we see from the above that the integral cohomology rings of the linear free T^2-actions on $S^3 \times S^5$ are up to isomorphism the following: $Z[x, y]/(x^2, xy^2 + y^3)$ and $Z[x, y]/(x^2 + lxy, y_3)$, $l \geqq 0$. Now, we consider the homotopy types of the CW complexes having these rings as integral cohomology rings.

Theorem 2.5. *Let X and Y be simply connected finite CW complexes such that $X \underset{Z}{\sim} Y$. If $H^*(X; Z)$ is isomorphic to one of the following:*

1) $Z[x, y]/(x^2, xy^2 + y^3)$ *and*
2) $Z[x, y]/(x^2 + lxy, y^3)$, $\quad l \geqq 0$,

where $\deg x = \deg y = 2$, *then* Y *is homotopy equivelent to* X.

Proof. First we give a proof for the rings of type 2). Let X be a simply connected finite CW complex such that $X \underset{Z}{\sim} Z[x, y]/(x^2 + lxy, y^3)$. We can take X as a following normal complex:

$$X = S_1^2 \vee S_2^2 \cup_\alpha e_1^4 \cup_\beta e_2^4 \cup_\gamma e_6,$$

where the dual cohomology classes of S_1^2, S_2^2, e_1^4, e_2^4 and e_6 are x, y, xy, y^2 and xy^2 respectively. Let ι_1 and ι_2 be generators of $\pi_2(S_1^2)$ and $\pi_2(S_2^2)$ respectively. Then $\pi_3(S_1^2 \vee S_2^2) \cong Z \oplus Z \oplus Z$ is generated by $\iota_1 \circ \eta$, $\iota_2 \circ \eta$ and $[\iota_1, \iota_2]$ where $\eta \in \pi_3(S^2)$ is the class of the Hopf map. From the cohomology ring of the 4-skelton $X^{(4)}$ of X, we see $\alpha = l(\iota_1 \circ \eta) + [\iota_1, \iota_2]$ and $\beta = \iota_2 \circ \eta$. In order to determine some homotopy groups of $X^{(4)}$ we consider the following linear free T^2-action on $S^3 \times S^5$: $\phi(s, t\,; z_0, z_1\,; w_0, w_1, w_2) = (sz_0, st^l z_1\,; tw_0, tw_1, tw_2)$. Let X_0 be the orbit space of ϕ. Then, $X_0 \cong X^{(4)} \cup_i e^6$, since $X_0 \underset{Z}{\sim} X$. From the fibration $T^2 \longrightarrow S^3 \times S^5 \longrightarrow X_0$, we see that $\pi_3(X^{(4)}) \cong Z$ and $\pi_4(X^{(4)}) \cong Z_2$ with generators $\iota_1 \circ \eta$ and $\iota_1 \circ \eta \circ S(\eta) = \zeta$ respectively. The homotopy exact sequence of the pairs $(X_0, X^{(4)})$ gives the following split exact sequence :

$$0 \longrightarrow \pi_6(X_0, X^{(4)}) \longrightarrow \pi_5(X^{(4)}) \longrightarrow \pi_5(X_0) \longrightarrow 0,$$

where $\pi_6(X_0, X^{(4)}) \cong Z$ and $\pi_5(X_0) \cong Z \oplus Z_2$. Hence, we have $\pi_5(X^{(4)}) \cong Z \oplus Z \oplus Z_2$ with generators i, j and $\zeta \circ S^2(\eta)$ where j is a generator of $\pi_5(S_2^2 \cup_\beta e_2^4)$ $(S_2^2 \cup_\beta e_2^4 \simeq CP(2))$. Now, from the integral cohomology ring of X, there is only two possibilities ; $\gamma = i$ or $i + \zeta \circ S^2(\eta)$. Let $f : X^{(4)} \longrightarrow X^{(4)}/(S_1^2 \vee S_2^2 \cup_\beta e_2^4) = S^4$ be the natural projection and $r : X^{(4)} \longrightarrow X^{(4)} \vee S^4$ a deformation of $id. \times f : X^{(4)} \longrightarrow X^{(4)} \times S^4$. $\pi_5(X^{(4)} \vee S^4) \cong \pi_5(X^{(4)}) \oplus \pi_5(S^4) \oplus \pi_6(X^{(4)} \times S^4, X^{(4)} \vee S^4)$, where $[\iota_1, \iota_4]$ and $[\iota_2, \iota_4]$ generate $\pi_6(X^{(4)} \times S^4, X^{(4)} \vee S^4)$ (ι_4 is a generator of $\pi_4(S^4)$). Then we have $r_*(i) = i + f_*(i) + a[\iota_1, \iota_4] + b[\iota_2, \iota_4]$ for some integers a and b. Inorder to determine $f_*(i)$ we consider the map $\bar{f} : X_0 \longrightarrow S^4 \cup_{f_*(i)} e^6$ which is an extention of f. Let u and v be the dual cohomology classes of S^4 and e^6 in the latter complex. It is well known that $Sq^2(u) = v$ or 0, if either $f_*(i) \neq 0$ or 0. Since $\bar{f}^*(u) = xy$, $\bar{f}^*(v) = xy^2$ and $Sq^2(xy) = x^2y + xy^2 = (l+1)xy^2$, it follows that $f_*(i) \neq 0$ or 0 according to either l is even or odd. Let u and v also the dual classes of S^4 and e^6 in the complex $(X^{(4)} \vee S^4) \cup_{r_*(i)} e^6$. Then, $xy = av$ and $yu = bv$. Let $\bar{r} : X_0 \longrightarrow (X^{(4)} \vee S^4) \cup_{r_*(i)} e^6$ be an extention of r. Then, $\bar{r}^*(xu) = a\,xy^3$ and $\bar{r}^*(xu) = \bar{r}^*(x)\bar{r}^*(u) = x^2y = -lxy^2$, thus $a = -l$ and similarly $b = 1$. Let $\psi = (id. \vee \zeta) \circ r$ be the homotopy equivalance of $X^{(4)}$. Since $[\iota_2, \eta \circ S(\eta)] = 0$ and Whitehead products are natural, $[\iota_1, \zeta] = 0$. On the orther hand, by the formulas in [2], we have

$$\begin{aligned} [\iota_2, \zeta] &= [\iota_2, (\iota_1 \circ \eta) \circ S(\eta)] \\ &= [\iota_2, \iota_1 \circ \eta] \circ S^2(\eta) \\ &= ([\iota_2, \iota_1] \circ S(\eta) - [[\iota_1, \iota_2], \iota_1]) \circ S^2(\eta) \end{aligned}$$

$$= l\zeta \circ S^2(\eta).$$

Then, we have the following :

$$\psi_*(i) = (id. \vee \zeta)_*(i + f_*(i) + l[\iota_1, \iota_4] + [\iota_2, \iota_4])$$
$$= i + \zeta \circ f_*(i) + l[\iota_1, \iota_4] + [\iota_2, \iota_4])$$
$$= i + \zeta \circ S^2(\eta).$$

This implies that the two complexes $X^{(4)} \cup_i e^6$ and $X^{(4)} \cup_{(i+\zeta\circ S^2(\eta))} e^6$ are homotopy equivalent and thus completes the proof for the ring in 2). The proof for the ring of type 1) is easy, and hence we do not give it.

q. e. d.

Corollary 2. 6. *The homotopy types of the orbit spaces of linear free T^2-actions on $S^3 \times S^5$ are determined by its integral cohomology rings.*

Since $CP(3) \# CP(3)$ is the orbit space of a free T^2-action on $S^3 \times S^5$ [4], its homotopy type depends only on the integral cohomology ring.

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