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A COMMUTATIVITY THEOREM FOR *s*-UNITAL RINGS

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A ring R is called an s-unital ring if every element a of R belongs to $aR \cap Ra$. It is the purpose of this paper to prove the following commutativity theorem.

Theorem. *If R is an s-unital ring, then the following are equivalent*:

1) *R is commutative.*

2) *For each pair of elements x, y of R there exist relatively prime, posititive integers k and l such that $(xy)^k = (yx)^k$ and $(xy)^l = (yx)^l$.*

3) *For each finite subset F of R there exists a positive integer n such that $(xy)^k = (yx)^k$ for all $x, y \in F$ and all $k \geq n$.*

4) *There exist relativey prime, positive integers k and l such that $[x^k, y^k] = 0 = [x^l, y^l]$ for all x, $y \in R$.*

5) *There exists a positive integer n such that $[x^n, y^n] = 0 = [x^{n+1}, y^{n+1}]$ for all x, $y \in R$.*

Obviously, our theorem includes [2, Theorem] and [3, Theorem 2]. However, we borrow heavily from the papers [2] and [3] at various points. Among other things, [3, Theorem 1] plays an important role in our proof.

In what follows, R will represent a ring. The center of R and the Jacobson radical of R will be denoted by C and J, respectively. Let R° be the set of all quasi-regular elements of R. As is well known, R° is a group (the adjoint group of R) with respect to the circle composition defined by $x \circ y = x + y - xy$. If R is an s-unital ring then for each finite subset F of R there exists an element e of R such that $ex = xe = x$ for all $x \in F$ (see [1, Lemma 1]). This result will be freely used in the subsequent study.

Now, we begin with the next lemma.

Lemma 1. *Let e, x and y be elements of R, m an integer, and n a positive integer.*

(a) *If $ex = xe = x$, $ey = ye = y$ and $mx^n[x, y] = 0 = m(x+e)^n[x+e, y]$, then $m[x, y] = 0$.*

(b) *If $x[x, y] = [x, y]x$ then $[x^n, y] = nx^{n-1}[x, y]$.*

Proof. (a) We have $0=mx^{n-1}(x+e)^n[x+e,y]=mx^{n-1}[x,y]$ and $0=(-1)^n m(x+e)^{n-1}\{-e+(x+e)\}^n[x,y]=m(x+e)^{n-1}[x+e,y]$. Continuing this process, we obtain eventually $m[x,y]=0$.

(b) Since $[x^{n+1},y]=x[x^n,y]+[x,y]x^n$, the assertion can be shown by induction method.

Lemma 2. *Assume that an s-unital ring R satisfies the condition* 3) *in Theorem.*

(a) *R° is included in C.*

(b) *R/J is commutative.*

(c) *For each pair of elements x, y of R there exists a positive integer n such that $[x,y^n]=0$.*

Proof. (a) Let a be an arbitrary element of R°, and a' the quasi-inverse of a. As is well known, the map $\sigma: R\to R$ defined by $y\mapsto y-a'y-ya+a'ya$ is an automorphism of R. Let x be an arbitrary element of R. Then there exists an element e of R such that $ea=ae=a$ (which implies $ea'=a'e=a'$) and $ex=xe=x$, and there exists also $e'\in R$ such that $e'a=ae'=a$ and $e'(x+e)=(x+e)e'=x+e$. By the condition 3), we can find then a positive integer n such that for all $k\geq n$ there holds the following:

$$\begin{aligned}(e-a')x^k(e-a)&=\sigma(x^k)=\sigma(x)^k\\&=\{(e-a')x(e-a)\}^k\\&=\{(e-a)(e-a')x\}^k=x^k,\\(e'-a')(x+e)^k(e'-a)&=(x+e)^k.\end{aligned}$$

From the above we have $x^k(e-a)=(e-a)x^k$ and $(x+e)^k(e'-a)=(e'-a)(x+e)^k$. Hence, we obtain $x^ka=ax^k$ and $(x+e)^ka=a(x+e)^k$. By making use of these relations, we get $x^n[x,a]=x^{n+1}a-x^nax=x^{n+1}a-ax^{n+1}=0$ and $(x+e)^n[x+e,a]=0$. By Lemma 1 (a), it follows then that $[x,a]=0$.

(b) In virtue of (a), the proof is quite similar to that of [2, Claim 3].

(c) Choose an element e of R such that $ex=xe=x$ and $ey=ye=y$. By the condition 3), there exists a positive integer n such that $(xy)^n=(yx)^n$ and $\{(x+e)y\}^n=\{y(x+e)\}^n$. Then, $(xy)^nx=x(yx)^n=x(xy)^n$, and therefore $[(xy)^n,x]=0$. Noting that $(xy)^n-x^ny^n\in J\subseteq C$ by (a) and (b), we have $x^n[x,y^n]=[x,x^ny^n]=[x,(xy)^n]=0$. Similarly, we have $(x+e)^n[x+e,y^n]=0$. Hence, by Lemma 1 (a), it follows that $[x,y^n]=0$.

Lemma 3. *Assume that an s-unital ring R satisfies the condition* 5) *in Theorem.*

(a) R° *generates a commutative (multiplicative) semigroup.*
(b) R/J *is commutative.*
(c) J^2 *is included in* C.
(d) $[a, y^{n+1}] = 0$ *for all* $a \in J$ *and* $y \in R$.

