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COMMUTATIVITY THEOREMS OF OUTCALT-YAQUB TYPE

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Throughout R will represent a ring. Let A be an additive subsemigroup of R, and N the set of all nilpotent elements in R. Given a non-empty subset S of R, we set $V_R(S)=\{x\in R \mid sx=xs$ for all $s\in S\}$ and $V_R^-(S)=\{x\in R \mid sx=-xs$ for all $s\in S\}$, and the subring generated by S will be denoted by $\langle S\rangle$.

In this paper, we consider the following conditions :

A$_1$) For every $x\in R$, $x-x^2x'\in A$ with some $x'\in\langle x\rangle$.

A$_2$) For every $x\in R$ and every positive integer n, $x-x^n x'\in A$ with some $x'\in\langle x\rangle$.

B) $x-y\in A$ implies $x^q=y^q$ with some prime number $q=q(x,y)$ or $xy=yx$.

B$_0$) $x-y\in A$ impies $x^2=y^2$ or $xy=yx$.

B$_1$) $x-y\in A$ and $y-z\in A$ imply $x^2=z^2$ or $xy=yx$.

B$_2$) $x-y\in A$ implies $x^2=y^2$ or both x, $y\in V_R(A)$.

B$_3$) Either R is commutative or $R=V_R^-(A)$ and $a^2=0$ for all $a\in A$.

C$_1$) For every $x\in R$, $x-x^{n+1}\in N$ with some positive integer $n=n(x)$ or $x\in V_R(A)$.

C$_2$) For every $x\in R$, $x^m=x^n$ with some distinct positive integers $m=m(x)$ and $n=n(x)$ or $x\in V_R(A)$.

D) $x-y\in A$ implies $[[x,y],y]=0$.

Remark. (a) The conditions A$_1$), B$_2$) and C$_2$) have been considered in [4]. Obviously, C$_2$) implies C$_1$).

(b) Let A' be the additive subgroup generated by A. If every element of A commutes with each other, then B$_2$) is equivalent to the same with A' instead of A.

(c) Let f be a ring homomorphism of R onto R^*. If R satisfies one of the conditions A$_1$) — D), then R^* does the same with $f(A)$ instead of A.

It is the purpose of this paper to prove the following commutativity theorems.

Theorem 1. *Suppose every element of* A *commutes with each other. If* A$_1$), B), C$_1$) *and* D) *are satisfied, then* R *is commutative.*

Theorem 2. *Let* A *be an additive subgroup of* R. *If* A$_1$) *and* C$_1$) *are satisfied, then* B$_1$)—B$_3$) *are equivalent.*

We shall present also a corollary to Theorem 1, which will improve [4, Theorem 1]. Moreover, by [2, Lemma 3 (1) and (2)], the corollary includes [2, Lemma 4 (3)] and Theorem 2 deduces [2, Theorem 2].

In preparation for the proofs of our theorems, we establish the following lemmas.

Lemma 1. (a) A$_1$) *and* A$_2$) *are equivalent. If* A$_1$) *is satisfied, then* N *is included in* A.

(b) *In general,* B$_3$) $\Longrightarrow$ B$_2$) $\Longrightarrow$ B$_1$) $\Longrightarrow$ B$_0$) $\Longrightarrow$ B). *If every element of* A *commutes with each other, then* B$_0$) – B$_2$) *are equivalent.*

(c) *If* B) *is satisfied, then for each* $x \in R$ *and* $a \in A$ *there exists a prime number* q *such that* $[x^q, a] = 0$.

(d) *If* R *contains* 1, *then* B$_0$) *implies* D).

Proof. (a) is almost evident.

(b) The first assertion can be easily seen. Now, suppose that A is commutative and B$_0$) is satisfied. Suppose $x-y \in A$ and $x^2 \neq y^2$. Then $xy = yx$. If $ax \neq xa$ with some $a \in A$, then $(x+a)x \neq x(x+a)$ implies $(x+a)^2 = x^2$. Since $(x-y)a = a(x-y)$, it follows that $ay \neq ya$, and therefore $(x+a)y \neq y(x+a)$. But then we have $y^2 = (x+a)^2 = x^2$. This contradiction means that $x, y \in V_R(A)$.

(c) Let $x \in R$, and $a \in A$. If $ax \neq xa$, then there exists a prime number q such that $(x+a)^q = x^q$. Hence, $x^q(x+a) = (x+a)^q(x+a) = (x+a)(x+a)^q = (x+a)x^q$, which simplifies to $x^q a = ax^q$.

(d) Suppose $x-y \in A$ and $xy \neq yx$. Then $x^2 = y^2$. Since $(x+1)-(y+1) \in A$, we have $(x+1)^2 = (y+1)^2$, whence it follows $2x = 2y$. Now, from those above, we readily obtain $[[x,y],y] = xy^2 + y^2x - 2yxy = 2x^3 - 2x^3 = 0$.

Lemma 2. *Suppose* A$_1$) *and* B) *are satisfied.*

(a) *Every idempotent of* R *is central.*

(b) *If every element of* A *commutes with each other, then* N *is an ideal of* R.

Proof. (a) Let e be an arbitrary idempotent of R. Then, by

Lemma 1 (a) and (c), $e \in V_R(A) \subseteq V_R(N)$. As is well known, the idempotent e commuting with all nilpotent elements is central.