Proof. (a) For $x \in R$, we define inductively $x^{(1)} = x$, $x^{(k)} = x^{(k-1)} \circ x$. Given $x, y \in R$, we choose an element e of R such that $ex = xe = x$ and $ey = ye = y$. As can be easily verified by induction method, there holds $x^{(k)} = e^k - (e-x)^k$ for all positive integers k. By the condition 5), we see that $x^{(n)} \circ y^{(n)} = y^{(n)} \circ x^{(n)}$ and $x^{(n+1)} \circ y^{(n+1)} = y^{(n+1)} \circ x^{(n+1)}$. Hence, by [3, Theorem 1], the adjoint group of R is commutative, i. e., $a + b - ab = b + a - ba$ for all a, $b \in R^{\circ}$. Now, it is evident that $ab = ba$.

(b) In virtue of (a), the proof is quite similar to that of Claim 2 in the proof of [3, Theorem 2].

(c) If a, $b \in J$ and $y \in R$, then by (a) we have $(ab)y = a(by) = (by)a = b(ya) = (ya)b = y(ab)$.

(d) Let a' be the quasi-inverse of a, and e an element of R such that $ea = ae = a$ and $ey = ye = y$. By (a), $[e - a, y^n]$ commutes with $e - a$. Then, by Lemma 1 (b), $0 = [(e-a)^n, y^n] = n(e-a)^{n-1}[e-a, y^n] = -n(e-a)^{n-1}[a, y^n]$. Hence, $0 = n(e-a')^{n-1}(e-a)^{n-1}[a, y^n] = ne^{2(n-1)}[a, y^n] = n[a, y^n]$, and similarly $0 = (n+1)[a, y^{n+1}]$. Since $J^2 \subseteq C$ by (c), the only terms in the expansion of $(y+a)^{n+1}$ which do not commute with y^{n+1} are those involving exactly one a. By making use of this fact and $n[a, y^n] = 0$, we see that

$$
\begin{aligned}
0 &= n\,[(y+a)^{n+1}, y^{n+1}] = n[\textstyle\sum_0^n y^{n-k} a y^k, y^{n+1}] \\
&= \textstyle\sum_0^n n y^{n-k} a y^{n+k+1} - \sum_0^n n y^{n-k+1} a y^{n+k} \\
&= nay^{2n+1} - ny^{n+1}ay^n = ny^{2n}[a, y].
\end{aligned}
$$

Hence, by Lemma 1 (a), it follows that $n[a, y] = 0$, and also $n[a, y^{n+1}] = 0$. We obtain therefore $[a, y^{n+1}] = n[a, y^{n+1}] + [a, y^{n+1}] = (n+1)[a, y^{n+1}] = 0$.

We can now complete the proof of our theorem.

Proof of Theorem. The proof of $2) \Longrightarrow 3)$ is given in [2, Claim 1], and the proof of $4) \Longrightarrow 5)$ is easy. It remains therefore to prove $3) \Longrightarrow 1)$ and $5) \Longrightarrow 1)$.

$3) \Longrightarrow 1)$ Given $x, y \in R$, we choose an element e of R such that $ex = xe = x$ and $ey = ye = y$. By Lemma 2 (c), we can easily see that there exists a positive integer m such that $[x, y^m] = 0 = [x, (y+e)^m]$. By the condition 3), there exists a positive integer n such that for each $k \geq n$

$$y^{mk}x^k = (y^m x)^k = \{(y^{m-1}x)y\}^k = y^{m-1}(xy^m)^{k-1}xy = y^{mk-1}x^k y,$$
$$(y+e)^{mk}x^k = (y+e)^{mk-1}x^k(y+e).$$

Then, $y^{mk-1}[y, x^k] = 0 = (y+e)^{mk-1}[y+e, x^k]$, and therefore $[y, x^k] = 0$ by Lemma 1 (a). Hence, $x^k[x, y] = x^{k+1}y - x^k yx = 0$. Now, repeat the above with x replaced by $x+e$ to obtain a positive integer $n' \geq n$ such that for each $h \geq n'$ there holds $(x+e)^h[x+e, y] = 0$. Again by Lemma 1 (a), we have then $[x, y] = 0$.

5) $\Longrightarrow$ 1) Let $x, y \in R$, and $a \in J$. By Lemma 3 (c) and (d), we have

$$\begin{aligned} 0 &= y\,[(y+a)^n, y^n]y = y[\textstyle\sum_0^{n-1} y^{n-k-1}ay^k, y^n]y \\ &= \textstyle\sum_0^{n-1} y^{2n-k+1}ay^k - \sum_0^{n-1} y^{2n-k}ay^{k+1} \\ &= y^{2n+1}a - y^{n+1}ay^n = y^{2n+1}a - ay^{2n+1}. \end{aligned}$$

Combining this with Lemma 3 (d), we obtain $0 = ay^{2n+2} - y^{2n+2}a = y^{2n+1}[a, y]$. Hence, by Lemma 1 (a), $[a, y] = 0$, which means $J \subseteq C$. Since R/J is commutative by Lemma 3 (b), there hold $x[x, y^n] = [x, y^n]x$ and $y[x, y] = [x, y]y$. Hence, by Lemma 1 (b), $0 = [x^n, y^n] = nx^{n-1}[x, y^n]$ and $[x, y^n] = ny^{n-1}[x, y]$. By the repeated use of Lemma 1 (a), we have then $0 = n[x, y^n] = n^2y^{n-1}[x, y]$ and $n^2[x, y] = 0$. Similarly, we have $(n+1)^2[x, y] = 0$. Since n^2 and $(n+1)^2$ are relatively prime, we conclude $[x, y] = 0$.

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