(b) Let $x \in R$, and $a \in N$ with $a^m = 0$. By Lemma 1 (c), there exists a prime number q such that $(xa)^q a = a(xa)^q$. Then, one will easily see that $(xa)^{qm} = \{(xa)^{q-1}x\}^m a^m = 0$, and similarly $(ax)^{qm} = 0$. We have therefore seen that $Ra \subseteq N$ and $aR \subseteq N$. Since N is commutative as a subset of A (Lemma 1 (a)), N is evidently an ideal of R.

Lemma 3. *Let A be an additive subgroup of R. If* B$_1$) *is satisfied, then either $A \subseteq V_R(A)$ or $A \subseteq V_{\bar{R}}(A)$ and $a^2 = 0$ for all $a \in A$.*

Proof. Suppose there exist $a, b \in A$ such that $ab \neq ba$. Since $a+b \equiv a \equiv 0 \pmod{A}$ and $(a+b)a \neq a(a+b)$, by B$_1$) we have $(a+b)^2 = a^2 = 0$, and similarly $b^2 = 0$. From these it follows $ab = -ba$. Now, by making use of Brauer's trick, we readily see that A is anti-commutative. If c is an arbitrary element of A commuting with all elements of A, then $ac = ca = -ac$. Hence, $c^2 = (a+c)^2 - 2ac - a^2 = 0$.

Proof of Theorem 1. Since R is a subdirect sum of subdirectly irreducible rings by Birkhoff's theorem, we may in view of Remark (c) assume that R is subdirectly irreducible. According to Herstein's theorem [1, Theorem 19], it is enough to prove that A is included in the center of R. Suppose on the contrary that there exist $a \in A$ and $x \in R$ such that $ax \neq xa$. Then $x \notin V_R(A)$, and hence $x \notin N$ (Lemma 1 (a)). We consider the reduced ring $\bar{R} = R/N$ (Lemma 2 (b)). By C$_1$), $\bar{x} = \bar{x}^{n+1}$ for some positive integer n. Since $\bar{x}^n$ is a non-zero idempotent of $\bar{R}$ and R is subdirectly irreducible, any idempotent lifted from $\bar{x}^n$ must be 1 (see Lemma 2 (a)). Hence, x is a unit of R. We claim that $a(kx) \neq (kx)a$ for some $k > 1$. In fact, a cannot commute with both $2x$ and $3x$. Then, again by C$_1$), $k\bar{x} = (k\bar{x})^{m+1}$ with some positive integer m. Here, without loss of generality, we may assume that $m = n$. Then $(k^{n+1} - k)\bar{x} = 0$, which means that the characteristic of R is non-zero. Recalling that R is subdirectly irreducible, we conclude that $p^\alpha R = 0$ with some prime number p. By B), there exists a prime number q such that $(a+x)^q = x^q$. Then there holds $[x^q, a] = 0$ (see the proof of Lemma 1 (c)). Since $0 = [[a+x, x], x] = [[a, x], x]$ by D), an easy induction proves that $[x^h, a] = hx^{h-1}[x, a]$ for all positive integers h. In particular, $0 = [x^q, a] = qx^{q-1}[x, a]$, whence it follows $q[x, a] = 0$. Combining this with $p^\alpha R = 0$, we conclude that $p = q$. Obviously, $\langle \bar{x} \rangle$ is a finite field of characteristic q : $\langle \bar{x} \rangle = \mathrm{GF}(r)$, where $r = q^\beta$ and $\beta > 0$. Since $x - x^r \in N \subseteq A \subseteq V_R(A)$ (Lemma 1 (a)) and $x^r a = ax^r$, it follows at once $xa = ax$, which is a contradiction.

Corollary 1. *Suppose every element of A commutes with each other. If* A_1), B_0) *and* C_1) *are satisfied, then R is commutative.*

Proof. By Lemma 1 (d), the proof proceeds in the same way as that of Theorem 1 did.

Proof of Theorem 2. In virtue of Lemma 1 (b), it remains only to prove that B_1) implies B_3). By Lemma 3, either $A \subseteq V_R(A)$ or $A \subseteq V_R^-(A)$ and $a^2 = 0$ for all $a \in A$. If $A \subseteq V_R(A)$ then R is commutative by Corollary 1. Henceforth, we assume that $A \subsetneq V_R(A)$ but $A \subseteq V_R^-(A)$. Let $x \in R$ and $b \in A$ such that $xb \neq bx$. Then $(x+b)^2 = x^2$ by B_1). Hence we get $xb = -bx$. Now, by making use of Brauer's trick, we can easily see that $R = V_R(A)$ or $V_R^-(A)$. Since $A \not\subseteq V_R(A)$, R must be $V_R^-(A)$.

REFERENCES

[1] I. N. HERSTEIN: The structure of a certain class of rings, Amer. J. Math. **75** (1953), 864—871.

[2] Y. HIRANO and H. TOMINAGA: Two commutativity theorems for rings, Math. J. Okayama Univ. **20** (1978), 67—72.

[3] S. IKEHATA and H. TOMINAGA: A commutativity theorem, Math. Japonica 24 (1979), 29—30.

[4] D. L. OUTCALT and Adil YAQUB; A commutativity theorem for rings with constraints involving an additive semigroup, submitted.

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Added in proof. Obviously, the condition D) is satisfied if and only if $[[x,y],y] = 0$ for all $x \in A$ and $y \in R$. The principal theorem of [3] is still valid under the weaker hypothesis that $[[x,y],y] = 0$ for all $x \in N$ and $y \in R$.